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ALGORITHM 7

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EVALUATION OF A TRIGONOMETRIC POLYNOMIAL

1. Function declaration.

real procedure *Foursum* (t, n, a, b);
  value t, n;
  integer n;
  real t;
  array a, b;
  comment the function *Foursum* calculates the value of the trigonometric polynomial

(1) $$\frac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kt+b_k\sin kt).$$

Data:
t — value of the variable in (1),
n — degree of the polynomial (1),
$a[0:n]$, $b[1:n]$ — arrays of the coefficients of (1);

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begin
  real an, an1, bn, ct, st;
  an := a[n];
  bn := b[n];
  ct := cos(t);
  st := sin(t);
  for n := n-1 step -1 until 1 do
    begin
    an1 := a[n] + an × ct + bn × st;
    bn := b[n] - an × st + bn × ct;
    an := an1
    end n;
  Foursum := .5 × a[0] + an × ct + bn × st
end Foursum
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2. Method used. Polynomial (1) is evaluated by use of the following recurrent formulae

$$\alpha_n = a_n, \quad \beta_n = b_n,$$

$$\left.\begin{aligned}\alpha_k &= a_k + \alpha_{k+1}\cos t + \beta_{k+1}\sin t\\ \beta_k &= b_k - \alpha_{k+1}\sin t + \beta_{k+1}\cos t\end{aligned}\right\} \quad (k = n-1,\ n-2, \ldots, 1),$$

$$f(t) = \frac{1}{2}\alpha_0 + \alpha_1\cos t + \beta_1\sin t. \tag{2}$$

For the proof we show by induction that

$$f(t) = \tfrac{1}{2}a_0 + \sum_{k=1}^{j}(a_k\cos kt + b_k\sin kt) + \big(\alpha_{j+1}\cos(j+1)t + \beta_{j+1}\sin(j+1)t\big) \tag{3}$$

for $j = n-1,\ n-2, \ldots, 0$.

For $j = n-1$ formula (3) is evident. Since

$$\begin{aligned}&a_j\cos jt + b_j\sin jt + \alpha_{j+1}\cos(j+1)t + \beta_{j+1}\sin(j+1)t\\ &\quad = (a_j + \alpha_{j+1}\cos t + \beta_{j+1}\sin t)\cos jt + (b_j - \alpha_{j+1}\sin t + \beta_{j+1}\cos t)\sin jt\\ &\quad = \alpha_j\cos jt + \beta_j\sin jt,\end{aligned}$$

it follows that if (3) holds for any fixed j, then it also holds for $j-1$. Formula (2) is obtained in the case $j = 0$.

If we write

$$X_k = (\alpha_k, \beta_k), \quad A_k = (a_k, b_k) \quad \text{for } k = 1, 2, \ldots, n,$$

$$A_0 = (a_0, 0), \quad M = \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix},$$

we obtain from (2) a scheme for the evaluation of a matrix polynomial $A_0 + MA_1 + M^2A_2 + \ldots + M^nA_n$, analogous to the well-known Horner scheme. Applying to it an argumentation being similar to that one used by Clenshaw [1] for his Chebyshev series summation algorithm, one may prove that there is no danger of any building-up errors in our recurrence process.

The algorithm given in section 1 was probably never published before, though Thacher in [3] had it certainly in mind. Algorithms for separate evaluation of

$$\tfrac{1}{2}a_0 + \sum_{k=1}^{n} a_k\cos kt \quad \text{and} \quad \sum_{k=1}^{n} b_k\sin kt$$

were given by Matthewman [2] and Wells [4].

3. Verification. The function *Foursum* was verified on the Odra 1204 computer for the polynomials

$$(4) \qquad p_n(t) = 1+\sum_{k=1}^{n}(\cos kt+\sin kt) \qquad (n = 1, 2, \ldots, 50)$$

and for the values of $t = 0.1, 0.2, 0.5, 1$ (see Thacher [3]). The results were compared with those calculated from the formulae

$$5) \qquad P_n(t) = 1+\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\left(\cos\tfrac{1}{2}(n+1)t+\sin\tfrac{1}{2}(n+1)t\right).$$

In 165 of the 200 cases the absolute difference of the values was found less than 10^{-10} of the mantissa. The greatest absolute difference was $348_{10}-11$ (for $n = 48$ and $t = 1$ the evaluation of (4) and (5) resulted in $-0.790\,271\,425\,62$ and $-0.790\,271\,422\,14$, respectively). The greatest relative absolute difference (calculated relative to the value of $P_n(t)$) was found $1382_{10}-11$ (for $n = 43$ and $t = 1$ formulae (4), (5) have given the values $0.757\,237\,212\,39_{10}-2$, $0.757\,237\,226\,21_{10}-2$, respectively).

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References

[1] C. W. Clenshaw, *A note on the summation of Chebyshev series*, Math. Tables Aids Comput. 9 (1955), pp. 118-120.

[2] J. H. Matthewman, *Note on the selective summation of Fourier series*, Computer J. 6 (1963), pp. 248-249.

[3] H. C. Thacher, Jr., *Certification of algorithm 128*, Comm. ACM 7 (1964), p. 421.

[4] M. Wells, *Algorithm 128. Summation of Fourier series*, Comm. ACM 5 (1962), p. 513.

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LUCJA SOBICH (Lublin) ALGORYTM 7

OBLICZANIE WARTOŚCI WIELOMIANU TRYGONOMETRYCZNEGO

STRESZCZENIE

Wartością funkcji *Foursum* jest wartość wielomianu trygonometrycznego (1).

Dane: t — wartość zmiennej w (1): n — stopień wielomianu (1); $a[0: n]$, $b[1: n]$ — tablice współczynników wielomianu (1).

Użyta metoda, korzystająca ze znanych związków rekurencyjnych dla sinusa i kosinusa, jest opisana w § 2. Obliczenia kontrolne (§ 3) wykonano na maszynie cyfrowej Odra 1204.

ЛЮЦИЯ СОБИХ (Люблин) АЛГОРИТМ 7

ВЫЧИСЛЕНИЕ ЗНАЧЕНИЙ ТРИГОНОМЕТРИЧЕСКОГО МНОГОЧЛЕНА

РЕЗЮМЕ

Значением функции *Foursum* является значение тригонометрического многочлена (1).

Данные: t — значение переменной в (1), n — степень многочлена (1), $a[0:n]$, $b[1:n]$ — массивы коэффициентов многочлена (1).

Применяется метод основанный на известных рекуррентных соотношениях для синуса и косинуса (§ 2). Контрольные вычисления (§ 3) выполнены на ЭВМ Одра 1204.
