ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
Hugo Steinhaus Jubilee Volume
X (1969)

Z. A. ŁOMNICKI (Wolverhampton)

ON THE DISTRIBUTION OF PRODUCTS OF INDEPENDENT BETA VARIABLES (1)

1. Introduction.

In a number of applications it is necessary to know the properties of the product of random variables: this occurs in particular when the random variables involved have dimensions of a ratio, like tolerance expressed in percentages of desired value, amplification ratios etc. The special situation of the product of a number of independent identically distributed random variables discussed in this paper arises, for instance, in the case when some devices designed to amplify a magnitude and having identical characteristics are connected in series. If x_i is the random variable describing the amplification by the i-th device, the total amplification $y = x_1 x_2 \ldots x_n$ is also a random variable and it is important to know the distribution of this product. Examples of a number of engineering applications involving products and quotients of random variables can be found, for instance in Donahue [4].

It was shown by Springer and Thompson [10] how to obtain the probability density function (p.d.f.) of products of n independent, identically distributed random variables by the application of the Mellin transform; they have obtained, among others, formulae (in the form of rapidly convergent infinite series) for the p.d.f. of the product of n independent normal variates and independent Cauchy variates and treated also a special case of beta variates [see below formula (7)]. Their method is a generalization to n factors of a method presented by Epstein [5] for $n = 2$. Following Epstein many authors applied the Mellin transform to the study of distribution of products and quotients of random variables; a detailed bibliography can be found in Springer and Thompson [9]; cf. also Kotlarski [6] and Zolotarev [11].

The Mellin transform of a function $f(x)$ where $x > 0$ is defined as

$$M[f(x)\,|\,s] = \int_0^\infty x^{s-1} f(x)\,dx.$$

(1) Research sponsored by U. S. Office of Naval Research.

Under certain regularity conditions (cf. Courant-Hilbert [3], pp. 103-105) this transform considered as a function of a complex variable s admits an inversion integral

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} M[f(x)\,|\,s]\,ds, \qquad c > 0, \tag{1}$$

where the path of integration is a line parallel to the imaginary axis and to the right of the origin.

An immediate consequence of the definition of the Mellin transform is the following multiplicative property: If x and y are independent random variables with the p.d.fs $f(x)$ and $g(y)$ respectively and if $h(z)$ is the p.d.f. of $z = xy$ then

$$M[h(z)\,|\,s] = M[f(x)\,|\,s]\cdot M[g(y)\,|\,s]. \tag{2}$$

Thus to find the p.d.f. of $y = x_1 x_2 \ldots x_n$ where the x_i's are independent identically distributed random variables with p.d.f. $f(x)$ it is sufficient to find the Mellin transform of $f(x)$, to take its n-th power and to find the inverse with the aid of formula (1).

The following property of the Mellin transform will be needed in Section 3: If $f(x)$ is the p.d.f. of the random variable x which has a finite second moment and if

$$F(x) = \int_0^x f(t)\,dt, \qquad \mathscr{F}(x) = \int_x^\infty f(t)\,dt = 1 - F(x),$$

then the Mellin transform of $\mathscr{F}(x)$ is equal to

$$\begin{aligned} M[\mathscr{F}(x)\,|\,s] = \int_0^\infty \mathscr{F}(x) x^{s-1}\,dx &= s^{-1} x^s \mathscr{F}(x)]_0^\infty + s^{-1} \int_0^\infty x^s f(x)\,dx \\ &= s^{-1} M[f(x)\,|\,s+1]. \end{aligned} \tag{3}$$

It has been shown [7] by the author of this paper that the results obtained by Springer and Thompson in [10] for normal random variables can be presented in a somewhat simpler form and that a general formula for any number of factors can be given (still rather unwieldy in the case of large values of n). It has also been shown that by a direct application of the Mellin transform similar infinite series expansions can be derived for the corresponding probability distribution functions; this is useful since the straightforward integration of the relevant infinite series representing the p.d.fs is not always easy. Finally, attention has been drawn to the fact (implicit in the Springer-Thompson treatment) that it is sufficient to evaluate the formulae for the p.d.f. of the product of independent exponentially distributed random variables and from such tables the p.d.fs for products of gamma, normal and Weibull random variables can be quickly evaluated by simple transformations.

