ZASTOSOWANIA MATEMATYKI
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E. PLESZCZYŃSKA (Warszawa)

FURTHER APPLICATIONS OF THE T^*-TEST TO TIME-SERIES ANALYSIS

1. Introduction. A test of the hypothesis that a discrete-parameter process is purely stochastic, i.e. has a zero expectation all the time, and that it is linear is described in [10], the alternative hypothesis being that the process is composed of a purely stochastic linear part and of a determinate periodic trend with an unspecified frequency and a zero temporal mean. The power of this test — which we shall call the *T^*-test* — tends to 1 as the sample size tends to infinity.

Section 3 of the present paper extends the class of alternative hypotheses for which the power of the T^*-test is asymptotically 1. These results are then used in Section 4 to describe a T^*-method of polynomial trend estimation in the case where the degree of the polynomial is not assumed to be known. It is shown (Theorem 4.12) that the proposed estimator is asymptotically distributed as the least-square estimator in the case of a known degree of the polynomial.

Methods of polynomial trend estimation published till now assume that the degree of the polynomial is known. In contrast with these methods, the procedure presently outlined not only dispenses with assumptions on the degree of the polynomial, but also includes a test for the presence, after its removal, of a determinate component of a wide class. Obviously, trend elimination is an essential prerequisite of spectral analysis.

2. The description of the T^*-test. For the convenience of the reader we shall repeat now the definition of the T^*-test given in [10].

It will be assumed that the investigated stochastic process $\{z_t\}$ is given by

$$z_t = x_t + y_t,$$

where $\{x_t\}$ is linear, i.e.

$$x_t = \sum_{q=-\infty}^{\infty} h_{t-q}\varepsilon_q \quad (t = 0, \pm 1, \pm 2, \ldots)$$

with h_k real for all k, $h_k = 0$ whenever $k < 0$, $\sum_{k=0}^{\infty}|h_k| < \infty$, $\sum_{k=1}^{\infty} kh_k^2 < \infty$, $\{\varepsilon_t\}$ being a sequence of independent equally distributed variables with zero means and finite moments up to the order $2M$ ($M \geqslant 1$), while $\{y_t\}$ is a real function defined on $t = 0, \pm 1, \ldots$

The above assumptions about $\{x_t\}$ will be denoted H_{2M} — as in [10].

Let $z_1, \ldots, z_N$ be a sample of size N, on the basis of which we want to test the hypothesis H_0: $\{z_t\} = \{x_t\}$ against the alternative hypothesis that $\{z_t\} = \{x_t + y_t\}$ ($y_t \not\equiv 0$).

For each N we chose two integers μ and ν in the following way: if there is precisely one pair of integers (p, q) such that $N = p \cdot q$ and $N^{2/5} \leqslant p < N^{1/2}$, we put $\nu = p$, $\mu = q$; if there is more than one such pair, we choose one with the smallest p; if there is none, we reduce the sample size as little as possible to find (p, q) satisfying the above conditions. So $\mu, \nu \to \infty$ with $N \to \infty$; in the sequel by N we denote the size of a sample after the reduction mentioned above, that is we shall assume $N = \mu\nu$.

We use the following notation introduced in [10]:

$$C^*_{s,N} = \frac{1}{N-|s|} \sum_{t=1}^{N-|s|} z_t z_{t+|s|} \qquad (|s| = 0, 1, \ldots, N),$$

$$S^*_N = \frac{1}{2} \sum_{s=1-N}^{N-1} \left(1 - \frac{|s|}{N}\right)^2 C^{*2}_{s,N},$$

$$C^*_{0,\nu,p} = \frac{1}{\nu} \sum_{t=1}^{\nu} z^2_{t+p\nu} \qquad (p = 0, 1, \ldots, \mu-1)$$

$$U^*_{\mu,\nu} = \frac{\nu}{\mu-1} \sum_{p=0}^{\mu-1} (C^*_{0,\nu,p} - C^*_{0,N})^2,$$

$$T^*_{\mu,\nu}(k) = \left(\frac{\mu}{2}\right)^{1/2} \left[1 - \frac{2S^*_N + kC^{*2}_{0,N}}{U^*_{\mu,\nu}}\right].$$

Similar symbols without asterisks will denote the same expressions formed with $\{x_t\}$ instead of $\{z_t\}$.

According to Proposition 7.1 in [10], if $\mathfrak{R}_r$ is the r-th cumulant of ε_t ($t = 1, 2, \ldots$), the distribution of $T(\mathfrak{R}_4 \mathfrak{R}_2^{-2})$ tends to be normal with a zero mean and a unit variance as $\mu, \nu \to \infty$. On the other hand,

$$T_{\mu,\nu}(k) = T_{\mu,\nu}(\mathfrak{R}_4 \mathfrak{R}_2^{-2}) - \left(\frac{\mu}{2}\right)^{1/2} \frac{(k - \mathfrak{R}_4 \mathfrak{R}_2^{-2})}{U_{\mu,\nu}} C^2_{0,N},$$

while $\operatorname*{plim}_{N\to\infty} C^2_{0,N}$ is a constant and, by Proposition 3.2 in [10], $\operatorname*{plim}_{\mu,\nu\to\infty} U_{\mu,\nu}$ is also a positive constant. Consequently

$$\operatorname*{plim}_{\mu,\nu\to\infty} T_{\mu,\nu}(k) = \begin{cases} +\infty & \text{for } k < \mathfrak{R}_4 \mathfrak{R}_2^{-2}, \\ -\infty & \text{for } k > \mathfrak{R}_4 \mathfrak{R}_2^{-2} \end{cases} \tag{2.1}$$

and, therefore, if ζ_a is defined by $\Phi(\zeta_a) = a$, where $\Phi(\cdot)$ is the standardized normal probability distribution function, we have

$$(2.2) \qquad \lim_{\mu,\nu\to\infty} P\big(T_{\mu,\nu}(k) < \zeta_a\big) = \begin{cases} 0 & \text{for } k < \mathfrak{R}_4\mathfrak{R}_2^{-2}, \\ a & \text{for } k = \mathfrak{R}_4\mathfrak{R}_2^{-2}, \\ 1 & \text{for } k > \mathfrak{R}_4\mathfrak{R}_2^{-2}. \end{cases}$$

A test in which the critical set is defined by $T^*_{\mu,\nu}(k) < \zeta_a$ $(-2 \leqslant k \leqslant \mathfrak{R}_4 \mathfrak{R}_2^{-2})$ is called a T^*-*test* (or, more precisely, a $T^*_{\mu,\nu}(k)$-*test with parameters* μ, ν *and* k). If there is no prior information about the value of $\mathfrak{R}_4 \mathfrak{R}_2^{-2}$, one can always choose $k = -2$, since -2 is an absolute lower bound for $\mathfrak{R}_4 \mathfrak{R}_2^{-2}$. According to (2.2), the level of significance of the T^*-test is asymptotically equal to a or 0 for $k = \mathfrak{R}_4 \mathfrak{R}_2^{-2}$ or $-2 \leqslant k < \mathfrak{R}_4 \mathfrak{R}_2^{-2}$, respectively.

It was shown ([10], Proposition 8.5) that if $\{y_t\}$ belongs to a set of periodic functions with a zero temporal mean (more precisely, to a set of functions satisfying the condition H' defined in [10] on p. 389) and if $\{x_t\}$ satisfies $\tilde{H}_8$, the power of the T^*-test tends to 1 with $N \to \infty$.

In Section 3 we shall define a more general set $\mathfrak{M}$ of functions $\{y_t\}$ for which the value of the asymptotic power of T^*-test is also 1.

3. The set of alternative hypotheses of the T^*-test. Let $\mathfrak{M}$ be a set of all real functions $\{y_t\}$ defined for all integral values of t and satisfying the following condition: there exists a positive constant Q such that

$$(3.1) \qquad \lim_{N\to\infty} N^{-1} L_N^{-4} \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 Q_{k,N}^2 = Q,$$

where

$$L_N = \max_{1\leqslant t\leqslant N} |y_t|, \qquad Q_{k,N} = \frac{1}{N-|k|} \sum_{t=1}^{N-|k|} y_t y_{t+|k|}.$$

If $|y_t| \leqslant L$ $(t = 0, \pm 1, \ldots)$, the factor L_N^{-4} in (3.1) may be omitted. We can take, as an example, functions $\{y_t\}$ satisfying H', which are bounded; it follows from (8.2) in [10] that these functions belong to $\mathfrak{M}$.

Class $\mathfrak{M}$ does not contain the sequence identically equal to zero, since in this case the left-hand side of (3.1) is not defined.

LEMMA 3.1. *All polynomials belong to* $\mathfrak{M}$.

Proof. Let $y_m(t) = a_0 + \ldots + a_m t^m$, with $a_m \neq 0$, be any polynomial. For $m = 0$, Q in (3.1) is equal to $2/3$. Clearly, for $m > 0$ there exists an integer N_0 with the property that $N > N_0$ entails

$$\max_{1\leqslant t\leqslant N} |y_m(t)| = |a_m N^m + \ldots + a_0|.$$

Computing the left-hand side of (3.1) we can, therefore, replace L_N^{-4} by $a_m^{-4}N^{-4m}$. Then, multiplied by a_m^4, this becomes a quartic form

$$\sum_{p,q,r,s=0}^{m} W_{p,q,r,s}a_p a_q a_r a_s$$

with

$$W_{p,q,r,s} = N^{-4m-3}\sum_{k=1-N}^{N-1}\sum_{t=1}^{N-|k|} t^p(t+|k|)^q \sum_{u=1}^{N-|k|} u^r(u+|k|)^s.$$

