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THE $PH/M/c$ QUEUE WITH VARYING ENVIRONMENT

We study the steady state queue length of the $PH/M/c$ queue in which the arrival time distribution and the service rate change in accordance with the change of state in a continuous-time irreducible Markov chain.

1. Introduction. Consider an m-state irreducible continuous-time Markov chain with infinitesimal generator Q which describes a randomly varying environment for a queue of $PH/M/c$ type. We specify that, whenever the Markov chain is in the state i, the service rate of the server is $\mu_i > 0$, $1 \leqslant i \leqslant m$. We always assume that only one customer is in the process of joining the queue for service and that the customer arrival time distribution when he starts to the queue in the environment i is of phase type $F_i(\cdot)$. If the environment changes from i to j during his arrival time to the queue, his new arrival time distribution is the stationary version of $F_j(\cdot)$. The random environment model under exponential assumptions was first considered by Eisen and Tainiter [1] who studied the particular cases with $m = 2$. Yechiali and Naor [11], Yechiali [10], and Purdue [8] also treated exponential queueing models with $m = 2$ and with an arbitrary value of m. Neuts [5], [6] treated the $M/M/1$ and $M/M/c$ queues with random environment, which included the above models as special cases. In this paper we consider such a model when the arrival time distribution is of phase type (PH) and we obtain the steady state probability vector of the queue length in the matrix geometric form and in the modified matrix geometric form for the $PH/M/1$ and $PH/M/c$ queues, respectively.

Phase-type arrival. Consider a continuous-time Markov process with state space $\{1, \ldots, n_i, n_i+1\}$ for which the states $1, \ldots, n_i$ are transient and the state n_i+1 is absorbing for $1 \leqslant i \leqslant m$. We assume that, starting at any transient state, absorption into the state n_i+1 is almost certain. The infinitesimal generator P_i of such a Markov process is of the form

$$P_i = \begin{vmatrix} \boldsymbol{T}_i & \boldsymbol{T}_i^0 \\ \boldsymbol{0} & 0 \end{vmatrix},$$

where T_i is an $(n_i \times n_i)$-matrix with $(T_i)_{jj} < 0$ and $(T_i)_{jk} \geqslant 0$ for $j \neq k$ such that T_i^{-1} exists. The vector $\boldsymbol{T}_i^0$ is non-negative and satisfies the equation $T_i \boldsymbol{e} + \boldsymbol{T}_i^0 = \boldsymbol{0}$, where $\boldsymbol{e} = (1\ 1\ 1\ \ldots\ 1)'$. Let $(\boldsymbol{a}_i, 0)$ denote the vector of initial probabilities. For the above-defined Markov process, the probability distribution $F_i(\cdot)$ of the time till the absorption in the state n_i+1 is given by

$$F_i(x) = 1 - \boldsymbol{a}_i \exp(T_i x)\boldsymbol{e}, \quad x \geqslant 0.$$

The pair $(\boldsymbol{a}_i, T_i)$ is called a *representation* of $F_i(\cdot)$. PH distributions were introduced and studied by Neuts in [2]-[4]. One may refer to [7] for the properties of PH distributions.

For any vector $\boldsymbol{c}$ and any number c we introduce the matrices

$$\Delta(\boldsymbol{c}) = \mathrm{diag}(c_1, c_2, \ldots, c_k) \quad \text{and} \quad \Delta(c) = \mathrm{diag}(c, c, \ldots, c).$$

Let T_i^0 be the $(n_i \times n_i)$-matrix with elements $(T_i^0)_{jk} = (\boldsymbol{T}_i^0)_j$. Consider

$$Q_i = T_i + T_i^0 \Delta(\boldsymbol{a}_i).$$

In [2] it is shown that without loss of generality one may assume that the representation $(\boldsymbol{a}_i, T_i)$ of $F_i(\cdot)$ is chosen so that the matrix Q_i is irreducible. The matrix Q_i is the infinitesimal generator of the PH renewal process studied by Neuts [3].

Let $\boldsymbol{\pi}_i$ be the invariant probability vector of Q_i for $1 \leqslant i \leqslant m$. It is clear from [4] that the stationary version of the PH renewal process is obtained by starting the Markov process Q_i with initial probability vector $\boldsymbol{\pi}_i$ and the stationary version of $F_i(\cdot)$ has the representation $(\boldsymbol{\pi}_i, T_i)$, $1 \leqslant i \leqslant m$.