The practical usefulness of the results described above is limited by the fact that all the corresponding distributions have infinite ranges while the physical devices to which our mathematical models are to be applied often have finite characteristics. Thus the problem arises to what extent the methods applied in the above-quoted papers can be modified to obtain similar results for random variables having finite ranges and the first family of random variables which suggests itself and seems to be tractable in this way is the family of beta distributions with parameters p, q, i.e. the random variables having the p.d.f.

$$(4)\qquad \beta(x; p, q) = \begin{cases} x^{p-1}(1-x)^{q-1}/B(p, q) & \text{for} \quad 0 \leqslant x \leqslant 1, \\ 0 & \text{for} \quad x < 0, x \geqslant 1, \end{cases}$$

where $p, q \geqslant 1$ and $B(p, q)$ is the Euler Integral of the First Kind

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

Notice that p and q are assumed to be not less than 1; for p or q less than 1 the p.d.f. would exhibit an infinite jump at $x = 0$ or at $x = 1$ and the inversion formula (1) would not be applicable. This is not a serious limitation since those U and J-shaped distributions are of negligible practical importance.

Springer and Thompson [10] gave an explicit formula for the p.d.f. of the product of n independent beta variables of a special class viz. with parameters $p = a+1, q = 1$ defined by the p.d.f.

$$(5)\qquad \beta(x; a+1, 1) = (a+1)x^a,$$

which they called "monomial" distributions; the rectangular distribution is a member of this class for $a = 0$. For this class of beta distributions the application of the Mellin transform gives an immediate answer: The Mellin transform of (5) is equal to $(a+1)(s+a)^{-1}$ and the Mellin transform of the p.d.f. of the product of n independent variables of that kind is equal to $(a+1)^n(s+a)^{-n}$. But the inverse Mellin transform of $(s+a)^{-n}$ can be found in the tables (see e.g. Baterman Manuscript [2], formula 7.1. (16)) so that

$$(6)\qquad \beta_n(x; a+1, 1) = \begin{cases} (a+1)^n x^a(-\log x)^{n-1}/(n-1)! & \text{for} \quad 0 < x \leqslant 1, \\ 0 & \text{for} \quad x \geqslant 1. \end{cases}$$

There is a wide-spread belief among working statisticians that the problem of distribution of products of independent random variables is not of a very great interest since "one can always find the distribution of $y = \log x$, apply the theorems on the addition of independent random variables $y_i = \log x_i$; and then revert to the product of the x's". It happens so that in the above particular case this procedure works and the application of the Mellin transform (although very attractive) is not needed. Indeed the distribution of $y = -\log x$ is given by the exponential distribution with parameter $\lambda = a+1$ so that its p.d.f. is given by $f(y) = (a+1)\exp[-y(a+1)]$. But it is well known that the sum of n independent random variables of this kind is a gamma variate with the shape parameter n and the scale parameter $a+1$ so that the p.d.f. of the sum of logarithms is

$$f_n(y) = (a+1)^n y^{n-1}\exp[-y(a+1)]/(n-1)!$$

Putting $x = \exp(-y)$ we obtain (6). However this seems to be an exceptionally simple situation and in more general cases the passage through the distribution of the sum of logarithms does not make the problem any easier.

2. Probability density function of the product of beta variables.

2.1. Simplification of the problem. The Mellin transform of (4) is equal to

$$M[\beta(x;p,q)|s] = \int_0^1 x^{p+s-2}(1-x)^{q-1}dx/B(p,q)$$
$$= B(p+s-1,q)/B(p,q) = \frac{\Gamma(p+q)}{\Gamma(p)}\cdot\frac{\Gamma(p+s-1)}{\Gamma(p+q+s-1)}.$$

Denoting by $\beta_n(x;p,q)$ the p.d.f. of the product of n such independent random variables we have, in view of (2),

$$(7)\quad M[\beta_n(x;p,q)|s] = [\Gamma(p+q)/\Gamma(p)]^n[\Gamma(p+s-1)/\Gamma(p+q+s-1)]^n$$

and (1) yields

$$\beta_n(x;p,q)$$
$$= \left[\frac{\Gamma(p+q)}{\Gamma(p)}\right]^n \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\{x^{-s}\Gamma^n(p+s-1)/\Gamma^n(p+q+s-1)\}ds, \qquad c>0.$$

On substitution $t = s+p-1$ this becomes

$$(8)\quad \beta_n(x;p,q) = \left[\frac{\Gamma(p+q)}{\Gamma(p)}\right]^n \frac{x^{p-1}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\{x^{-t}\Gamma^n(t)/\Gamma^n(t+q)\}dt, \qquad c>0,$$

the path of integration being still a line parallel to the imaginary axis and to the right of the origin since $p > 1$. But the above formula can be rewritten as

$$\beta_n(x; p, q) = \left[\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q+1)}\right]^n x^{p-1}\frac{\Gamma^n(q+1)}{2\pi i}\int_{c-i\infty}^{c+i\infty}\{x^{-t}\Gamma^n(t)/\Gamma^n(t+q)\}\,dt$$

which, in view of (8) with $p = 1$, gives

$$\beta_n(x; p, q) = \left[\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q+1)}\right]^n x^{p-1}\beta_n(x; 1, q). \tag{9}$$

This shows that it is sufficient to evaluate the p.d.f. of the product of n random variables of a family of one-parameter beta variables having $p = 1$ and arbitrary $q \geqslant 1$; for $p > 1$ the relevant p.d.fs are obtained by the above simple transformation.