But

$$0<\sum_{t=1}^{N-|k|} t^p(t+|k|)^q \leqslant \sum_{t=1}^{N-|k|}(t+|k|)^{p+q}\leqslant \sum_{t=1}^{N} t^{p+q} = \frac{N^{p+q+1}}{p+q+1}+O(N^{p+q}),$$

whence

$$0<W_{p,q,r,s}\leqslant \sum_{k=1-N}^{N-1}\left[\frac{N^{p+q+r+s-4m-1}}{(p+q+1)(r+s+1)}+O(N^{p+q+r+s-4m-2})\right]$$

$$<\frac{2N^{p+q+r+s-4m}}{(p+q+1)(r+s+1)}+O(N^{p+q+r+s-4m-1}).$$

Thus $W_{p,q,r,s}\to 0$ unless $p=q=r=s=m$. In other words, putting $a_0=\ldots=a_{m-1}=0$ does not alter Q, and

$$Q = \lim_{N\to\infty} N^{-4m-3}\sum_{k=1-N}^{N-1}\Big(\sum_{t=1}^{N-|k|} t^m(t+|k|)^m\Big)^2$$

$$=\lim_{N\to\infty} N^{-4m-3}\left\{\frac{N^{4m+2}}{(2m+1)^2}+2\sum_{k=1}^{N-1}\left[\sum_{j=0}^{m}\binom{m}{j}k^{m-j}\frac{(N-k)^{m+j+1}}{(m+j+1)}\right]^2\right\}$$

$$=\lim_{N\to\infty} 2N^{-4m-3}\sum_{i,j=0}^{m}\binom{m}{i}\binom{m}{j}\frac{1}{(m+i+1)(m+j+1)}\sum_{k=1}^{N-1}k^{2m-i-j}(N-k)^{2m+i+j+2}$$

$$=\lim_{N\to\infty} 2N^{-4m-3}\sum_{i,j=0}^{m}\binom{m}{i}\binom{m}{j}\frac{1}{(m+i+1)(m+j+1)}\times$$

$$\times\sum_{l=0}^{2m+i+j+2}\binom{2m+i+j+2}{l}(-1)^{i+j-l}N^l\sum_{k=1}^{N-1}k^{4m+2-l}$$

$$=\lim_{N\to\infty} 2\sum_{i,j=0}^{m}\binom{m}{i}\binom{m}{j}\frac{1}{(m+i+1)(m+j+1)}\times$$

$$\times\sum_{l=0}^{2m+i+j+2}\binom{2m+i+j+2}{l}(-1)^{i+j-l}\frac{1}{4m+3-l}.$$

The last sum is nothing else but

$$\sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{k+l},$$

where $n = 2m+i+j+2$ and $k = 2m+1-i-j$; it is easily proved by induction that for any natural $k \geqslant 1$ and $n \geqslant 0$

$$\sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{k+l} = \frac{n!}{k(k+1)\cdot\ldots\cdot(k+n)}.$$

It follows that

$$Q = 2\sum_{i,j=0}^{m} \binom{m}{i}\binom{m}{j} \frac{(2m+i+j+2)!}{(m+i+1)(m+j+1)(2m-i-j+1)\cdot\ldots\cdot(4m+3)};$$

hence $Q > 0$, which completes the proof of Lemma 3.1.

THEOREM 3.2. *Under $\tilde{H}_8$ for any $\{y_t\} \in \mathfrak{M}$ the power of the T^*-test tends to* 1 *with $N \to \infty$.*

Proof. We show that under $\tilde{H}_8$ for any $\{y_t\} \in \mathfrak{M}$

$$\operatorname*{plim}_{\mu,\nu\to\infty} T^*_{\mu,\nu}(k) = -\infty,$$

so that by the definition of the critical set in the T^*-test the power of this test is asymptotically 1.

Put

$$D_N = \frac{2S^*_N + kC^{*2}_{0,N}}{U^*_{\mu,\nu}}, \quad F_N = N^{-1}L_N^{-4}S^*_N, \quad G_N = N^{-1}L_N^{-4}C^{*2}_{0,N},$$
$$H_N = N^{-1}L_N^{-4}U^*_{\mu,\nu},$$

so that $D_N = (2F_N + kG_N)/H_N$ and $T^*_{\mu,\nu}(k) = (\mu/2)^{1/2}(1-D_N)$.

We shall show that

$$\operatorname*{plim}_{N\to\infty} F_N = \tfrac{1}{2}Q > 0, \quad \operatorname*{plim}_{N\to\infty} G_N = \operatorname*{plim}_{N\to\infty} H_N = 0$$

and, consequently, $\operatorname*{plim}_{N\to\infty} D_N = \infty$ and a fortiori

$$\operatorname*{plim}_{\mu,\nu\to\infty} T^*_{\mu,\nu}(k) = -\infty.$$

As in [10], we put

$$A_{k,N} = \frac{1}{N-|k|}\sum_{t=1}^{N-|k|} x_t y_{t+|k|}, \quad B_{k,N} = \frac{1}{N-|k|}\sum_{t=1}^{N-|k|} y_t x_{t+|k|},$$
$$R_k = E(x_t x_{t+k}),$$

$$(3.2)\qquad V_s[i(1),\ldots,i(s)] = \sum_{q=-\infty}^{\infty} h_{i(1)-q}\cdot\ldots\cdot h_{i(s)-q}\qquad (s=2,3,\ldots),$$

$$(3.3)\qquad v_{0,0} = \Re_4\Re_2^{-2}R_0^2+2\sum_{q=-\infty}^{\infty} R_q^2.$$

We use Proposition 2.1 of [10], which states that under $\tilde{H}_{2M}$ $(M \geqslant 1)$ for every r $(1 \leqslant r \leqslant 2M)$ and for every set of integers $i(1),\ldots,i(r)$

$$(3.4)\qquad E(x_{i(1)}\cdot\ldots\cdot x_{i(r)}) = \sum \Re_{r_1}V_{r_1}\big(i(l_1),\ldots,i(l_{r_1})\big)\cdot\ldots\cdot\Re_{r_k}V_{r_k}\big(i(l_{r-r_k+1}),\ldots,i(l_r)\big),$$

where the sum is extended over all different groupings of the set $I = \langle i(1),\ldots,i(r)\rangle$, i.e. over all sets of disjoint subsets of I whose union is I, two groupings being regarded as distinct if, for some m and n $(m,n=1,\ldots,r)$, $i(m)$ and $i(n)$ belong to the same subset in one grouping, and to two different ones in another. Here $\langle l_1,\ldots,l_r\rangle$ denotes any permutation of the numbers $\langle 1,\ldots,r\rangle$ corresponding to a given grouping, i.e. such that the sets $\langle i(l_1),\ldots,i(l_{r_1})\rangle,\ldots,\langle i(l_{r-r_k+1}),\ldots,i(l_r)\rangle$ form this particular grouping. In view of the symmetry of the V's with respect to their arguments, any two permutations corresponding to the same grouping would yield the same term, so that, for any grouping, the choice of the permutation representing it does not matter, and, therefore, the sum is well defined. The variables $i(1),\ldots,i(r)$ must be regarded as distinct although in the end any values, not necessarily distinct ones, can be substituted for them. It should be noted that in the right-hand side of (3.4) we can confine ourselves to groupings in which each set is composed of more than one element, since $\Re_1 = 0$.

We shall also need the following formula immediately obtained from (3.2):

$$(3.5)\qquad \Big|\sum_{j(1)=1}^{N_1}\ldots\sum_{j(l)=1}^{N_l} V_l\big(j(1),\ldots,j(l)\big)\Big| \leqslant \min(N_1,\ldots,N_l)\Big(\sum_{k=-\infty}^{\infty}|h_k|\Big)^l.$$

Turning to the sequence $\{E(F_N)\}$, we have ([10], p. 401):

$$(3.6)\qquad S_N^* = S_N+\sum_{k=1-N}^{N-1}\Big(1-\frac{|k|}{N}\Big)^2(A_{k,N}+B_{k,N})C_{k,N}+ \\ +\frac{1}{2}\sum_{k=1-N}^{N-1}\Big(1-\frac{|k|}{N}\Big)^2(A_{k,N}+B_{k,N})^2+\sum_{k=1-N}^{N-1}\Big(1-\frac{|k|}{N}\Big)^2 C_{k,N}Q_{k,N}+ \\ +\sum_{k=1-N}^{N-1}\Big(1-\frac{|k|}{N}\Big)^2(A_{k,N}+B_{k,N})Q_{k,N}+\frac{1}{2}\sum_{k=1-N}^{N-1}\Big(1-\frac{|k|}{N}\Big)^2 Q_{k,N}^2.$$

Since for $\{y_t\} \in \mathfrak{M}$ the limit of the last term divided by NL_N^4 is equal to $\frac{1}{2}Q > 0$, it suffices to show that the expectations of the remaining terms divided by NL_N^4 vanish in the limit.

We have

$$\lim_{N\to\infty} E(S_N/NL_N^4) = 0,$$

because ([2], p. 159)

$$\lim_{N\to\infty} ES_N = \sum_{q=-\infty}^{\infty} R_q^2.$$

Furthermore,

$$N^{-1}L_N^{-4}\left|E\left(\frac{1}{2}\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 A_{k,N}C_{k,N}\right)\right|$$
$$\leqslant \frac{1}{2}N^{-3}L_N^{-3}|\mathfrak{R}_3| \sum_{k=1-N}^{N-1}\sum_{s,t=1}^{N-|k|} |V_3(s-t, 0, |k|)|$$
$$\leqslant N^{-2}L_N^{-3}|\mathfrak{R}_3| \sum_{k=1-N}^{N-1}\sum_{j=1-N-|k|}^{N-|k|-1} |V_3(j, 0, |k|)|$$
$$\leqslant N^{-2}L_N^{-3}|\mathfrak{R}_3| \left(\sum_{k=0}^{\infty}|h_k|\right)^3 = O(N^{-2}L_N^{-3}),$$

and we obtain a similar result replacing $A_{k,N}$ by $B_{k,N}$; it follows that the expectation of the second term in (3.6) divided by NL_N^4 tends to zero. The same is also true for the third term, because

$$N^{-1}L_N^{-4}E\left(\frac{1}{2}\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 A_{k,N}^2\right) \leqslant \frac{1}{2}N^{-3}L_N^{-2}\sum_{k=1-N}^{N-1}\sum_{s,t=1}^{N-|k|}|R_{s-t}|$$
$$\leqslant N^{-1}L_N^{-2}\sum_{q=-\infty}^{\infty}|R_q| = O(N^{-1}L_N^{-2})$$

and the result is the same when $A_{k,N}^2$ is replaced by $B_{k,N}^2$ or $A_{k,N}B_{k,N}$.

Since for the fourth term we have

$$N^{-1}L_N^{-4}\left|E\left(\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 C_{k,N}Q_{k,N}\right)\right| \leqslant N^{-1}L_N^{-2}\sum_{q=-\infty}^{\infty}|R_q| = O(N^{-1}L_N^{-2})$$

and since the fifth term has a zero expectation, it follows that

$$\lim_{N\to\infty} EF_N = \tfrac{1}{2}Q > 0.$$

Now we show that var $F_N = O(N^{-1}L_N^{-2})$. We will examine variances of every term of (3.6) divided by NL_N^4, starting with the fifth term. Put

$$\gamma_t = N^{-1}\sum_{k=t-N}^{N-t} y_{t+|k|}\left(1-\frac{|k|}{N}\right)Q_{k,N}.$$

Then $|\gamma_t| \leqslant L_N^3$ for $0 \leqslant t \leqslant N$ and

$$\operatorname{var}\left(N^{-1}L_N^{-4}\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 A_{k,N}Q_{k,N}\right)$$
$$= N^{-2}L_N^{-8}E\left(\sum_{t=1}^{N}\gamma_t x_t\right)^2 = N^{-2}L_N^{-8}\sum_{s,t=1}^{N}\gamma_t\gamma_s R_{s-t}$$
$$\leqslant N^{-1}L_N^{-2}\sum_{q=-\infty}^{\infty}|R_q| = O(N^{-1}L_N^{-2});$$

a similar result is obtained when $A_{k,N}$ is replaced by $B_{k,N}$ and it follows by the Schwarz inequality that the covariance of the terms in $A_{k,N}$ and $B_{k,N}$ is at most of the order $N^{-1}L_N^{-2}$, so that the variance of the fifth term divided by NL_N^4 is $O(N^{-1}L_N^{-2})$.

