ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVI, 4 (1980)

W. OKRASIŃSKI (Wrocław)

SOME REMARKS ABOUT THE INFILTRATION OF WATER FROM A CYLINDRICAL RESERVOIR

Let us consider a cylindrical reservoir, with the bottom touching a horizontal impermeable layer, which is filled with water and surrounded by dry soil. It is well known that under some simplifying assumptions of the hydrogeological kind (see [1], [3] and [4]) the process of percolation may be described by the Boussinesq equation which in the case of radial symmetry is of the form

$$(1)\qquad (hh_r)_r+\frac{1}{r}hh_r = h_t.$$

The function $h = h(r, t)$ describes the free surface of the moistened region which, obviously, depends on time t. In paper [2] the authors were studying a class of approximative solutions of the problem, which may be obtained – after the transformation of equation (1) – into an ordinary one. They gave also an estimate of the error under certain suppositions concerning the free surface.

The aim of the present paper is to give another estimate of the error which seems to have a nicer form. Our result is obtained by a method which is different from that used in [2]. Instead of the comparison of the solutions of two ordinary differential equations (the approximative and exact ones) we bring the exact equation to an integral form, analogous to that obtained in [2] for the approximative equation.

1. Reduction of the considered problem to the integral equation. In [2] it is assumed that the reach r_0 of the infiltrating water at time $t \geqslant 0$ is described by the formula

$$r_0(t) = \sqrt{2ct+1},$$

where c is a positive constant. We put $A = r_0^2(t)$. In [2] the approximative solution h_0 of (1) is sought in the form

$$h_0(r, t) = p(s), \quad \text{where } s = \frac{r}{r_0(t)} \ \left(s \in \left[\frac{r}{r_0(t)}, 1\right]\right).$$

The function p satisfies the equation

(2) $$R(p) = 0,$$

where

$$R(p) \equiv (pp')' + \frac{1}{s} pp' + csp'.$$

Equation (2) is reduced to the integral equation

(3) $$u_0^2(x) = 2\int_0^x K(x, \tau) u_0(\tau) d\tau \qquad (x \in [0, 1]),$$

where

$$K(x, \tau) = A^{-\tau}\big(1 + (x-\tau)\ln A\big) \quad \text{and} \quad u_0(x) = \left[\frac{c}{2}\ln A\right]^{-1} p(A^{-x/2}).$$

Equation (3) has a unique continuous solution u_0 such that $u_0(0) = 0$ and $\dot{u}_0(x) > 0$ for $x \in (0, 1]$. The function u_0 depends on time t according to the meaning of the parameter A. Therefore, we write $u_0 = u_0(x, A)$. Thus we obtain

(4) $$h_0(r, t) = c\ln r_0(t) u_0\left(1 - \frac{\ln r}{\ln r_0(t)}, A\right).$$

Let h be the exact solution of the considered problem. According to the physical meaning we can suppose that h is a non-increasing function with respect to r for every $t > 0$. In new variables s and A the solution h takes the form

(5) $$h(r, t) = P_A(s).$$

In [2] it is shown that P_A satisfies the equation

(6) $$R(P_A) = F(s, A),$$

where

$$F(s, A) = cs\sqrt{A}\, h_r\left(s\sqrt{A}, \frac{A-1}{2c}\right) + Ah_t\left(s\sqrt{A}, \frac{A-1}{2c}\right).$$

Using the same method as in [2] we can reduce (6) to the integral equation

(7) $$u^2(x, A) = 2\int_0^x K(x, \tau) u(\tau, A) d\tau + \frac{2}{c^2}\int_0^x \int_0^\tau A^{-\sigma} F(A^{-\sigma/2}, A) d\sigma d\tau,$$

where

(8) $$u(x, A) = \left[\frac{c}{2}\ln A\right]^{-1} P_A(A^{-x/2}).$$

From (5) and (8) we obtain

$$h(r,t) = c\ln r_0(t)\, u\left(1-\frac{\ln r}{\ln r_0(t)}, A\right). \tag{9}$$

In the next section we give an estimate of the difference between h_0 and h.