In the subsequent two subsections we shall discuss the derivation of $\beta_n(x; 1, q)$ separately for the cases of integer and non-integer q.

2.2. The case of an integer q. From (7) we have

$$M[\beta_n(x; 1, q)\,|\,s] = \Gamma^n(q+1)\Gamma^n(s)/\Gamma^n(q+s).$$

But, with an integer q,

$$\Gamma(q+s) = (q+s-1)(q+s-2)\cdot\ldots\cdot s\Gamma(s),$$

so that

$$M[\beta_n(x; 1, q)\,|\,s] = \Gamma^n(q+1)\prod_{k=0}^{q-1}(s+k)^{-n},$$

and the inverse formula yields

$$\beta_n(x; 1, q) = \Gamma^n(q+1)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{x^{-s}\prod_{k=0}^{q-1}(s+k)^{-n}\right\}ds, \qquad c > 0. \tag{10}$$

The integrand in (10) has q poles of the n-th order for $s = -j$ $(j = 0, 1, \ldots, q-1)$ and the integral can be easily evaluated by contour integration yielding

$$\beta_n(x; 1, q) = \begin{cases}\Gamma^n(q+1)\sum_{j=0}^{q-1}R(x; n, j) & \text{for} \quad 0 < x \leqslant 1,\\ 0 & \text{for} \quad x \geqslant 1,\end{cases}$$

where

$$R(x; n, j) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\left\{x^{-s}(s+j)^n\prod_{k=0}^{q-1}(s+k)^{-n}\right\}\Big|_{s=-j} \tag{11}$$

is the residue of the integrand of (10) at the n-th order pole $s = -j$.

Let us write $\prod_{k=0}^{q-1}{}^{(j)} a_k$ for $\prod_{k=0}^{q-1} a_k/a_j$ (i.e. for the product in which the j-th factor has been omitted) and similarly $\sum_{k=0}^{q-1}{}^{(j)} a_k$ for $\sum_{k=0}^{q-1} a_k - a_j$ (i.e. for the sum in which the j-th term has been omitted). Denoting by $G_j(s)$ the function in the brackets in (11) we have

$$G_j(s) = x^{-s} \prod_{k=1}^{q-1}{}^{(j)} (s+k)^{-n}. \tag{12}$$

Before treating the case of arbitrary integer q let us discuss two simple cases of $q = 1$ and $q = 2$. *For* $q = 1$ we have to consider only one pole and its residue. Here (12) reduces to $G_0(s) = x^{-s}$ and (11) gives

$$R(x; n, 0) = \frac{1}{(n-1)!} x^{-s}(-\log x)^{n-1}|_{s=0} = \frac{1}{(n-1)!}(-\log x)^{n-1},$$

so that, in view of (9), we obtain

$$\beta_n(x; p, 1) = \begin{cases} \dfrac{p^n}{(n-1)!} x^{p-1}(-\log x)^{n-1} & \text{for} \quad 0 < x \leqslant 1, \\ 0 & \text{for} \quad x \geqslant 1, \end{cases}$$

which agrees with (6) when $p = a+1$. *For* $n = 2$ we have to consider two residues for $j = 0$ and $j = 1$ at the poles $s = 0$ and $s = -1$. Formula (12) yields

$$G_0(s) = x^{-s}(s+1)^{-n}, \qquad G_1(s) = x^{-s}s^{-n},$$

while formula (11) gives

$$R(x; n, 0) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\{x^{-s}(s+1)^{-n}\}|_{s=0},$$

$$R(x; n, 1) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\{x^{-s}s^{-n}\}|_{s=-1}.$$

Applying the Leibniz formula for the $(n-1)$-th derivative of the product we find

$$\beta_n(x; 1, 2) \tag{13}$$
$$= \begin{cases} 2^n \displaystyle\sum_{k=0}^{n-1} \binom{2n-k-2}{n-1} \frac{(-1)^n}{k!}(-\log x)^k[x+(-1)^{k+1}] & \text{for} \quad 0 < x \leqslant 1, \\ 0 & \text{for} \quad x \geqslant 1. \end{cases}$$