From [8] we know that var $S_N = O(N^{-1})$ and hence the variance of the first term of F_N is $O(N^{-3}L_N^{-8})$. Now we shall investigate the third term. We have

$$\text{(3.7)}\quad \operatorname{var}\left(\frac{1}{2}\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 A_{k,N}^2\right)^2$$
$$= \frac{1}{4}N^{-4}\sum_{k,l=1-N}^{N-1}\sum_{s,t=1}^{N-|k|}\sum_{u,w=1}^{N-|l|} y_{s+|k|}y_{t+|k|}y_{u+|l|}y_{w+|l|}[E(x_sx_tx_ux_w)-$$
$$-E(x_sx_t)E(x_ux_w)].$$

The expression in square brackets is equal to

$$\mathfrak{R}_4 V_4(s,t,u,w)+\mathfrak{R}_2^2\big(V_2(s,u)V_2(t,w)+V_2(s,w)V_2(t,u)\big);$$

therefore, the left-hand side of (3.7) is not greater than

$$\tfrac{1}{4}N^{-4}L_N^4\sum_{k,l=1-N}^{N-1}\left[|\mathfrak{R}_4|(N-|k|)\Big(\sum_{q=0}^{\infty}|h_q|\Big)^4+2\mathfrak{R}_2^2(N-|k|)^2\Big(\sum_{q=\infty}^{\infty}|R_q|\Big)^2\right] = O(L_N^4).$$

A similar result is obtained when $A_{k,N}^2$ is replaced by $B_{k,N}^2$ and therefore, by applying the Schwarz inequality to the term in $A_{k,N}B_{k,N}$ we infer that the variance of the third term of (4.6) divided by NL_N^4 is $O(N^{-2}L_N^{-4})$.

Reasoning in the same way as in [10], p. 404, we can show that the variance of the second term divided by NL_N^4 is also $O(N^{-2}L_N^{-4})$. Similarly, we can apply the argument of [10], p. 405 (with L_N^2 taken as an upper bound of $Q_{k,N}$ for all k and N, $|k| < N$), to show that the variance of the fourth term divided by NL_N^4 is $O(N^{-2}L_N^{-4})$. The variance of the last term of (3.6) is zero; hence, compared with the variance of the fifth term divided by NL_N^4, the variances of the other similarly divided terms are asymptotically negligible and $\operatorname{var} F_N = O(N^{-1}L_N^{-2})$. Consequently, in view of $\lim_{N\to\infty} EF_N = \frac{1}{2}Q$, we have

$$\operatorname*{plim}_{N\to\infty} F_N = \tfrac{1}{2}Q. \tag{3.8}$$

Now we shall show that $\operatorname*{plim}_{N\to\infty} G_N = 0$; as is well known, it is sufficient to show that $\operatorname*{l.i.m.}_{N\to\infty} G_N = 0$. We can write

$$G_N = N^{-1}L_N^{-4}(C_{0,N}+2A_{0,N}+Q_{0,N})^2.$$

But

$$N^{-2}L_N^{-8}EC_{0,N}^4 = O(N^{-2}L_N^{-8}), \qquad N^{-2}L_N^{-8}Q_{0,N}^4 \leqslant N^{-2},$$

$$N^{-2}L_N^{-8}EA_{0,N}^4 = N^{-6}L_N^{-8}\sum_{s,t,u,w=1}^{N} y_s y_t y_u y_w E(x_s x_t x_u x_w)$$

$$\leqslant N^{-6}L_N^{-4}\sum_{s,t,u,w=1}^{N}\{|\mathfrak{R}_4||V_4(s,t,u,w)|+\mathfrak{R}_2|V_2(s,u)V_2(t,w)|+$$

$$+\mathfrak{R}_2^2|V_2(s,w)V_2(t,u)|+\mathfrak{R}_2^2|V_2(s,t)V_2(u,w)|\}$$

$$\leqslant N^{-6}L_N^{-4}\Big(N|\mathfrak{R}_4|\Big(\sum_{k=0}^{\infty}|h_k|\Big)^4+3\mathfrak{R}_2^2N^2\Big(\sum_{k=-\infty}^{\infty}|R_k|\Big)^2\Big) = O(N^{-4}L_N^{-4}).$$

Applying the Schwarz inequality several times, we find $EG_N^2 = O(N^{-2})$; consequently,

$$\operatorname*{l.i.m.}_{N\to\infty} G_N = 0. \tag{3.9}$$

In order to prove that $\operatorname*{plim}_{N\to\infty} H_N = 0$, it suffices to show that EH_N and $\operatorname{var} H_N$ tend to zero with $\mu,\nu\to\infty$. But according to formulas in the proof of Lemma 5.2 in [10] we can write

$$H_N = N^{-1}L_N^{-4}\mathscr{U}_{\mu,\nu}^* = N^{-1}L_N^{-4}\frac{\mu}{\mu-1}\Big[\mathscr{U}_{\mu,\nu}^* - \nu(C_{0,N}^*-EC_{0,N}^*)^2+ \tag{3.10}$$

$$+\frac{\nu}{\mu}\sum_{p=0}^{\mu-1}(Q_{0,\nu,p}-Q_{0,N})^2+2\frac{\nu}{\mu}\sum_{p=0}^{\mu-1}(C_{0,\nu,p}^*-EC_{0,\nu,p}^*)(Q_{0,\nu,p}-Q_{0,N})\Big],$$

where

$$Q_{0,\nu,p} = \nu^{-1}\sum_{t=1}^{\nu} y_{t+p\nu}^2,$$

$$A_{0,\nu,p} = \nu^{-1}\sum_{t=1}^{\nu} y_{t+p\nu}x_{t+p\nu} \quad (p = 0, 1, \ldots, \mu-1),$$

$$\mathscr{U}_{\mu,\nu}^* = \mathscr{U}_{\mu,\nu}+4\mathscr{U}_{\mu,\nu}^{(1)}+4\mathscr{U}_{\mu,\nu}^{(2)},$$

$$\mathscr{U}_{\mu,\nu} = \frac{\nu}{\mu}\sum_{p=0}^{\mu-1}(C_{0,\nu,p}-R_0)^2,$$

$$\mathscr{U}_{\mu,\nu}^{(1)} = \frac{\nu}{\mu}\sum_{p=0}^{\mu-1}A_{0,\nu,p}^2,$$

$$\mathscr{U}_{\mu,\nu}^{(2)} = \frac{\nu}{\mu}\sum_{p=0}^{\mu-1}(C_{0,\nu,p}-R_0)A_{0,\nu,p}.$$

According to Lemma 3.1 in [10], $E\mathscr{U}_{\mu,\nu} = O(1)$; furthermore,

$$|E\mathscr{U}_{\mu,\nu}^{(1)}| \leqslant \frac{\nu}{\mu}\sum_{p=0}^{\mu-1}L_N^2\nu^{-2}\sum_{s,t=1}^{\nu}|R_{s-t}| \leqslant L_N^2\sum_{q=-\infty}^{\infty}|R_q| = O(L_N^2),$$

$$|E\mathscr{U}_{\mu,\nu}^{(2)}| \leqslant \mu^{-1}\nu^{-1}\sum_{p=0}^{\mu-1}L_N\sum_{t,u=1}^{\nu}|Ex_{t+p\nu}^2x_{u+p\nu}|$$

$$< 2\mu^{-1}\sum_{p=0}^{\mu-1}L_N|\mathfrak{R}_3|\sum_{q=-\infty}^{\infty}|V_3(0,0,q)| = O(L_N)$$

and, consequently, $E\mathscr{U}_{\mu,\nu}^* = O(L_N^2)$. Now

$$(3.11) \quad E\nu(C_{0,N}^*-EC_{0,N}^*)^2 = \nu E(C_{0,N}-R_0)^2+4\nu E(C_{0,N}-R_0)A_{0,N}+4\nu EA_{0,N}^2.$$

Since

$$E(C_{0,N}-R_0)^2 = O(N^{-1}), \qquad EA_{0,N}^2 \leqslant L_N^2N^{-2}\sum_{s,t=1}^{N}|R_{s-t}| = O(N^{-1}L_N^2),$$

it follows that $|E(C_{0,N}-R_0)A_{0,N}|$ is $O(N^{-1}L_N)$; hence the left-hand side of (3.11) is $O(L_N^2\mu^{-1})$. The third term in brackets in (3.10) is not random and it is at most $4\nu L_N^4$, while the mean of the fourth term is zero. Consequently, $EH_N = O(\mu^{-1})$. Now we turn to var H_N. From Lemma 3.1

in [10] we know that $\operatorname{var} \mathscr{U}_{\mu,\nu} = O(\mu^{-1})$. On the other hand,

$$\operatorname{var} \mathscr{U}^{(1)}_{\mu,\nu} = N^{-2} \sum_{p,q=0}^{\mu-1} \sum_{s,t,u,w=1}^{\nu} y_{s+p\nu} y_{t+p\nu} y_{u+q\nu} y_{w+q\nu} \times$$

$$\times [E(x_{s+p\nu} x_{t+p\nu} x_{u+q\nu} x_{w+q\nu}) - E(x_{s+p\nu} x_{t+p\nu}) E(x_{u+q\nu} x_{w+q\nu})]$$

$$\leqslant N^{-2} L_N^4 \sum_{p,q=0}^{\mu-1} \sum_{s,t,u,w=1}^{\nu} |\mathfrak{K}_4 V_4(s-u+(p-q)\nu, t-u+(p-q)\nu, 0, w-u) +$$

$$+ 2R_{s-u+(p-q)\nu} R_{t-w+(p-q)\nu}| .$$

Now if $s-u+(p-q)\nu = k$ and $t-w+(p-q)\nu = l$, this leaves us with at most 2μ joint choices of p and q, and when p and q are given, we can choose each pair (s, u) and (t, w) in at most ν ways. Thus a term $R_k R_l$ repeats itself at most $2\mu\nu^2$ times. Similarly, if the arguments in V_4 are given, we have at most 2μ choices for the pair (p, q). Furthermore, we have at most ν choices for the pair (w, u) and this choice determines s and t. Thus a V_4 with a given set of arguments repeats itself at most $2\mu\nu$ times. Thus

$$\operatorname{var} \mathscr{U}^{(1)}_{\mu,\nu} \leqslant L_N^4 \Big\{ [2_\mu^{-1} \nu^{-1} |\mathfrak{K}_4| \Big(\sum_{k=0}^{\infty} |h_k| \Big)^4 + 4_\mu^{-1} \Big(\sum_{k=-\infty}^{\infty} |R_k| \Big)^2 \Big\} = O(\mu^{-1} L_N^4).$$