In what follows we treat the steady state queue length of the $PH/M/1$ queue with random environment, its special cases, and the $PH/M/c$ queue.

2. Steady state queue length of the $PH/M/1$ queue. Let $\boldsymbol{a}$ be the invariant probability vector of Q which is the solution of the equation $\boldsymbol{a}Q = \boldsymbol{0}$, $\boldsymbol{a}\boldsymbol{e} = 1$. Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)$. The queueing model of interest can be studied by a continuous-time Markov chain on the state space

$$\{(k, i, j)\colon\ k \geqslant 1,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n_i\}.$$

The chain is in the *state* (k, i, j) when k customers are present, the Q-process is in the state i, and the arrival phase is j.

To describe the infinitesimal generator of the above Markov process of the queueing system we need $(N \times N)$-matrices ($N = \sum_{i=1}^{m} n_i$) of the form

$$\|M_{ij}\| = \begin{vmatrix} M_{11} & M_{12} & & M_{1m} \\ M_{21} & M_{22} & & M_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ M_{m1} & M_{m2} & & M_{mm} \end{vmatrix},$$

where M_{ij} is an $(n_i \times n_j)$-matrix for $1 \leqslant i, j \leqslant m$. Let

(1) $$A_0 = \|M_{ij}\|,$$

where $M_{ii} = \Delta(\mu_i)$ for $1 \leqslant i \leqslant m$ and M_{ij} are zero matrices for $i \neq j$. Let

(2) $$A_1 = \|M_{ij}\|,$$

where $M_{ii} = T_i + \Delta(Q_{ii}) - \Delta(\mu_i)$ and M_{ij} is an $(n_i \times n_j)$-matrix with all rows defined as $\boldsymbol{\pi}_j Q_{ij}$ for $i \neq j$. Let

(3) $$A_2 = \|M_{ij}\|,$$

where $M_{ii} = T_i^0 \Delta(\boldsymbol{a}_i)$ for $1 \leqslant i \leqslant m$ and M_{ij} are zero matrices for $i \neq j$.

The infinitesimal generator of the queueing system can now be written as

$$Q^* = \begin{vmatrix} A_0+A_1 & A_2 & 0 & 0 & \cdots \\ A_0 & A_1 & A_2 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & A_0 & A_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

The matrix Q^* is of the form studied by Neuts [5]. Define $A = A_0 + A_1 + A_2$ and a $(1 \times N)$-vector

$$\boldsymbol{\Pi} = (a_1 \boldsymbol{\pi}_1, a_2 \boldsymbol{\pi}_2, \ldots, a_m \boldsymbol{\pi}_m).$$

It can be seen that A is the irreducible and infinitesimal generator of a continuous-time Markov chain with $\boldsymbol{\Pi}$ as its invariant probability vector. Therefore, $\boldsymbol{\Pi} A = \boldsymbol{0}$ and $\boldsymbol{\Pi e} = 1$. Denote by $\boldsymbol{x}$ the vector of steady state probabilities associated with Q^* so that $\boldsymbol{x} Q^* = \boldsymbol{0}$ and $\boldsymbol{xe} = 1$. We partition $\boldsymbol{x}$ as

$$\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots),$$

where $\boldsymbol{x}_i$ $(i \geqslant 0)$ are $(1 \times N)$-vectors. We examine below the existence of a solution of the form $\boldsymbol{x}_i = \boldsymbol{x}_0 R^i$ for $i \geqslant 1$, where R has a spectral radius strictly less than one. To get such a solution we must have

$$\begin{aligned} \boldsymbol{x}_0(A_0+A_1)+\boldsymbol{x}_0 R A_0 &= \boldsymbol{0}, \\ \boldsymbol{x}_0 R^i (A_2+RA_1+R^2A_0) &= \boldsymbol{0}, \quad i \geqslant 0. \end{aligned} \tag{4}$$

From (4) it is clear that we need the matrix R which is the unique solution of the equation

$$A_2+RA_1+R^2A_0 = 0 \tag{5}$$

in the set of non-negative matrices of order N having a spectral radius less than one. Observe that, by [9], A_1 is non-singular since $A_1\boldsymbol{e} < \boldsymbol{0}$ and A_1^{-1} is non-positive with strictly negative diagonal elements. Repeating the arguments of Neuts [5] we find that such a matrix R exists with spectral radius less than one if $\boldsymbol{\Pi} A_0 \boldsymbol{e} > \boldsymbol{\Pi} A_2 \boldsymbol{e}$. Simplifying we get

$$\sum_{i=1}^{m} a_i \mu_i > \sum_{i=1}^{m} a_i \mu_i^{*-1}, \tag{6}$$

where $\mu_i^* = (\boldsymbol{\pi}_i \boldsymbol{T}_i^0)^{-1}$ is the mean of $F_i(\cdot)$ (see [2] and [3]).