Thus in the cases of $q = 1$ and $q = 2$ an explicit formula for $\beta_n(x; 1, q)$ has been given. For higher values of q the application of the Leibniz rule would lead to formulae which would be more and more complicated and it is suggested to take advantage of the fact that the logarithmic derivatives of (12) can be easily obtained; once they are found the problem is solved since it is known how to obtain the n-th derivative of a function from its logarithmic derivatives. Indeed, if

$$A(s) = \frac{d}{ds}[\log G(s)], \qquad A^{(r)}(s) = \frac{d_1^r}{ds^r} A(s) \qquad (r = 1, 2, \ldots),$$

then

$$\frac{d^n}{ds^n} G(s) = G(s) \cdot Z_n[A(s), A^{(1)}(s), \ldots, A^{(n-1)}(s)],$$

where

$$Z_n[A, A^{(1)}, \ldots, A^{(n-1)}]$$
$$= \sum \frac{n!}{(n_1!)^{k_1}(n_2!)^{k_2}\ldots(n_r!)^{k_r} k_1!\, k_2! \ldots k_r!} [A^{(n_1-1)}]^{k_1}[A^{(n_2-1)}]^{k_2}\ldots[A^{(n_r-1)}]^{k_1};$$

here the sum is extended to all the partitions of number n such that $n = n_1 k_1 + n_2 k_2 + \ldots n_r k_r$, $A^{(0)}$ denotes $A(s)$ and $A^{(r)}$ the r-th derivative of $A(s)$ with respect to s. The polynomials Z_n are known as *polynomials of E. T. Bell* (cf. e. g. [8], chapter 2, section 8). Clearly

$$Z_1 = A, \qquad Z_2 = A^2 + A^{(1)}, \qquad Z_3 = A^3 + 3AA^{(1)} + A^{(2)},$$

and further formulae can be derived recursively by applying

$$Z_{n+1} = AZ_n + Z_n^{(1)}.$$

(Cf. for example [7].)

In our case $G_j(s)$ is given by (12) and

$$\log G_j(s) = -s\log x - n \sum_{k=0}^{q-1}{}^{(j)} \log(s+k).$$

The logarithmic derivative of (12) is

$$A_j(s) = -\log x - n \sum_{k=0}^{q-1}{}^{(j)} (s+k)^{-1}$$

and

$$A_j^{(r)}(s) = (-1)^{r+1} nr! \sum_{k=0}^{q-1}{}^{(j)} (s+k)^{-(r+1)} \qquad (r = 1, 2, \ldots).$$

Since, according to (11), these derivatives are to be taken for $s = -j$ we have

$$G_j(-j) = x_j \prod_{k=0}^{q-1}{}^{(j)} (k-j)^{-n},$$

(14) $$A_j(-j) = -\log x - n \sum_{k=0}^{q-1}{}^{(j)} (k-j)^{-1},$$

$$A_j^{(r)}(-j) = (-1)^{r+1} nr! \sum_{k=0}^{q-1}{}^{(j)} (k-j)^{-(r+1)} \qquad (r = 1, 2, \ldots)$$

and the final formula for the p.d.f. of the product is

(15)
$$\beta_n(x;1,q) = \frac{1}{(n-1)!}\,\Gamma^n_{(q+1)} \sum_{j=0}^{q-1} x^j \prod_{k=0}^{q-1}{}^{(j)} (k-j)^{-n} Z_{n-1}[A_j, A_j^{(1)}, \ldots, A_j^{(n-2)}],$$

where the arguments of the Z_{n-1}-function are given by (14).

In the case of a rectangular distribution $p = q = 1$

$$G_0(s) = x^{-s}, \quad \log G_0(s) = -s\log x, \quad A_0(s) = -\log x,$$

all the derivatives of $A_0(s)$ with respect to s vanishing. Hence

$$Z_{n-1} = A_0^{n-1} \quad \text{and} \quad \beta_n(x;1,1) = \frac{1}{(n-1)!}(-\log x)^{n-1}$$

which agrees with (6). In the case of $n = 2$ it is easy to verify for low values of n that formulae (13) and (15) give the same results. For higher values of n it is preferable to use formula (13).