An argument similar to that of p. 397 in [10] leads to $\operatorname{var} \mathscr{U}^{(2)}_{\mu,\nu} = O(\mu^{-1} L_N^2)$, so that the variances of $\mathscr{U}_{\mu,\nu}$, $\mathscr{U}^{(1)}_{\mu,\nu}$ and $\mathscr{U}^{(2)}_{\mu,\nu}$ (and, consequently, all covariances) are at most $O(\mu^{-1} L_N^4)$; hence $\operatorname{var} \mathscr{U}^*_{\mu,\nu} = O(\mu^{-1} L_N^4)$. It is known (see e.g. (20) in [11]) that $E(C_{0,N} - R_0)^4 = O(N^{-2})$, while it has been noted that $EA_{0,N}^4 = O(N^{-2} L_N^4)$, so that by the Schwarz inequality $E(C^*_{0,N} - EC^*_{0,N})^4 = E(C_{0,N} - R_0 + 2A_{0,N})^4 = O(N^{-2} L_N^4)$, and, therefore, according to (3.11),

$$\operatorname{var} [\nu(C^*_{0,N} - EC^*_{0,N})^2] = O(\mu^{-2} L_N^4).$$

The variance of the third term in square brackets in (3.10) is zero. The last term can be treated in the same way as a similar expression in [10], p. 400: we can decompose it into two terms,

$$2\frac{\nu}{\mu} \sum_{p=0}^{\mu-1} (C_{0,\nu,p} - R_0)(Q_{0,\nu,p} - Q_{0,N}) \quad \text{and} \quad 4\frac{\nu}{\mu} \sum_{p=0}^{\mu-1} A_{0,\nu,p} (Q_{0,\nu,p} - Q_{0,N}),$$

where

$$Q_{0,\nu,p} = \nu^{-1} \sum_{t=1}^{\nu} y^2_{t+p\nu} \quad (p = 0, \ldots, \mu-1).$$

The expectations of the squares of these terms will not be smaller in absolute value if $\{h_k\}$ is replaced by $\{|h_k|\}$, $\mathfrak{R}_4$ by $|\mathfrak{R}_4|$ and $|Q_{0,\nu,p}-Q_{0,N}|$ by $2L_N^2$, because we have, for instance, for the first term

$$E\left[2\frac{\nu}{\mu}\sum_{p=0}^{\mu-1}(C_{0,\nu,p}-R_0)(Q_{0,\nu,p}-Q_{0,N})\right]^2$$

$$\leqslant 4\nu^2\mu^{-2}\sum_{p,q=0}^{\mu-1}|Q_{0,\nu,p}-Q_{0,N}|\cdot|Q_{0,\nu,q}-Q_{0,N}|\cdot|E(C_{0,\nu,p}-R_0)(C_{0,\nu,q}-R_0)|,$$

and $E(C_{0,\nu,p}-R_0)(C_{0,\nu,q}-R_0)$ is a sum of products of $\mathfrak{R}_4$ or $\mathfrak{R}_2^2$ and $V_4(\cdot)$ or $V_2(\cdot)V_2(\cdot)$. Denote by $\tilde{C}_{0,\nu,p}$, $\tilde{C}_{0,N}$, $\tilde{A}_{0,\nu,p}$, $\tilde{A}_{0,N}$, $\tilde{R}_0$ and $\{\tilde{x}_t\}$ the expressions arising out of $C_{0,\nu,p}$, $C_{0,N}$, $A_{0,\nu,p}$, $A_{0,N}$, R_0 and $\{x_t\}$ respectively as a result of these changes. The process $\{\tilde{x}_t\}$ is still linear and, therefore, $\operatorname{var}\tilde{C}_{0,N} = O(N^{-1})$ and

$$E\left(\frac{\tilde{x}_1+\ldots+\tilde{x}_N}{N}\right)^2 = O(N^{-1});$$

it follows that the expectation of the square of the first term is bounded by

$$16L_N^4\nu^2\mu^{-2}E(\mu\tilde{C}_{0,N}-\mu\tilde{R}_0)^2 = O(\nu\mu^{-1}L_N^4),$$

and the expectation of the square of the second term is bounded by

$$E\left(4\frac{\nu}{\mu}(2L_N^2)\sum_{p=0}^{\mu-1}\tilde{A}_{0,\nu,p}\right)^2 \leqslant 64L_N^4\mu^{-2}E\Big(\sum_{p=0}^{\mu-1}\sum_{t=1}^{\nu}\tilde{x}_{t+p\nu}\Big)^2$$

$$= 64L_N^4\nu^2E\left(\frac{\tilde{x}_1+\ldots+\tilde{x}_N}{N}\right)^2 = O(\nu\mu^{-1}L_N^4).$$

Consequently, the variance of the last term in brackets of (3.10) is $O(\nu\mu^{-1}L_N^4)$, and since the variances of other terms in (3.10) compared with it are asymptotically negligible, we have $\operatorname{var}H_N = O(\nu^{-1}\mu^{-3}L_N^{-4})$; therefore $\operatorname*{plim}_{N\to\infty}H_N = 0$. In view of (3.8) and (3.9) it follows that $\operatorname*{plim}_{N\to\infty}D_N = +\infty$ and Theorem 3.2 is a direct consequence of it. Now we shall prove a very intuitive property of the power function of the T^*-test in the case of functions $\{y_t\}$ which differ very slightly from zero: the power of the test for finite samples differs very slightly from the level of significance.

Let $\{\delta_n\}$ be an increasing sequence of natural numbers and denote by $T^*_{\mu,\nu}(k|\{\delta_n\})$ the function $T^*_{\mu,\nu}(k)$ with z_t replaced by x_t+y_t/δ_n.

THEOREM 3.3. *For any given N, k, $\{y_t\}$ and $\{\delta_n\}$, the sequence of the distribution functions of $T^*_{\mu,\nu}(k|\{\delta_n\})$ tends, with $n\to\infty$, to the distribution function of $T_{\mu,\nu}(k)$ at every continuity point of the latter.*

Proof. For each $1 \leqslant t \leqslant N$

$$\lim_{n\to\infty} \frac{y_t'}{\delta_n} = 0,$$

so that by the extension to the multivariate case of the convergence theorem given by Cramér ([3], 20.6) the sequence of the distribution functions of $(x_1+y_1/\delta_n, \ldots, x_N+y_N/\delta_n)$ tends to the distribution function of $(x_1, \ldots, x_N)$. The function $T_{\mu,\nu}(k|\{\delta_n\})$ being a continuous function of $\{x_1+y_1/\delta_n, \ldots, x_N+y_N/\delta_n\}$, Theorem 3.3 follows, then, from Theorem 2 in [6]. In a similar way it can be proved that by making the difference between any given functions $\{y_t^{(1)}\}$ and $y_t^{(2)}\}$ sufficiently small, the absolute difference of the powers of T^*-test for both functions can be made arbitrarily small.

It should be pointed out that in the case of an exponential trend $y_t = a^{at}$ $(a \neq 0)$ the power of the T^*-test is asymptotically equal to zero, which can be easily checked. This restricts the applications of the test, but nevertheless the area of such applications is ample in view of the fact that polynomials and periodic functions belong to the class $\mathfrak{M}$.

4. The T^*-method of polynomial trend estimation in the case of an unknown degree. We shall consider the case of $z_t = x_t+y_m(t)$, where the degree m of the polynomial $y_m(t)$ is finite but unknown. This form of $\{z_t\}$ will be assumed throughout this section. It was shown in part 3 that $\{y_m(t)\} \in \mathfrak{M}$.

When m is assumed to be known, the least-square method is usually recommended (see [3], p. 126). A definition of a particular least-square estimator using orthogonal polynomials was given in [6], p. 143. Now we shall repeat this definition with a few minor alternations dictated by the slightly different nature of the problem tackled. Let $\{\Phi_q(t)\}$, $q = 0, \ldots, m$, be the sequence of the $m+1$ first Chebyshev polynomials orthogonal on $1, 2, \ldots, N$, that is

$$\Phi_q(t) = \sum_{s=0}^{q} (-1)^{q-s} \frac{(q!)^2(q+s)!(N-s-1)!}{(2q)!(s!)^2(q-s)!(N-q-1)!} (t-1)^{(s)}, \tag{4.1}$$

where $u^{(s)} = u(u-1) \cdot \ldots \cdot (u-s+1)$.

If

$$b_q = \frac{(q!)^4}{(2q)!(2q+1)!} N(N^2-1^2) \cdot \ldots \cdot (N^2-q^2), \tag{4.2}$$

then

$$\sum_{t=1}^{N} \Phi_q(t)\Phi_i(t) = \begin{cases} 0 & \text{for } i \neq q, \\ b_q & \text{for } i = q. \end{cases} \tag{4.3}$$

The polynomial $y_m(t)$ can be represented in the form

$$y_m(t) = \sum_{q=0}^{m} A_q \Phi_q(t),$$

where

$$A_q = \frac{1}{b_q} \sum_{t=1}^{N} y_m(t) \Phi_q(t), \tag{4.4}$$

and the least-square unbiased estimators $\hat{A}_q$ of A_q are

$$\hat{A}_q = \frac{1}{b_q} \sum_{t=1}^{N} z_t \Phi_q(t) = A_q + a_q, \tag{4.5}$$

where

$$a_q = \frac{1}{b_q} \sum_{t=1}^{N} x_t \Phi_q(t). \tag{4.6}$$

Therefore

$$\hat{y}_m(t) = \sum_{q=0}^{m} \hat{A}_q \Phi_q(t) \tag{4.7}$$

is a least-square estimator of $y_m(t)$ in the case of a known m.

Denote by $\{z_t(j)\}$ the j-th residuum of the process $\{z_t\}$,

$$z_t^{(j)} = z_t - \hat{y}_j(t) \quad (t = 1, \ldots, N;\ j = 0, 1, 2, \ldots),$$

where $\hat{y}_j(t)$ is defined by (4.7) with m replaced by j, and let $T_{\mu,\nu}^{(j)}(k)$ be defined as $T_{\mu,\nu}^{*}(k)$ with the original sample $(z_1, \ldots, z_N)$ replaced by a "residual" sample $(z_1^{(j)}, \ldots, z_N^{(j)})$. To unify the notation it will be convenient to put $\hat{y}_{-1}(t) \equiv 0$, $T_{\mu,\nu}^{(-1)}(k) = T_{\mu,\nu}^{*}(k)$, and $T_{\mu,\nu}^{(j)}(k) = 0$ for $z_1 = \ldots = z_N = 0$.