We shall now find $\boldsymbol{x}_0$. The vector $\boldsymbol{x}_0$ must be chosen so that it satisfies

$$\sum_{i=0}^{\infty} \boldsymbol{x}_i \boldsymbol{e} = \boldsymbol{x}_0 (I-R)^{-1} \boldsymbol{e} = 1 \tag{7}$$

and (4). From (4) and (5) we obtain

$$\begin{aligned} \boldsymbol{x}_0(A_0+A_1+RA_0) &+ \sum_{\nu=0}^{\infty} \boldsymbol{x}_0 R^{\nu} (R^2A_0+RA_1+A_2) \\ &= \boldsymbol{x}_0(I-R)^{-1}(A_0+A_1+A_2) = \boldsymbol{x}_0(I-R)^{-1}A = \boldsymbol{0}. \end{aligned}$$

The uniqueness of the vector $\boldsymbol{\Pi}$ and (6), (7) imply that $\boldsymbol{x}_0 = \boldsymbol{\Pi}(I-R)$. This proves the following

THEOREM 1. *If* (6) *holds, then the queue is stable. The invariant probability vector of* Q^* *is given by* $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)$, $\boldsymbol{x}_k = \boldsymbol{\Pi}(I-R)R^k$ *for* $k \geqslant 0$. *The matrix* R *is the unique solution of equation* (5) *in the set of non-negative matrices of order* N, *which have a spectral radius less than one.*

Special cases. The above model is based on the analysis given by Neuts [5] for the $M/M/1$ queue.

(i) $H_n/M/1$ *queue*. Consider the hyperexponential distribution with representation $(\boldsymbol{a}, T)$, where $\boldsymbol{a} = n^{-1}\boldsymbol{e}$, $T = \operatorname{diag}(-\boldsymbol{\lambda})$, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Let $F_i(\cdot)$ be represented by $(\boldsymbol{a}, iT)$ for $1 \leqslant i \leqslant m$. Assume that there is no change in the arrival phase of the customer, who is in the process of joining the queue, due to a change in the environment. In this model, $\boldsymbol{\pi}_i = \boldsymbol{\pi}$ for $1 \leqslant i \leqslant m$, where

$$\boldsymbol{\pi} = \Big(\sum_{j=1}^{n} \lambda_j^{-1}\Big)^{-1} (\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}).$$

The matrices M_{ij} $(i \neq j)$ of A_1 become ΔQ_{ij}. Using the analysis given above one can prove that the steady state probability vector has the stated matrix geometric form. In spite of the change in A_1, the invariant probability vector $\boldsymbol{\Pi}$ of A takes the form

(8)
$$\boldsymbol{\Pi} = \boldsymbol{a} \oplus \boldsymbol{\pi} = (a_1 \boldsymbol{\pi}, a_2 \boldsymbol{\pi}, \ldots, a_m \boldsymbol{\pi}),$$

where $\oplus$ denotes the Kronecker multiplication. The steady state requirement can be seen as

$$\sum_{i=1}^{m} a_i \mu_i > n \Big(\sum_{j=1}^{n} \lambda_j^{-1}\Big)^{-1} \sum_{i=1}^{m} i a_i.$$

(ii) *Discrete PH*. Consider an $(n+1)$-state Markov chain with transition probability matrix S given by

$$S = \begin{vmatrix} K & \boldsymbol{K}^0 \\ \boldsymbol{0} & 1 \end{vmatrix}.$$