2.3. The case when q is not an integer. From (7)

$$M[\beta_n(x;1,q)\,|\,s] = \Gamma^n(q+1)\Gamma^n(s)/\Gamma^n(q+s)$$

and

$$(16) \quad \beta_n(x;1,q) = \Gamma^n(q+1)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\Gamma(s)}{\Gamma(s+q)}\right]^n x^{-s}ds, \qquad c>0.$$

Here the integrand has an infinity of poles of the n-th order at $s = -j$ $(j = 0, 1, \ldots)$ and the integral in (16) can be evaluated by contour integration yielding

$$(17) \qquad \beta_n(x;1,q) = \begin{cases} \Gamma^n(q+1)\sum\limits_{j=0}^{\infty} R(x;n,j) & \text{for} \quad 0<x<1, \\ 0 & \text{for} \quad x \geqslant 1, \end{cases}$$

where

$$R(x;n,j) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\left\{x^{-s}\frac{\Gamma^n(s)}{\Gamma^n(q+s)}(s+j)^n\right\}\bigg|_{s=-j}$$

is the residue of the integrand of (16) at the n-th order pole at $s = -j$.

Remark. When establishing the validity of the first line of (17) we integrate along a contour composed of the segment of integration path of (16) between $c-iA$ and $c+iA$, of two horizontal segments joining (in the s-plane) the pair of points $c+iA$ and $-A+iA$ and the pair $c-iA$ and $-A-iA$ and finally of a vertical segment joining points $-A-iA$ and $-A+iA$. It is important to choose a sequence of values A tending to infinity so that the vertical segment on the left side of imaginary axis should avoid the successive poles of the integrand of (16); this can be done, for instance, by putting $A = m+\frac{1}{2}$ $(m = 1, 2, \ldots)$. For more detailed account of all the contour integrations applied in this paper see the author's report [7a].

Clearly

$$\Gamma(s)(s+j) = \frac{\Gamma(s+j+1)}{(s+j-1)(s+j-2)\dots s} = \Gamma(s+j+1)\prod_{k=0}^{j-1}(s+k)^{-1}$$

and for $j = 0$ the above identity is still valid if we agree to interpret an "empty" product $\prod_{k=0}^{-1}$ as unity. Hence

(18)
$$R(x;n,j) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\left\{x^{-s}\Gamma^{n}(s+j+1)\Gamma^{-n}(q+s)\prod_{k=0}^{j-1}(s+k)^{-n}\right\}\Bigg|_{s=-j}.$$

Denoting by $G_j(s)$ the function in the brackets in (18)

$$G_j(s) = x^{-s}\Gamma^{n}(s+j+1)\Gamma^{-n}(q+s)\prod_{k=0}^{j-1}(s+k)^{-n},$$

we have

$$\log G_j(s) = -s\log x + n\log\Gamma(s+j+1) - n\log\Gamma(q+s) - n\sum_{k=0}^{j-1}\log(s+k)$$

(interpreting again an "empty" sum $\sum_{k=0}^{-1}$ as zero to retain the validity of the above for $j = 0$).

The successive logarithmic derivatives of $G_j(s)$ are

$$A_j(s) = -\log x + n\psi(s+j+1) - n\psi(q+s) - n\sum_{k=0}^{j-1}(s+k)^{-1},$$

$$A_j^{(1)}(s) = n\psi^{(1)}(s+j+1) - n\psi^{(1)}(q+s) + n\sum_{k=0}^{j=1}(s+k)^{-2},$$

$$A_j^{(r)}(s) = n\psi^{(r)}(s+j+1) - n\psi^{(r)}(q+s) + (-1)^{r+1}r!\sum_{k=0}^{j-1}(s+k)^{-(r+1)},$$

where $\psi(\cdot)$ denotes the Euler Psi-function the logarithmic derivative of $\Gamma(\cdot)$ and $\psi^{(r)}(\cdot)$ its successive derivatives.

Consequently

$$G_j(-j) = x^j\prod_{k=0}^{j-1}(k-j)^{-n}/\Gamma^{n}(q-j) = \frac{x^j(-1)^{jn}}{(j!)^n\Gamma^n(q-j)},$$

(19)
$$A_j(-j) = -\log x + n\psi(1) - n\psi(q-j) + n\sum_{k=1}^{j}k^{-1},$$

$$A_j^{(r)}(-j) = n\psi^{(r)}(1) - n\psi^{(r)}(q-j) + nr!\sum_{k=1}^{k=1}k^{-(r+1)},$$

and, by the argument which has been applied in the derivation of formula (15) we find

(20)

$$\beta_n(x;1,q) = \frac{1}{(n-1)!}\Gamma^n(q+1)\sum_{j=0}^{\infty} x^j \frac{(-1)^{jn}}{(j!)^n\Gamma^n(q-j)} Z_{n-1}[A_j, A_j^1, \ldots, A^{(n-2)}],$$

where the values to be put into Z_{n-1}-function are given by (19).