We define as follows the random variable $\hat{m}$, which is a function of $(z_1, \ldots, z_N)$ (and also of μ, ν, k and α, where $\alpha < 0{,}5$):

$$\hat{m} = \begin{cases} -1 & \text{if } T_{\mu,\nu}^{(-1)}(k) > \zeta_\alpha, \\ j & \text{if } T_{\mu,\nu}^{(i)}(k) < \zeta_\alpha \quad \text{for } i = -1, \ldots, j-1 \text{ and } T_{\mu,\nu}^{(j)}(k) > \zeta_\alpha. \end{cases}$$

The sequence $\{\hat{y}_{\hat{m}}(t)\}$, $t = 1, \ldots, N$, is the proposed estimator of $\{y_m(t)\}$ in the case of an unknown m. In other terms, we take as an estimator of $y_m(t)$ a polynomial $\hat{y}_j(t)$ of the least degree j for which the hypothesis that the residual process $\{z_t^{(j)}\}$ is purely stochastic is not rejected *by* the T^*-test; thus we can say that $\{z_t^{(\hat{m})}\}$ is a good approximation of the linear process $\{x_t\}$ in the sense of the T^*-test.

The presently described method will be called the *T^*-method of polynomial trend estimation.* The asymptotic distribution of $\hat{y}_{\hat{m}}(t)$ is given in

Theorem 4.12, while Theorem 4.13 formulates an important property of $\max\limits_{1\leqslant t\leqslant N} |\hat{y}_{\hat{m}}(t)-y_m(t)|$.

Put

$$\delta_j(t) = y_m(t)-\hat{y}_j(t) \quad (j\geqslant -1), \tag{4.8}$$

$$\gamma_j(t) = -\sum_{q=0}^{j} a_q\Phi_q(t) \quad (j\geqslant 0), \tag{4.9}$$

$$W_j(t) = A_{j+1}\Phi_{j+1}(t)+\ldots+A_m\Phi_m(t) \quad (-1\leqslant j< m). \tag{4.10}$$

It follows from the definition of $\hat{y}_j(t)$ that

$$\delta_j(t) = \begin{cases} \gamma_j(t) & \text{for } j\geqslant m, \\ W_j(t)+\gamma_j(t) & \text{for } 0\leqslant j< m \end{cases}$$

and $\delta_{-1}(t) = W_{-1}(t) = y_m(t)$.

Let $C_{k,N}^{(j)}$, $C_{0,\nu,p}^{(j)}$, $U_{\mu,\nu}^{(j)}$ and $S_N^{(j)}$ stand for the $C_{k,N}^*$, $C_{0,\nu,p}^*$, $U_{\mu,\nu}^*$ and S_N^* with $(z_1,\ldots,z_N)$ replaced by the residual sample $(z_1^{(j)},\ldots,z_N^{(j)})$, and put

$$A_{k,N}^{(j)} = \frac{1}{N-|k|}\sum_{t=1}^{N-|k|} x_t\,\delta_j(t+|k|),$$

$$B_{k,N}^{(j)} = \frac{1}{N-|k|}\sum_{t=1}^{N-|k|} x_{t+|k|}\,\delta_j(t),$$

$$Q_{k,N}^{(j)} = \frac{1}{N-|k|}\sum_{t=1}^{N-|k|} \delta_j(t)\,\delta_j(t+|k|),$$

$$A_{0,\nu,p}^{(j)} = \nu^{-1}\sum_{t=1}^{\nu} x_{t+p\nu}\,\delta_j(t+p\nu),$$

$$Q_{0,\nu,p}^{(j)} = \nu^{-1}\sum_{t=1}^{\nu} \delta_j^2(t+p\nu) \quad (p = 0,\ldots,\mu-1).$$

In subsequent considerations we shall use the following formula: under $\tilde{H}_{2s}$, for any $s = 1, 2, \ldots$ and any set of integers $\big(i(1),\ldots,i(2s)\big)$,

(4.11)

$$\Big|\sum_{i(1)=1}^{N_1}\cdots\sum_{i(2s)=1}^{N_{2s}} E\big(x_{i(1)}\cdot\ldots\cdot x_{i(2s)}\big)\Big| \leqslant I_{2s}\mathfrak{R}^{(2s)}\Big(\sum_{k=0}^{\infty}|h_k|\Big)^{2s}\cdot N_{u_1}N_{u_3}\cdot\ldots\cdot N_{u_{2s-1}},$$

where I_r is the number of possible groupings of the set $(i_1,\ldots,i_r)$,

$$\mathfrak{R}^{(r)} = \max_{(r_1,\ldots,r_k)}\big(|\mathfrak{R}_{r_1}|,\ldots,|\mathfrak{R}_{r_k}|\big), \quad r_i\geqslant 2,\ r_1+\ldots+r_k = r,$$

and $N_{u_1},\ldots,N_{u_{2s}}$ is a permutation of $N_1,\ldots,N_{2s}$ such that $N_{u_1}\leqslant N_{u_2}\leqslant\ldots\leqslant N_{u_{2s}}$. Formula (4.11) is a simple consequence of (3.2), (3.4) and

(3.5), because in view of (3.4) the left-hand side of (4.11) is bounded by $\mathfrak{R}^{(2s)}$ multiplied by the absolute value of a sum extended to all groupings with terms being products of factors of the same form as the left-hand side of (3.5), so that (3.5) may be applied to them.

We shall also use the relation (see [6], p. 145)

$$\max_{1\leqslant t\leqslant r} |\Phi_q(t)| \leqslant M_q(q+1)r^q, \tag{4.12}$$

where

$$M_q = \max_{0\leqslant j\leqslant q} \frac{(q!)^2(q+j)!}{(2q)!(j!)^2(q-j)!}.$$

LEMMA 4.1. *Under $\tilde{H}_8$, for $j \geqslant m$*

$$\operatorname*{plim}_{\mu,\nu\to\infty} \mu^{1/2}(U_{\mu,\nu}^{(j)} - U_{\mu,\nu}) = 0.$$

Proof. By the definition of $U_{\mu,\nu}^{(j)}$,

$$\mu^{1/2}(U_{\mu,\nu}^{(j)} - U_{\mu,\nu}) = \frac{\nu\mu^{1/2}}{\mu-1}\sum_{p=0}^{\mu-1} S_p,$$

where

$$S_p = 2(C_{0,\nu,p} - C_{0,N})(2A_{0,\nu,p}^{(j)} + Q_{0,\nu,p}^{(j)} - 2A_{0,N}^{(j)} - Q_{0,N}^{(j)}) + \\ + (2A_{0,\nu,p}^{(i)} + Q_{0,\nu,p}^{(j)} - 2A_{0,N}^{(j)} - Q_{0,N}^{(j)})^2. \tag{4.13}$$

We investigate the order of magnitude of

$$E\big(\mu^{1/2}(U_{\mu,\nu}^{(j)} - U_{\mu,\nu})\big)^2 = \frac{\nu^2\mu}{(\mu-1)^2}\sum_{p,r=0}^{\mu-1} E(S_p S_r).$$

For convenience we shall introduce the notation which will be used only in the course of this proof:

$$C_p = 2(C_{0,\nu,p} - C_{0,N}),$$
$$B_p = 2A_{0,\nu,p}^{(j)} + Q_{0,\nu,p}^{(j)} - 2A_{0,N}^{(j)} - Q_{0,N}^{(j)},$$
$$A_p = 2A_{0,\nu,p}^{(j)}, \quad A = 2A_{0,N}^{(j)}, \quad Q_p = Q_{0,\nu,p}^{(j)}, \quad Q = Q_{0,N}^{(j)}.$$

Consequently, $ES_pS_r = EC_pC_rB_pB_r + EC_pB_pB_r^2 + EC_rB_rB_p^2 + EB_p^2B_r^2$. First we evaluate the order of EA_p^4, EQ_p^4, EA^4 and EQ^4. In view of (4.2), (4.11) and (4.12)

$$EA_p^4 = E\Big[\nu^{-1}\sum_{t=1}^{\nu} x_{t+p\nu\gamma_j}(t+p\nu)\Big]^4$$
$$= \nu^{-4}\sum_{q_1,\ldots,q_4=0}^{j}\;\sum_{t_1,\ldots,t_4=1}^{\nu}\;\sum_{s_1,\ldots,s_4=1}^{N} E(x_{t_1+p\nu}\cdot\ldots\cdot x_{t_4+p\nu}\cdot x_{s_1}\cdot\ldots\cdot x_{s_4})\times$$
$$\times\frac{\Phi_{q_1}(s_1)\cdot\ldots\cdot\Phi_{q_4}(s_4)\cdot\Phi_{q_1}(t_1+p\nu)\cdot\ldots\cdot\Phi_{q_4}(t_4+p\nu)}{b_{q_1}\cdot\ldots\cdot b_{q_4}};$$

hence for $0 \leqslant p \leqslant \mu-1$ EA_p^4 is $O(\nu^{-2}N^{-2})$. Similarly, we find that EQ_p^4, EA^4 and EQ^4 are $O(N^{-4})$, so that by the Hölder inequality EB_p^4 is $O(\nu^{-2}N^{-2})$; consequently, $EB_p^2B_r^2$ is $O(\nu^{-2}N^{-2})$.

It is known [8] that $E(C_{0,\nu,p}-R_0)^4$ is $O(\nu^{-2})$ and $E(C_{0,N}-R_0)^4$ is $O(N^{-2})$ so that $EC_p^4 = O(\nu^{-2})$. Therefore, by the Schwarz inequality repeatedly used $E(C_pC_rB_pB_r)$ is $O(\nu^{-2}N^{-1})$ for $0 \leqslant p, r \leqslant \mu-1$, and $E(C_pB_pB_r^2)$ is $O(\nu^{-2}N^{-3/2})$. Hence ES_pS_r is $O(\nu^{-2}N^{-1})$. It follows that $E\big(\mu^{1/2}(U_{\mu,\nu}^{(j)}-U_{\mu,\nu})\big)^2$ is $O(\nu^{-1})$, which completes the proof.

COROLLARY 4.2. *Under $\tilde{H}_8$, for $j \geqslant m$*

$$\operatorname*{plim}_{\mu,\nu\to\infty} U_{\mu,\nu}^{(j)} = v_{0,0}.$$

The proof follows from Lemma 4.1 and Proposition 3.2 in [10].

It is often taken for granted that the coefficients of a polynomial trend fitted to a linear process by the method of least squares are asymptotically normally distributed. However, since in further considerations the knowledge of the form and parameters of this distribution is essential and since a diligent search of the existing literature of the subject failed to produce a ready answer, a detailed investigation of this matter was called for, and its results are set out in the following.

LEMMA 4.3. *Under $\tilde{H}_{2M}(M \geqslant 1)$, the distribution of $b_q^{1/2}a_q$ tends, with $N \to \infty$ and with an arbitrary q, to be normal with a zero mean and a variance equal to $\sum_{k=-\infty}^{\infty} R_k$, the moments up to the order $2M$ tending to the respective moments of this distribution.*

Proof. According to (4.7),

$$a_q = b_q^{-1}\sum_{t=1}^{N}\Phi_q(t)x_t.$$

Hence $Ea_q = 0$.