2. Estimate of the error. For $A \geqslant e$ and $a \in (\frac{1}{2}, 1)$ we put

$$I = \left(0, \left(2-\frac{1}{a}\right)\ln^{-1}A\right] \tag{10}$$

and we write

$$d(u_0, u) = \sup_{\tau\in I} \tau^{-1}|u_0(\tau, A) - u(\tau, A)|.$$

THEOREM 1. *If the function F satisfies the inequality*

$$-M(\sqrt{A}-1)^{-1} \leqslant F(s, A) \leqslant 0 \quad (A > 1,\ s \in [1/A, 1]) \tag{11}$$

with a positive constant M, then for every $A \geqslant e$ and $a \in (\frac{1}{2}, 1)$ the inequality

$$d(u_0, u) \leqslant \frac{1}{1-a} G(A) \frac{\ln A - 1 + A^{-1}}{(1-A^{-1})\ln A} \tag{12}$$

holds, where $G(A) = 2Mc^{-2}(\sqrt{A}-1)^{-1}$.

Before proving Theorem 1 we give the following

LEMMA. *For every $A > 1$ and $a \in (\frac{1}{2}, 1)$ we have*

$$\int_0^x K(x,\tau)\tau\, d\tau \leqslant ax\frac{1-A^{-x}}{\ln A} \quad \textit{for } x \in I.$$

Proof. Let

$$\psi(x) = ax\frac{1-A^{-x}}{\ln A} - \int_0^x K(x,\tau)\tau\, d\tau.$$

We have $\psi(0) = \psi'(0) = 0$ and

$$\psi''(x) = aA^{-x}\ln A\left[\left(2-\frac{1}{a}\right)\ln^{-1}A - x\right].$$

Thus $\psi''(x) \geqslant 0$ and, therefore, $\psi(x) \geqslant 0$ for $x \in I$.

Proof of Theorem 1. In [2] it is shown that the solution u_0 of (3) satisfies the inequality

$$\frac{1-A^{-x}}{\ln A} \leqslant u_0(x, A) \leqslant x \quad \text{for } x \in [0, 1]. \tag{13}$$

Using a similar method we can show that

(14) $$0 \leqslant u(x, A) \leqslant x \quad \text{for } x \in [0, 1].$$

By (3) and (7) we have

(15) $$|u_0(x, A) - u(x, A)| = [u_0(x, A) + u(x, A)]^{-1} \Big| \int_0^x K(x, \tau)\big(u_0(\tau, A) - u(\tau, A)\big) d\tau - \frac{2}{c^2} \int_0^x \int_0^\tau A^{-\sigma} F(A^{-\sigma/2}, A) d\sigma d\tau \Big|.$$

From (13) and (14) we obtain

(16) $$u_0(x, A) + u(x, A) \geqslant \frac{1 - A^{-x}}{\ln A}.$$

Applying (11) and (16) to (15) we get

(17) $$|u_0(x, A) - u(x, A)| \leqslant \frac{\ln A}{1 - A^{-x}} d(u_0, u) \int_0^x K(x, \tau) d\tau + G(A) \frac{\ln A}{1 - A^{-x}} \int_0^x \int_0^\tau A^{-\sigma} d\sigma d\tau \quad \text{for } x \in I.$$

Using (17) and the Lemma we obtain

(18) $$|u_0(x, A) - u(x, A)| \leqslant a x d(u_0, u) + G(A) \frac{x \ln A - 1 + A^{-x}}{(1 - A^{-x}) \ln A}.$$

By (18) we have

(19) $$d(u_0, u) \leqslant a d(u_0, u) + G(A) \frac{1}{\ln A} \sup_{\tau \in I} \frac{\tau \ln A - 1 + A^{-\tau}}{1 - A^{-\tau}}.$$

After simple calculations we get

(20) $$\sup_{\tau \in (0,1]} \frac{\tau \ln A - 1 + A^{-\tau}}{1 - A^{-\tau}} = \frac{\ln A - 1 + A^{-1}}{1 - A^{-1}}.$$

Applying (20) to (19) we obtain (12).