The infinite series (20) is absolutely convergent for $0 < x < 1$. Indeed, the coefficient $\Gamma^n(q+1)(j!)^{-n}\Gamma^{-n}(q-j)$ is equal to the n-th power of $Q_j = q(q-1)\ldots(q-j)/j!$ $(j = 0, 1, \ldots)$. For $j = 0, 1, \ldots, [q]$ there are $[q]+1$ positive values of Q_j; let Q be their maximum. For $j > [q]+1$, $|q-j|/j = (j-q)/j < 1$ so that $|Q_j| < Q$. Further, the values to be put into the Z_{n-1}-function are given by (19) and since q is not an integer the series expansions of $\psi^{(r)}(y)$ valid for $y \neq 0, -1, -2, \ldots$ can be applied to evaluate $\psi^{(r)}(q-j)$ (see e.g. [1] formula (6.4.10)). Thus

$$|\psi^{(r)}(q-j)| \leqslant r!\sum_{k=0}^{\infty}|q-j+k|^{-(r+1)} \leqslant r!\Big\{(q-[q])^{-(r+1)}+ \\ +[[q]+1-q]^{-(r+1)}+2\sum_{k=1}^{\infty}k^{-(r+1)}\Big\}.$$

This is bounded from above by a constant independent of j and the same applies to the remaining terms of $A_j^{(r)}$ (see e.g. [1], formula (6.4.2)). The terms of A_j are not bounded but, by a similar argument, it can be shown that they are of order of $O(\log j)$ for any fixed $x > 0$ (see e.g. [1] formula (6.3.2) and the recurrence formula (6.3.6)). Since the values (19) enter into formula (20) as the n-th powers at the most, the series (20) is dominated by $\sum_{j=1}^{\infty} x^j(\log j)^n$ multiplied by a constant and this shows that it is convergent for positive x smaller than 1.

3. Probability distribution function of the product of beta variables.

3.1. General remarks. Let the probability distribution function corresponding to the p.d.f. (4) be

$$B_n(x;p,q) = \int_0^x \beta(t;p,q)\,dt = B^{-1}(p,q)\int_0^x t^{p-1}(1-t)^{q-1}dt.$$

In view of (3) and (7) we have

$$M[1-B_n(x;p,q)|s] = s^{-1}[\Gamma(p+q)/\Gamma(p)]^n\cdot[\Gamma(p+s)/\Gamma(p+q+s)]^n$$

and the inversion formula (1) yields

$$(21)\quad 1-B_n(x; p, q)$$
$$= \left[\frac{\Gamma(p+q)}{\Gamma(p)}\right]^n \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \{x^{-s}s^{-1}\Gamma^n(p+s)/\Gamma^n(p+q+s)\}\,ds, \qquad c>0.$$

It is not possible to simplify the problem by reducing the discussion to the one-parameter family of probability distribution functions in the way similar to that carried out in Section 2.1 for the p.d.fs. We have to investigate the two-parameter family of probability distribution functions and we shall again discuss separately the cases of integer and non-integer q.

3.2. The case of integer q. Since for integer q

$$\Gamma(p+q+s) = (p+q+s-1)(p+q+s-2)\cdot\ldots\cdot(p+s)\Gamma(p+s),$$

(21) becomes

$$(22)\quad 1-B_n(x; p, q)$$
$$= \frac{\Gamma^n(p+q)}{\Gamma^n(p)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{x^{-s}s^{-1} \prod_{k=0}^{q-1} (p+s-k)^{-n}\right\} ds, \qquad c>0.$$

The integrand has one single pole at $s=0$ and q poles of the n-th order at $s=-j-p$ $(j=0,1,\ldots,q-1)$. The integral in (22) can be evaluated again by contour integration as in the case if integral in formula (10) and since the residue at $s=0$ multiplied by $\Gamma^n(p+q)/\Gamma^n(p)$ is equal to 1 we have

$$B_n(x; p, q) = \begin{cases} -\dfrac{\Gamma^n(p+q)}{\Gamma^n(p)} \displaystyle\sum_{j=0}^{q-1} R^*(x; n, j) & \text{for} \quad 0<x<1, \\ 1 & \text{for} \quad x \geqslant 1, \end{cases}$$

where $R^*(x; n, j)$ is the residue of the integral appearing in (22) at the n-th order pole at $s=-j-p$ $(j=0,1,\ldots,q-1)$:

(23)

$$R^*(x; n, j) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{x^{-s}s^{-1}(s+j+p)^n \prod_{k=0}^{q-1} (p+s+k)^{-n}\right\}\Big|_{s=-p-j}.$$