Furthermore,

$$\operatorname{var}(b_q^{1/2}a_q) = b_q^{-1}\sum_{t,u=1}^{N}\Phi_q(t)\Phi_q(u)R_{t-u}.$$

For a given $t-u=k$, the coefficient of R_k can be represented as

$$b_q^{-1}\sum_{t=1}^{N-|k|}\Phi_q(t)\Phi_q(t+|k|).$$

But $b_q^{-1} = O(N^{-(2q+1)})$ and, for $1 \leqslant t \leqslant N-|k|$ and for a fixed k, $\Phi_q(t)$ is at most $O(N^q)$ and $\Phi_q(t+|k|)-\Phi_q(t)$ is at most $O(N^{q-1})$.

Hence

$$b_q^{-1}\sum_{t=1}^{N-|k|}\Phi_q(t)\Phi_q(t+|k|)-b_q^{-1}\sum_{t=1}^{N-|k|}\Phi_q^2(t) = O(N^{-1}),$$

while, clearly,

$$b_q^{-1}\sum_{t=N-|k|+1}^{N}\Phi_q(t)\Phi_q(t+|k|) = O(N^{-1}).$$

Hence, in the limit, the coefficient of R_k tends to

$$b_q^{-1}\sum_{t=1}^{N}\Phi_q^2(t) = 1;$$

therefore, by Lemma 18.1 in [8]

$$\lim_{N\to\infty}\operatorname{var}(b_q^{1/2}a_q) = \sum_{k=-\infty}^{\infty}R_k,$$

the coefficients of R_k being, clearly, collectively bounded. Furthermore, the coefficient of x_t in $b_q^{1/2}a_q$ is $b_q^{-1/2}\Phi_q(t) = O(N^{-1/2})$, and Lemma 4.3 follows from Lemma 2.2 in [10].

Lemma 4.4. *Under $\tilde{H}_{2M}$ ($M\geqslant 1$), for any finite set of values of q, the joint distribution of the various $b_q^{1/2}a_q$ tends to be normal with zero correlation, i.e. they are independent in the limit, and all the moments up to the order $2M$ tend to the appropriate moments of this distribution.*

Proof. For arbitrary q_1 and q_2 ($q_1\neq q_2$),

$$E(b_{q_1}^{1/2}a_{q_1}b_{q_2}^{1/2}a_{q_2}) = \sqrt{b_{q_1}b_{q_2}}\sum_{t,u=1}^{N}\Phi_{q_1}(t)\Phi_{q_2}(u)R_{t-u}.$$

As in the previous lemma, it can be seen that in the limit, for any fixed k, the coefficient of R_k can be replaced by

$$\sum_{t=1}^{N}\Phi_{q_1}(t)\Phi_{q_2}(t)\sqrt{b_{q_1}b_{q_2}} = 0.$$

Taking into account Lemma 4.3, it follows that any linear combination of $b_{q_1}^{1/2}a_{q_1},\ldots,b_{q_s}^{1/2}a_{q_s}$ tends to be normally distributed, since the coefficients of the x's still satisfy the condition of Lemma 2.2 in [10]. Lemma 4.4 follows, then, by Proposition 7.1 in [7].

Lemma 4.5. *Under $\tilde{H}_8$, for $j\geqslant m$*

$$\operatorname*{plim}_{\mu,\nu\to\infty}\mu^{1/2}(S_N^{(j)}-S_N) = 0.$$

Proof. According to (3.6), when $j \geqslant m$,

$$\mu^{1/2}(S_N^{(j)} - S_N) = \sqrt{\mu} \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 (A_{k,N}^{(j)} - B_{k,N}^{(j)}) C_{k,N} +$$

$$+ \frac{\sqrt{\mu}}{2} \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 (A_{k,N}^{(j)} + B_{k,N}^{(j)})^2 + \sqrt{\mu} \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 C_{k,N} Q_{k,N}^{(j)} +$$

$$+ \sqrt{\mu} \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 (A_{k,N}^{(j)} + B_{k,N}^{(j)}) Q_{k,N}^{(j)} + \frac{\sqrt{\mu}}{2} \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 Q_{k,N}^{(j)2}.$$

It suffices to prove that each of these five terms tends to 0 in the mean as $N \to \infty$. We begin with the last. We have

$$\left(1 - \frac{|k|}{N}\right) Q_{k,N}^{(j)} = N^{-1} \sum_{q_1,q_2=0}^{j} a_{q_1} a_{q_2} \sum_{t=1}^{N-|k|} \Phi_{q_1}(t) \Phi_{q_2}(t + |k|)$$

and, further,

$$\mu E\left[\left(1 - \frac{|k|}{N}\right) Q_{k,N}^{(j)} \left(1 - \frac{|l|}{N}\right) Q_{l,N}^{(j)}\right]^2 \tag{4.14}$$

$$= \mu N^{-4} \sum_{q_1,\ldots,q_8=0}^{j} \{E(a_{q_1} \cdot \ldots \cdot a_{q_8}) \cdot \sum_{t_1,t_2=1}^{N-|k|} \sum_{t_3,t_4=1}^{N-|l|} \Phi_{q_1}(t_1) \Phi_{q_2}(t_1 + |k|) \Phi_{q_3}(t_2) \times$$

$$\times \Phi_{q_4}(t_2 + |k|) \Phi_{q_5}(t_3) \Phi_{q_6}(t_3 + |l|) \Phi_{q_7}(t_4) \Phi_{q_8}(t_4 + |l|)\}.$$

But the product of the Φ's is $O(N^{q_1+\ldots+q_8})$ and since, for any $q_1, \ldots, q_8$, there are at most N^4 such products, the multiple sum with respect to t_1, t_2, t_3, t_4 is $O(N^{q_1+\ldots+q_8+4})$. On the other hand, by Lemma 4.3

$$E(a_{q_1} \cdot \ldots \cdot a_{q_8}) = O[(b_{q_1} \cdot \ldots \cdot b_{q_8})^{-1/2}] = O(N^{-(q_1+\ldots+q_8+4)}).$$

Taking into account the finite and constant range of the q's, we find that the left-hand side of (4.14) is $O(\mu N^{-4})$. Hence

$$E\left[\frac{\sqrt{\mu}}{2} \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 Q_{k,N}^{(j)2}\right]^2 = O(\mu N^{-2}). \tag{4.15}$$

Passing to the second term, it suffices to consider

$$\sqrt{\mu} \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 A_{k,N}^{(j)2},$$

since the same arguments will apply to

$$\sqrt{\mu} \sum_{k=1-N}^{N-1} \left(1 - \frac{|k|}{N}\right)^2 B_{k,N}^{(j)2}$$

and the Schwarz inequality will take care of the covariance of these two terms. Now by the definition of $A_{k,N}^{(j)}$ and by (4.9)

$$\left(1-\frac{|k|}{N}\right)A_{k,N}^{(j)} = -N^{-1}\sum_{q=0}^{j} a_q \sum_{t=1}^{N-|k|} x_t \Phi_q(t+|k|).$$

In view of the finite and constant range of j, when dealing merely with orders of magnitude, we can omit the summation with respect to j, calling the simplified expression $A(k, n, q)$. Then by (4.6), (4.11) and (4.12)

$$(4.16)\quad \mu E\left[\left(1-\frac{|k|}{N}\right)A(k, N, q)\left(1-\frac{|l|}{N}\right)A(l, N, q)\right]^2$$

$$= \mu N^{-4}\sum_{t_1,t_2=1}^{N-|k|}\sum_{t_3,t_4=1}^{N-|l|} E(a_q^4 x_{t_1}x_{t_2}x_{t_3}x_{t_4})\Phi_q(t_1+|k|)\Phi_q(t_2+|k|)\Phi_q(t_3+|l|)\Phi_q(t_4+|l|)$$

$$= \mu N^{-4} b_q^{-4}\sum_{u_1,\dots,u_4=1}^{N}\sum_{t_1,t_2=1}^{N-|k|}\sum_{t_3,t_4=1}^{N-|l|} E\left(\prod_{i=1}^{4} x_{u_i}x_{t_i}\right)\Phi_q(u_1)\cdot\ldots\cdot\Phi_q(u_4)\Phi_q(t_1+|k|)\times$$

$$\times\Phi_q(t_2+|k|)\Phi_q(t_3+|l|)\cdot\Phi_q(t_4+|l|) = O(\mu N^{-4})$$

and, consequently, uniformly

$$\mu E\left[\left(1-\frac{|k|}{N}\right)A_{k,N}^{(j)}\left(1-\frac{|l|}{N}\right)A_{l,N}^{(j)}\right]^2 = O(\mu N^{-4}).$$

Taking the sum of $O(N^2)$ such terms, we find

$$E\left[\frac{\sqrt{\mu}}{2}\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 A_{k,N}^{(j)2}\right]^2 = O(\mu N^{-2})$$

and further

$$(4.17)\quad E\left[\frac{\sqrt{\mu}}{2}\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 (A_{k,N}^{(j)}+B_{k,N}^{(j)})^2\right]^2 = O(\mu N^{-2}).$$

By (4.15) and (4.17), the Schwarz inequality yields

$$(4.18)\quad E\left[\frac{\sqrt{\mu}}{2}\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 (A_{k,N}^{(j)}+B_{k,N}^{(j)})Q_{k,N}^{(j)}\right] = O(\mu N^{-2}).$$

But we know from the asymptotic distribution of S_N that

$$E\left[\sqrt{\mu}\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 C_{k,N}^2\right]^2 = O(\mu);$$

hence by (4.17)

$$(4.19)\quad E\left[\sqrt{\mu}\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 (A_{k,N}^{(j)}+B_{k,N}^{(j)})C_{k,N}\right]^2 = O(\mu N^{-1})$$

and by (4.15)

$$E\left[\sqrt{\mu}\sum_{k=1-N}^{N-1}\left(1-\frac{|k|}{N}\right)^2 C_{k,N}Q_{k,N}^{(j)}\right]^2 = O(\mu N^{-1}). \tag{4.20}$$

Now Lemma 4.5 is a direct consequence of (4.15) and (4.17)-(4.20).

LEMMA 4.6. *Under $\tilde{H}_8$, for $j \geqslant m$*

$$\operatorname*{plim}_{\mu,\nu\to\infty} \mu^{1/2}(C_{0,N}^{(j)2} - C_{0,N}^2) = 0.$$

Proof. The argument is the same as in the previous proof, except that the summation with respect to k is omitted and k is replaced by 0.