The square matrix K is of order n and $\boldsymbol{K}_i^0 > 0$ for $1 \leqslant i \leqslant n$. Note that $(I-K)^{-1}$ exists. This guarantees that the eventual absorption from any initial state into the state $n+1$ is certain. Let the initial probabilities of the Markov chain be $(\boldsymbol{a}, 0)$, $\boldsymbol{a}_i > 0$ for $1 \leqslant i \leqslant n$. This discrete PH distribution has the representation $(\boldsymbol{a}, K)$ (see [2]). Let the stochastic matrix $\hat{K}$ be of the form $\hat{K} = K + K^0 \Delta(\boldsymbol{a})$, where $K_{ij}^0 = \boldsymbol{K}_i^0$ have $\boldsymbol{\pi}$ as its invariant probability vector. Assume that $F_i(\cdot)$ for $1 \leqslant i \leqslant m$ have the representations $(\boldsymbol{a}, (-I+K)\lambda_i)$, where $\lambda_i > 0$. If there is a change in the environment from i to j, we assume that the arrival time distribution changes from the i-th to the j-th type but the arrival starts from the same arrival phase just before the change in the environment. In this case it may be noted that $\boldsymbol{\pi}_i = \boldsymbol{\pi}$ for $1 \leqslant i \leqslant m$. The matrices M_{ij}, $i \neq j$, $1 \leqslant i, j \leqslant m$, in A_1 become $\Delta(Q_{ij})$. In spite of this change the above analysis can be used to obtain the invariant probability vector of the queue length in the matrix geometric form. The vector $\boldsymbol{\Pi}$ of A has the same form given by (8) and it is seen to be $\boldsymbol{\Pi} = \boldsymbol{a} \oplus \boldsymbol{\pi}$, and the corresponding steady state requirement (6) is

$$\Big(\sum_{i=1}^{m} a_i \lambda_i\Big)\Big(k \sum_{i=1}^{m} a_i \mu_i\Big)^{-1} < 1,$$

where k is the mean of the underlying discrete PH distribution.

3. Steady state queue length of the $PH/M/c$ queue. The $PH/M/c$ queue with randomly varying environment defines a Markov process with the infinitesimal generator P^* given by

$$P^* = \left|\begin{matrix} \hat{A}_{00} & \hat{A}_{01} & & & & & & \\ \hat{A}_{10} & \hat{A}_{11} & \hat{A}_{12} & & & & & \\ & \hat{A}_{20} & \hat{A}_{21} & \hat{A}_{22} & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \hat{A}_{c-1,0} & \hat{A}_{c-1,1} & \hat{A}_{c-1,2} & & \\ & & & & \hat{A}_0 & \hat{A}_1 & \hat{A}_2 & \\ & & & & & \hat{A}_0 & \hat{A}_1 & \hat{A}_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}\right|,$$

where $\hat{A}_{01} = \hat{A}_{i2} = \hat{A}_2 = A_2$, $\hat{A}_{i0} = iA_0$, $\hat{A}_{00} = A_0+A_1$, $\hat{A}_{i1} = A_1-(i-1)A_0$ for $1 \leqslant i \leqslant c-1$, $\hat{A}_0 = cA_0$, $\hat{A}_1 = A_1-(c-1)A_0$, and A_0, A_1, A_2 are given by (1)-(3). Using [6] one can prove the following

THEOREM 2. *If*

$$c\sum_{i=1}^{m} a_i\mu_i > \sum_{i=1}^{m} a_i\mu_i^{*-1},$$

then the queue is stable. The steady state vector

$$\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{c-1}, \boldsymbol{x}_c, \ldots)$$

is given by $\boldsymbol{x}_k = \boldsymbol{x}_{c-1}R_c^{k-c+1}$ *for* $k \geqslant c$. *The matrix* R_c *is the unique solution of the equation*

$$R_c^2\hat{A}_0+R_c\hat{A}_1+\hat{A}_2 = 0$$

in the set of non-negative matrices of spectral radius less than one.

The matrix P^{**} given by

$$P^{**} = \left|\begin{matrix} \hat{A}_{00} & \hat{A}_{01} & & & & \\ \hat{A}_{10} & \hat{A}_{11} & \hat{A}_{12} & & & \\ & \hat{A}_{20} & \hat{A}_{21} & \hat{A}_{22} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & \hat{A}_{c-1,0} & \hat{A}_{c-1,1}+R_c\hat{A}_0 \end{matrix}\right|$$

is an irreducible semistable matrix of order cN. The vector $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{c-1})$ is its left eigenvector corresponding to the eigenvalue zero. It is normalized so that

$$\boldsymbol{x}_0\boldsymbol{e}+\ldots+\boldsymbol{x}_{c-2}\boldsymbol{e}+\boldsymbol{x}_{c-1}(I-R_c)^{-1}\boldsymbol{e} = 1.$$

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