Retaining the notation introduced in formula (12) we can write the function $G_j^*(s)$ in the brackets of (23) as

$$(24)\qquad G_j^*(s) = x^{-s}s^{-1} \prod_{k=0}^{q-1}{}^{(j)} (p+s+k)^{-n}.$$

In the *case of* $q = 1$ we have only one residue for $j = 0$. From (24) $G_0^*(s) = x^{-s}s^{-1}$ and

$$R^*(x;\, n,\, 0)$$
$$= \frac{1}{(n-1)!} \sum_{r=0}^{n-1} \binom{n-1}{r} x^{-s}(-\log x)^r(-1)^{n-r-1}(n-r-1)!\, s^{-(n-r)}\big|_{s=-p}$$

so that

$$B_n(x;\, p,\, 1) = x^p \sum_{r=0}^{n-1} \frac{p^r}{r!}(-\log x)^r. \tag{25}$$

To verify (25)we differentiate it with respect to x and obtain (6).

In the *case of* $q = 2$ we have to consider two residues for $j = 0$ and $j = 1$. Thus

$$R^*(x;\, n,\, 0) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \{x^{-s}s^{-1}(p+s+1)^{-n}\}|_{s=-p},$$

and a similar expression can be written for the residue at $s = -p-1$. Applying the Leibniz formula for the $(n-1)$-th derivative of the product and evaluating these derivatives for $s = -p$ and $s = -p-1$ respectively we find, after simple algebra,

$$B_n(x;\, p,\, 2) \tag{26}$$
$$= (-1)^n p^n (p+1)^n \sum_{r=0}^{n-1} \binom{2n-r-2}{n-1} \sum_{k=0}^{r} \frac{(-\log x)^k}{k!} \times$$
$$\times x^p[p^{-r+k-1}(-1)^{r+1} + x(p+1)^{-r+k-1}].$$

By differentiating (26) with respect to x we verify that its derivative is equal to (13) as it should be.

Again, as in Section 2.2 for higher values of n the application of the Leibniz rule would lead to complicated expressions and we shall again evaluate the logarithmic derivative of (24). Here

$$\log G_j^*(s) = -s\log x - \log s - n \sum_{k=0}^{q-1}{}^{(j)} \log(p+s+k). \tag{27}$$

Evaluating the derivative $A(s)$ of (27) with respect to s and higher derivatives $A^r(s)$ and bearing in mind that they have to be taken for $s = -j-p$ we obtain

$$G_j^*(-j-p) = x^{j+p}(j+p)^{-1} \prod_{k=0}^{q-1}{}^{(j)} (k-j)^{-n},$$

$$A_j^*(-j-p) = -\log x + (j+p)^{-1} - n \sum_{k=0}^{q-1}{}^{(j)} (k-j)^{-1}, \tag{28}$$

$$A_j^{*(r)}(-j-p) = r!(j+p)^{-(r+1)} + nr!(-1)^{(r+1)} \sum_{k=0}^{q-1}{}^{(j)} (k-j)^{-(r+1)},$$

and the final formula for the probability distribution function of the product is given by

$$(29)\qquad B_n(x;p,q) = \frac{1}{(n-1)!}\left[\frac{\Gamma(p+q)}{\Gamma(p)}\right]^n \sum_{j=0}^{q-1} \frac{x^{p+j}}{p+j} \prod_{k=0}^{q-1}{}^{(j)} \times$$

$$\times (k-j)^{-n} Z_{n-1}[A_j^*, A_j^{*1}, \ldots, A_j^{*(n-2)}],$$

where the arguments of Z_{n-1}-function are given by (28).

EXAMPLE. If $q = 1$ then, according to (29),

$$B_n(x;p,1) = \frac{p^{n-1}}{(n-1)!} x^p Z_{n-1}\left[\left(-\log + \frac{1}{p}\right), \frac{1}{p^2}, \frac{2}{p^3}, \ldots, \frac{(n-2)!}{p}\right].$$

This formula is, for high values of n, more complicated than (25) but it can be verified that for $n = 2, 3$ etc. it is equivalent to (25).