THEOREM 4.7. *Under $\tilde{H}_8$, for any k and $m \geqslant 0$*

$$\operatorname*{plim}_{\mu,\nu\to\infty} [T_{\mu,\nu}^{(j)}(k) - T_{\mu,\nu}(k)] = 0 \quad \text{for } j \geqslant m,$$

$$\operatorname*{plim}_{\mu,\nu\to\infty} T_{\mu,\nu}^{(j)}(k) = -\infty \quad \text{for } -1 \leqslant j < m.$$

Proof. According to Lemmas 4.1, 4.5 and 4.6, to Corollary 4.2 and to a convergence theorem given by Cramer ([3], 20.6), we have under $\tilde{H}_8$ for $j \geqslant m \geqslant 0$ and for any k

$$\begin{aligned}
&\operatorname*{plim}_{\mu,\nu\to\infty} [T_{\mu,\nu}^{(j)}(k) - T_{\mu,\nu}(k)] \\
&= \operatorname*{plim}_{\mu,\nu\to\infty}\left[\left(\frac{\mu}{2}\right)^{1/2}\frac{U_{\mu,\nu}^{(j)} - 2S_N^{(j)} - kC_{0,N}^{(j)2}}{U_{\mu,\nu}^{(j)}} - T_{\mu,\nu}(k)\right] \\
&= \operatorname*{plim}_{\mu,\nu\to\infty}\left\{\frac{\sqrt{\mu}[U_{\mu,\nu} - 2S_N - kC_{0,N}^2]}{\sqrt{2}\,U_{\mu,\nu}^{(j)}} + \right. \\
&\qquad \left. + \frac{\sqrt{\mu}[U_{\mu,\nu}^{(j)} - U_{\mu,\nu} - 2(S_N^{(j)} - S_N) - k(C_{0,N}^{(j)2} - C_{0,N}^2)]}{\sqrt{2}\,U_{\mu,\nu}^{(j)}} - T_{\mu,\nu}(k)\right\} \\
&= \operatorname*{plim}_{\mu,\nu\to\infty}\left\{\left(\frac{\mu}{2}\right)^{1/2}\frac{U_{\mu,\nu} - 2S_N - kC_{0,N}^2}{U_{\mu,\nu}^{(j)}} - T_{\mu,\nu}(k)\right\} \\
&= \operatorname*{plim}_{\mu,\nu\to\infty}\left\{\left(\frac{T_{\mu,\nu}(k)}{\sqrt{\mu}}\right)\frac{\sqrt{\mu}(U_{\mu,\nu} - U_{\mu,\nu}^{(j)})}{U_{\mu,\nu}^{(j)}}\right\} = 0,
\end{aligned}$$

and the first part of the theorem is proved.

If $j < m$, then $z_t^{(j)} = x_t + \gamma_j(t) + W_j(t)$ and $W_j(t)$ is a polynomial of degree m. From the definition of $\gamma_j(t)$ and $W_j(t)$ it follows that the order of magnitude of $E(U_{\mu,\nu}^{(j)})^2$ is for $j < m$ the same as that of $E(U_{\mu,\nu}^*)^2$, because the terms of the highest order are those with sums of products of W_j's in the case of $E(U_{\mu,\nu}^{(j)})^2$ and of y_m's in the case of $E(U_{\mu,\nu}^*)^2$. It follows from the proof of Theorem 3.2 that $\lim_{N\to\infty} EH_N^2 = 0$, where $H_N = U_{\mu,\nu}^*/NL_N^4$.

Therefore the counterpart of H_N in $T^{(j)}_{\mu,\nu}(k)$ still tends to 0 in probability. In a similar way one can see that the counterparts of F_N and G_N tend in probability to $Q/2$ and 0 respectively, and the second part of the theorem follows at once.

COROLLARY 4.8. *Under $\tilde{H}_8$, for $k < \mathfrak{R}_4\mathfrak{R}_2^{-2}$ and $m \geqslant 0$*

$$\lim_{N\to\infty} P(\hat{m} = m) = 1.$$

It is an immediate consequence of Theorem 4.7, formulas (2.1) and (2.2) and the definition of $\hat{m}$. In a similar way we can show that for $k = \mathfrak{R}_4\mathfrak{R}_2^{-2}$

$$\lim_{N\to\infty} P(\hat{m} = m) = 1-a.$$

LEMMA 4.9. *Under $\tilde{H}_2$, for any $j \geqslant 0$ and for any a $(0 \leqslant a \leqslant 1)$*

$$\lim_{N\to\infty; t/N\to a} \operatorname{var}\Big\{N^{1/2}\sum_{q=0}^{j} a_q \Phi_q(t)\Big\} = \sum_{k=-\infty}^{\infty} R_k \sum_{q=0}^{j}(2q+1)\Big\{\sum_{s=0}^{q}\frac{(-1)^s(q+s)!\,a^s}{(s!)^2(q-s)!}\Big\}^2. \tag{4.21}$$

Proof. According to Lemmas 4.3 and 4.4,

$$\operatorname{var}\Big\{N^{1/2}\sum_{q=0}^{j} a_q\Phi_q(t)\Big\} = N\sum_{k=-\infty}^{\infty} R_k \sum_{q=0}^{j}\frac{\big(\Phi_q(t)\big)^2}{b_q} + o(1).$$

By (4.1) and (4.2)

$$\frac{N\big(\Phi_q(t)\big)^2}{b_q} = \frac{(2q+1)}{[(N-q-1)!]^2}\times$$

$$\times \sum_{s,u=0}^{q}\frac{(-1)^{s+u}(q+s)!(q+u)!(N-s-1)!(N-u-1)!(t-1)^{(s)}(t-1)^{(u)}}{(s!)^2(u!)^2(q-s)!(q-u)!(N^2-1^2)\cdot\ldots\cdot(N^2-q^2)}.$$

But, when $N\to\infty$ and $t/N\to a$, we have

$$(N-q-1)! \sim N!N^{-(q+1)}, \qquad (t-1)^{(s)} \sim N^s a^s,$$
$$(N^2-1^2)\cdot\ldots\cdot(N^2-q^2) \sim N^{2q},$$

and substituting these values in the previous formula, we find

$$\lim_{N\to\infty,\frac{t}{N}\to a} \operatorname{var}\Big\{N^{1/2}\sum_{q=0}^{j} a_q\varphi_q(t)\Big\} = \sum_{k=-\infty}^{\infty} R_k\sum_{q=0}^{j}(2q+1)\sum_{s,u=0}^{q}\frac{(-1)^{s+u}(q+s)!(q+u)!\,a^{s+u}}{(s!)^2(u!)^2(q-s)!(q-u)!},$$

and Lemma 4.9 is proved.

COROLLARY 4.10. *Under $\tilde{H}_{2M}$ $(M \geqslant 1)$, for any $j \geqslant 0$ and any a $(0 \leqslant a \leqslant 1)$, when $N \to \infty$ and $t/N \to a$, the distribution of*

$$N^{1/2} \sum_{q=0}^{j} a_q \Phi_q(t)$$

tends to be normal with a zero mean and a variance given by (4.21), *the moments up to the order $2M$ tending to the respective moments of the limiting distribution.*

The proof follows from Lemma 4.9 and Lemma 2.2 in [10].

Remark. There would be no difficulty in establishing the joint asymptotic distribution of

$$\sum_{q=0}^{j} a_q \Phi_q(t)$$

for any finite number of values of t, say $t_1, \ldots, t_r$, with $t_i/N \to a^{(i)}$. Also the values of j could be varied.

LEMMA 4.11. *Under $\tilde{H}_2$, for any $j \geqslant 0$,*

$$\underset{N\to\infty}{\text{l.i.m.}} \max_{1\leqslant t\leqslant N} \Big| \sum_{q=0}^{j} a_q \Phi_q(t) \Big| = 0.$$

Proof. Since

$$\max_{1\leqslant t\leqslant N} |\Phi_q(t)|/b_q^{1/2} \to 0 \quad \text{as } N \to \infty,$$

it follows from Lemma 4.3 that $\underset{N\to\infty}{\text{l.i.m.}} |a_q \Phi_q(t)| = 0$ uniformly in t for any particular q, and since

$$\max_{1\leqslant t\leqslant N} \Big| \sum_{q=0}^{j} a_q \Phi_q(t) \Big| \leqslant \sum_{q=0}^{j} |a_q| \max_{1\leqslant t\leqslant N} |\Phi_q(t)|,$$

Lemma 4.11 follows at once.

THEOREM 4.12. *Under $\tilde{H}_{2M}$ $(M \geqslant 4)$, for $k < \mathfrak{R}_4 \mathfrak{R}_2^{-2}$ and $m = 0, 1, 2, \ldots$ the distribution of $N^{1/2}[\hat{y}_{\hat{m}}(t) - y_m(t)]$ tends with $N \to \infty$ and $t/N \to a$ $(0 \leqslant a \leqslant 1)$ to a normal distribution with a zero mean and a variance equal to*

$$\sum_{k=-\infty}^{\infty} R_k \sum_{q=0}^{m} (2q+1) \left\{ \sum_{s=0}^{q} \frac{(-1)^s (q+s)!\, a^s}{(s!)^2 (q-s)!} \right\}^2.$$

Proof. The statement with $\hat{y}_{\hat{m}}(t)$ replaced by $\hat{y}_m(t)$ is a direct consequence of Lemma 4.9 and Corollary 4.10. On the other hand, in view of Corollary 4.8, for any t uniformly

$$\underset{N\to\infty}{\text{plim}} \{N^{1/2}[\hat{y}_{\hat{m}}(t) - \hat{y}_m(t)]\} = 0,$$

and the theorem follows at once.

THEOREM 4.13. *Under* $\tilde{H}_{2M}$ $(M \geqslant 4)$

$$\operatorname*{plim}_{N\to\infty} \max_{1\leqslant t\leqslant N} |\hat{y}_{\hat{m}}(t) - y_m(t)| = 0.$$

Proof. The argument is similar to the preceding one, the statement for $\hat{y}_m(t)$ being a consequence of Lemma 4.11.

5. Discussion of the proposed method and of related statistical methods. The problem of estimating a function is never an easy task and the required properties of such estimators are not so generally established and accepted as in classical estimation problems; it is still more difficult to construct estimators satisfying these requirements. The polynomial trend estimator proposed here is asymptotically a good estimator in the maximum absolute error sense (Theorem 4.13) and is asymptotically unbiased and consistent for each $1 \leqslant t \leqslant N$ (Theorem 4.12); but it was constructed according to another criterion concerning the residual process: with a suitably chosen test, the probability of accepting the hypothesis that the residual process is a purely stochastic process had to be big enough. In the T^*-method with $k < \mathfrak{R}_4\mathfrak{R}_2^{-2}$ this probability is asymptotically equal to 1.

In principle, any test may be used instead of the T^*-test if only its power has desirable properties; but till now no other tests are known except for some tests which assume a great deal of prior information about the linear process $\{x_t\}$, which usually is not available in practice.

As far as we know, no other definition of a polynomial trend estimator in case of an unknown degree has so far been published; models usually discussed (see for instance [3] or [9]) are models with a determinate component in form of a linear combination of some given functions, where the number of terms is assumed to be known. A polynomial of a known degree is a special case of such components.