THEOREM 2. *For* $\gamma \in (e^{-1/2}, 1)$ *we have*

(21) $$\sup_{r \in J} |h_0(r, t) - h(r, t)| = O\big(r_0^{-1}(t)\big) \quad \text{as } t \to +\infty,$$

where $J = \big[\gamma r_0(t), r_0(t)\big)$.

Proof. Since

$$d(u_0, u) \geqslant \left[\left(2-\frac{1}{a}\right)\ln^{-1}A\right]^{-1} \sup_{\tau\in I}|u_0(\tau, A)-u(\tau, A)|,$$

from Theorem 1 we obtain

$$\sup_{\tau\in I}|u_0(\tau, A)-u(\tau, A)| \leqslant \frac{1}{1-a}G(A)\frac{\ln A-1+A^{-1}}{(1-A^{-1})\ln^2 A}. \tag{22}$$

Let

$$a = \frac{1}{2(\ln\gamma+1)}.$$

For this number a, applying (4) and (9) to (22) we get

$$\sup_{r\in J}|h_0(r, t)-h(r, t)| \leqslant \frac{2M(\ln\gamma+1)}{c(2\ln\gamma+1)}\,\frac{r_0^2(t)}{\left(r_0(t)-1\right)\left(r_0^2(t)-1\right)}.$$

Thus we infer that (21) is true.

It can be computed that $e^{-1/2} \approx 0.6065307$.

Formula (21) shows that h_0 approximates well the free surface on the interval $[0.6065307\,r_0(t), r_0(t)]$ as the time t is sufficiently large.

3. Final remarks. Assumption (10) is of a physical meaning. Let us consider, as in [2], a simple model of our physical problem. Assuming that

$$h(r, t) = \frac{r_0(t)-r}{r_0(t)-1} \tag{23}$$

we obtain

$$h_r = -\frac{1}{r_0(t)-1} \quad \text{and} \quad h_t = \frac{c(r-1)}{\left(r_0(t)-1\right)^2 r_0(t)}$$

(let us note that (23) does not satisfy equation (1)). In this model the function F takes the form

$$F(s, A) = -c\frac{\sqrt{A}(1-s)}{(\sqrt{A}-1)^2}.$$

After simple calculations we see that F satisfies (11) with the positive constant $M = 2c$.

References

[1] J. Bear, D. Zaslavsky and S. Irmay, *Physical principles of water percolation and seepage*, UNESCO 1968 (Russian translation: Moscow 1971).

[2] J. Goncerzewicz, H. Marcinkowska, W. Okrasiński and K. Tabisz, *On the percolation of water from a cylindrical reservoir into the surrounding soil*, Zastosow. Matem. 16 (1978), p. 249-261.

[3] P. J. Polubarinova-Kochina (П. Я. Полубаринова-Кочина), *Теория движения грунтовых вод*, Москва 1952.
[4] A. Rybarski and C. Ryll-Nardzewski, *O filtracji beznaporowej w ośrodku jednorodnym i izotropowym, I*, Matematyka Stosowana 5 (1975), p. 157-166.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
50-384 WROCŁAW

Received on 30. 6. 1977

W. OKRASIŃSKI (Wrocław)

PEWNE UWAGI O FILTRACJI WODY ZE ZBIORNIKA W KSZTAŁCIE WALCA

STRESZCZENIE

W pracy przedstawiono oszacowanie różnicy między rozwiązaniem dokładnym a pewnym rozwiązaniem przybliżonym równania różniczkowego, opisującego proces nawilżania gruntu przez wodę infiltrującą ze zbiornika w kształcie walca.