3.3. The case when q is not an integer. Since q is not an integer the integral (21) cannot be written in a simpler form as in the case of (22). The integrand of (21) has a single pole at $s = 0$ and an infinity of poles of n-th order at $s = -p-j$ $(j = 0, 1, \ldots)$. The integral in (21) can be again evaluated by contour integration by a method similar to that applied in the evaluation of the integral of (16). Since the residue at $s = 0$ multiplied by $\Gamma^n(p+q)/\Gamma^n(p)$ is equal to 1 we have

$$B_n(x;p,q) = \begin{cases} -\left[\dfrac{\Gamma(p+q)}{\Gamma(p)}\right]^n \displaystyle\sum_{j=0}^{\infty} R^*(x;n,j) & \text{for} \quad 0 < x < 1, \\ 1 & \text{for} \quad x \geqslant 1, \end{cases}$$

where

$$R^*(x;n,j) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{ x^{-s} s^{-1} (p+s+j)^n \frac{\Gamma^n(p+s)}{\Gamma^n(p+q+s)} \right\}\Bigg|_{s=-p-j}$$

is the residue of the integrand of (21) at the n-th order pole at $s = -p-j$ $(j = 0, 1, \ldots)$. Clearly

$$\Gamma(p+s)(p+s+j) = \frac{\Gamma(s+p+j+1)}{(p+s+j-1)(p+s+j-2)\cdot\ldots\cdot(p+s)} =$$

$$= \Gamma(s+p+j+1) \prod_{k=0}^{j-1} (p+s+k)^{-1},$$

where again an "empty" product $\prod_{k=0}^{-1}$ should be interpreted as unity.

Hence

$$(30)\qquad R^*(x;n,j) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\times$$

$$\times\left\{x^{-s}s^{-1}\Gamma^n(s+p+j+1)\prod_{k=0}^{j-1}(p+s+k)^{-n}/\Gamma^n(p+q+s)\right\}\Bigg|_{s=-p-j}.$$

Denoting by $G_j^*(s)$ the function in the brackets in (30) we have

$$(31)\qquad \log G_j^*(s) = -s\log x-\log s+n\log\Gamma(s+p+j+1)-$$

$$-n\log\Gamma(p+q+s)-n\sum_{k=0}^{j-1}\log(p+s+k).$$

Evaluating the successive derivatives of (31) and bearing in mind that they have to be taken at $s=-p-j$ we find

$$G_j^*(-p-j) = -x^{p+j}(p+j)^{-1}(-1)^{nj}(j!)^{-n}\Gamma^{-n}(q-j),$$

$$(32)\qquad A_j^*(-p-j) = -\log x+(p+j)^{-1}+n\psi(1)-n\psi(q-j)+n\sum_{k=1}^{j}k^{-1},$$

$$A_j^{*(r)}(-p-j) = (-1)^{r+1}r!(p+j)^{-(r+1)}+$$

$$+n\psi^{(r)}(1)-n\psi^{(r)}(q-j)+nr!\sum_{k=1}^{j}k^{-(r+1)},$$

and finally

$$(33)\qquad B_n(x;p,q) = \left[\frac{\Gamma(p+q)}{\Gamma(p)}\right]^n\frac{1}{(n-1)!}\sum_{j=0}^{\infty}\times$$

$$\times\frac{x^{p+j}}{p+j}\frac{(-1)^{nj}}{(j!)^n}\frac{1}{\Gamma^n(q-j)}Z_{n-1}[A_j^*, A_j^{*(1)}, \ldots, A_j^{*(n-2)}],$$

where the arguments of Z_{n-1}-function are given by (32).

The convergence of the infinite series (33) follows from a similar argument as that applied in the case of the infinite series (20).

4. Distribution of products of independent random variables having the same beta distribution with different scale parameters.

The generalization of the above results to the case important in practical applications when $y = x_1x_2\ldots x_n$ and the i-th factor has the p.d.f. $a_i\beta(a_ix_i;p,q)$ is immediate. The p.d.f. $g_n(y)$ of such a product is equal to

$$(34)\qquad q_n(y) = a_1a_2\cdot\ldots\cdot a_n\beta_n(a_1a_2\ldots a_ny;p,q),$$

and consequently the probability distribution function $G_n(y)$ is equal to

$$G_n(y) = B_n(a_1 a_2 \ldots a_n y; p, q).$$

Indeed, since

$$M[f(ax)|s] = a^{-s} M[f(x)|s],$$

we have

$$\begin{aligned} M[q_n(y)] &= [a_1 a_2 \ldots a_n]^{-s+1} \{M[\beta(x; p, q)|s]\}^n \\ &= [a_1 a_2 \ldots a_n]^{-s+1} M[\beta_n(y; p, q)|s] \\ &= a_1 a_2 \ldots a_n M[\beta_n(a_1 a_2 \ldots a_n y; p, q)|s] \\ &= M[a_1 a_2 \ldots a_n \beta_n(a_1 a_2 \ldots a_n y)|s], \end{aligned}$$

which completes the proof of (34) in view of the uniqueness property of the Mellin transform (cf. [3], p. 104, Th. 2).

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BOULTON PAUL AIRCRAFT LTD.
WOLVERHAMPTON, ENGLAND

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON, USA

Received on 15. 9. 1967