The practical advice given by M. G. Kendall ([5], p. 357) in the case of a polynomial of an unknown degree is to evaluate least-square estimators of degree $0, 1, 2, \ldots$ and to calculate on each stage the sum of squared residuals; the procedure is to be stopped when sufficiently good "fit" is obtained, that is — as it was suggested in previous editions of [5] — when the difference between two subsequent sums does not diminish "significantly". However, no rule is given to decide when the difference does not diminish significantly; hence the objection against this procedure is that it is subjective. Big difficulties arise specially in cases where there is no trend at all or where the trend is not in the form of a polynomial; the above procedure may lead in such cases to non-sensical decisions.

Although Hannan ([3], p. 126) investigates a model with a known degree, he introduces a test of the hypothesis that the coefficient of the greatest power of t is equal to zero. This test provides an opportunity to check to some extent the assumed model: in fact with this test it is rather an upper bound of the true degree than this degree itself which must be assumed to be known. Moreover, the existence of this test suggests an idea of a sequential procedure similar to one proposed by us, that is of estimating subsequent polynomials of degree $0, 1, 2, \ldots$ and basing the stopping rule on the results of this test applied on each stage to the respective coefficient. However, for reasons given below this suggestion must be rejected.

In the first place, to justify the method described above one should rather prove that if the degree of the polynomial is m, then the power of this "coefficient test" for the coefficient a_j is for $j < m$ asymptotically equal to 1. The corresponding proofs have been given in case of the T^*--method; but it is doubtful whether the "coefficient test" satisfies this requirement. Secondly, there must exist a "start" test of the whole sequential procedure which will verify the hypothesis that there exists a trend at all; luckily this difficulty might be overcome by the use of the T^*-test. The next difficulty — admittedly a small one — concerns the level of significance α, which will have to be constant, so that presumably asymptotically with a probability approximately equal to α we would overestimate m. In the case of the T^*-method this difficulty is at least alleviated insomuch as for $k < \mathfrak{R}_4\mathfrak{R}_2^{-2}$ the level of significance tends to zero.

But the main difficulty lies in the fact that the "coefficient test" requires some prior knowledge about the process $\{x_t\}$, in particular the knowledge of the value of the spectral density function of $\{x_t\}$ at the zero frequency. To fulfil this requirement, Hannan suggests that one should find an estimator of the polynomial for the considered degree j, evaluate the residual sample $\{z_t^{(j)}\}$, find the estimator of the required value of the spectral density function and perform the test using the estimated value instead of the true one. In view of the known difficulties connected with the estimation of the spectral density function of a linear process, it is not easy to investigate the statistical consequences of such a multistage procedure, especially for $j < m$.

6. Numerical illustrations. Table 1 summarises the result of some numerical work done on artificial time-series in order to show how the method described works with finite samples.

We have explored series 10 and 16 published in [4], which are realizations of autoregressive processes $\{x_t\}$ defined respectively by

$$x_{t+3} - 1.8x_{t+2} + 1.27x_{t+1} - 0.35x_t = \varepsilon_{t+3}$$

and

$$x_{t+3} - 1.2x_{t+2} + 0.61x_{t+1} - 0.05x_t = \varepsilon_{t+3},$$

where $\{\varepsilon_t\}$ is a sequence of independent normal standardized variables. These two series, or only parts of them, were used to form samples $(x_1, \ldots, x_N)$; for each $1 \leqslant t \leqslant N$ the value of a chosen polynomial $y_m(t)$ (column 3 of Table 1) was added to x_t to form z_t, and then the estimator $\hat{y}_{\hat{m}}(t)$ was calculated repeatedly for $k = 0$, -1 and -2, with $\alpha = 0.01$.

Some examples with the same $y_m(t)$ and the same N $(N = \mu \cdot \nu)$ were worked out for both series.

For the convenience of the reader Table 1 contains values of $\hat{m}$ in separate columns, although they can be found from $\hat{y}_{\hat{m}}(t)$; after comparing $\hat{m}$ with m, we can immediately see whether our procedure succeeded or failed in this example. We mark successes with "+" and failures with "—". Columns 9 and 12 headed "conclusion" contain these marks. Successes take place of course if $\hat{m} = m$, failures are if $\hat{m} < m$ or $\hat{m} > m$. For technical reasons we have continued our calculations only to the stage $j = m+1$, so that we had only verified that $\hat{m} > m+1$ and it is registered in that form in Table 1 (see m in examples 3 and 11 for series 16).

It should be pointed out that the distribution of $\hat{m}$ for finite N has not been investigated, and we cannot predict the behaviour of $\hat{m}$ if $\hat{m} > m$. It is hoped that the matter will be fully investigated in due course. However, it is believed that inordinately large values of $\hat{m}$ should tend to be less and less probable.

In the majority of examples in Table 1, the estimators for $k = 0$, -1 and -2 are identical (it should be noted that for both series 10 and 16 we have $\mathfrak{R}_4\mathfrak{R}_2^{-2} = 0$, because $\{\varepsilon_t\}$ are normal); differences arose in examples 3, 8 and 11 for series 16. In examples 3 and 11 the cases $k = -2$ or -1 are more favourable than the case $k = 0$, because $\hat{m} > m$ for $k = 0$ and $\hat{m} = m$ for $k = -2$ or -1, while, in example 8, $\hat{m} = m-1$ for $k = 0$ and $\hat{m} = m-2$ for $k = -1$ or -2.

Obviously, in practice we seldom have any information about $\mathfrak{R}_4\mathfrak{R}_2^{-2}$ and therefore we must choose $k = -2$. Our estimation procedure works very well for $k = -2$, as confirmed by majority of examples in Table 1. Also, this choice of k diminishes the risk of $\hat{m} > m$, as illustrated by examples 3 and 11. However, if the trend is rather faint as is the case in example 8, the value -2 for k may be less favourable than some value nearer to $\mathfrak{R}_4\mathfrak{R}_2^{-2}$.

Several failures of the estimation method for k near to $\mathfrak{R}_4\mathfrak{R}_2^{-2}$ in the case of series 16 could be forecast because the consistent estimator $k_{\mu,\nu}$ of $\mathfrak{R}_4\mathfrak{R}_2^{-2}$, defined in [10], p. 387, has for this series a highly negative value: $k_{25,20} = -1.32$, which causes $T_{\mu,\nu}(\mathfrak{R}_4\mathfrak{R}_2^{-2})$ to take also a negative value beyond ζ_α; this feature applies also to shorter parts of the series, as treated in examples 3 and 11. On the contrary, series 10 does not belong to this kind of sample and cases with $\hat{m} > m$ were not observed.

TABLE 1. Estimators $\hat{y}_{\hat{m}}(t)$

(2)	(3)	(4)	(5)	(6)	Series 10 (7)	Series 10 (8)	Series 10 (9)	Series 16 (10)	Series 16 (11)
m	$y_m(t)$	μ	ν	k	$\hat{m}$	$\hat{y}_{\hat{m}}(t)$	Conclusion	$\hat{m}$	$\hat{y}_{\hat{m}}(t)$
2	$0.1t^2$	25	20	0				2	$0.100001t^2+$
				-1				2	$-0.0016t$
				-2				2	$+0.397$
2	$0.01t^2$	25	20	0				2	$0.010001t^2$
				-1				2	$-0.0016t$
				-2				2	$+0.406$
2	$0.01t^2$	30	10	0	2	$0.009987t^2$	+	>3	—
				-1	2	$+0.0140t$	+	2	$0.00996\ t^2+$
				-2	2	-1.556	+	2	$+0.0087t-0.051$
2	$0.01t^2$	20	10	0	2	$0.009933t^2$	+		
				-1	2	$+0.0235t$	+		
				-2	2	-1.836	+		
2	$0.001t^2$	20	10	0	1		−		
				-1	1	$0.2108t+$	−		
				-2	1	-8.110	−		
2	$0.0001t^2$	20	10	0	-1	0	−		
				-1	-1	0	−		
				-2	-1	0	−		
2	$0.0001t^2$	25	20	0				2	$0.000101t^2+$
				-1				2	$-0.0016t+$
				-2				2	$+0.407$
2	$0.00001t^2$	25	20	0				1	$0.0039t-0.055$
				-1				0	0.915
				-2				0	
1	$0.04t+750$	25	20	0				1	$0.0389t+$
				-1				1	$+750.364$
				-2				1	
1	$0.04t+750$	20	10	0	1		+		
				-1	1	$0.0500t+$	+		
				-2	1	$+748.620$	+		
0	2	30	10	0	0		+	>1	—
				-1	0	2.143	+	0	2.163
				-2	0		+	0	

Examples in Table 1 were planned in order to show how the results of estimation gradually become worse when N or the scale of $y_m(t)$ diminishes. For instance, for series 10 and $N = 200$ the estimators are satisfactory if $y_m(t) = 0.01t^2$, but we have $\hat{m} = 1$ for $0.001t^2$ and $m = -1$ for $0.0001t^2$ (examples 4, 5 and 6).

Unfortunately, we cannot investigate theoretically the speed with which $\{T^{(j)}_{\mu,\nu}(k)\}$ tend to their limits without detailed assumptions about $\{x_i\}$ and $y_m(t)$ which usually are not available. However, the examples which have been worked out give us at least some rough idea of what can be expected, and the results are rather optimistic: usually, for $j < m$, $T^{(j)}_{\mu,\nu}(k)$ tends quickly to $-\infty$ for all k, so that the choice of $k = -2$ seems quite sensible. More details concerning Examples 4 and 11 may be of some interest; they are shown in Table 2.

TABLE 2. Values of $T^{(j)}_{\mu,\nu}(k)$

Example	Series	$y_m(t)$	μ	ν	j	$T^{(j)}_{\mu,\nu}(k)$		
						$k = 0$	$k = -1$	$k = -2$
					-1	-7.9	-7.7	-7.5
					0	-13.9	-13.7	-13.4
4	10	$0.01t^2$	20	10	1	-11.7	-11.4	-11.1
					2	$\boxed{-0.2}$	$\boxed{0.3}$	$\boxed{0.9}$
					3	-1.4	-0.7	-0.1
					-1	-81.8	-80.6	-79.4
11	16	2	30	10	0	-3.1	$\boxed{-1.9}$	$\boxed{-0.6}$
					2	-3.2	-1.9	-0.7

Values of $\hat{m}$ given in Table 1 are for examples 4 and 11 easily obtained from Table 2 in view of $\zeta_{0.01} = -2.33$.

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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E. PLESZCZYŃSKA (Warszawa)

DALSZE ZASTOSOWANIE TESTU T^* W ANALIZIE SZEREGÓW CZASOWYCH

STRESZCZENIE

W pracy podano algorytm estymacji trendu wielomianowego nieznanego stopnia nałożonego na liniowy proces stochastyczny i zbadano asymptotyczne własności proponowanego estymatora (twierdzenia 4.12 i 4.13). Metoda ta oparta jest na zastosowaniu testu T^* opisanego w [10]. Ponadto twierdzenie 3.2 uogólnia klasę hipotez alternatywnych, przy których test T^* ma asymptotycznie moc równą jedności.
