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LEXICOGRAPHIC AND TIME MINIMIZATION IN THE TRANSPORTATION PROBLEM

1. Introduction. We deal with the cost transportation problem (Problem C), the time transportation problem (Problem T) and the lexicographic transportation problem (Problem L). All these three problems have the system of constraints

$$\begin{cases} \sum_{j=1}^{n} x_{ij} = a_i, & i = 1, \ldots, m, \\ \sum_{i=1}^{m} x_{ij} = b_j, & j = 1, \ldots, n, \\ x_{ij} \geqslant 0, & i = 1, \ldots, m;\ j = 1, \ldots, n, \end{cases} \tag{1.1}$$

where $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ are positive numbers satisfying

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

Problem C, which is a classical transportation problem, can be stated as follows:

PROBLEM C. Minimize

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{1.2}$$

subject to (1.1).

Problem T can be stated as follows:

PROBLEM T. Minimize

$$t(X) = \max_{(i,j)\in\Phi} t_{ij} \operatorname{sgn} x_{ij} \tag{1.3}$$

subject to (1.1), where $T = (t_{ij})$ is assumed to be a non-negative matrix, and $\Phi = \{1, \ldots, m\} \times \{1, \ldots, n\}$.

Suppose that there are given non-empty disjoint sets $\Phi_0, \ldots, \Phi_K$ such that

$$\bigcup_{k=0}^{K} \Phi_k = \Phi. \tag{1.4}$$

For any matrix $X = (x_{ij})$ satisfying (1.1) we introduce a vector

$$y(X) = \big(y_0(X), \ldots, y_K(X)\big), \tag{1.5}$$

where

$$y_k(X) = \sum_{(i,j)\in\Phi_k} x_{ij}, \quad k = 0, \ldots, K. \tag{1.6}$$

We state Problem L as follows:

PROBLEM L. Minimize $y(X)$ subject to (1.1).

Obviously, $y(X)$ is lexicographically minimized.

In Section 3 we show that we can construct such a matrix $C = (c_{ij})$ that Problem L and the associated Problem C are equivalent.

In Section 4 we construct Problem L such that an optimal solution to Problem L is also an optimal solution to Problem T. This way (*via* Problem L) Problem T can be solved by solving the associated Problem C.

An outline of the method reducing Problem T to Problem C was given in [4]. The proofs were given in [5]. However, Problem L was not posed and the proofs were more complicated than the proofs which are given in this paper.

The method as proposed in [4] and [5] might yield a matrix C whose some elements were 0 or 1 while others were of great magnitude. This disadvantage can be easily avoided by finding "good" bounds for a minimum value of (1.3). The simple method of finding a "good" lower bound is given in Section 5. This was first proposed in [5]. The value of (1.3) for an initial feasible solution is taken as an upper bound. The method for finding a relatively good initial solution is given in Section 6. The matrix C is constructed after these bounds have been found.

The method is illustrated by a numerical example in Section 7.

Before we proceed to the description of the method we briefly recall the history of Problem T.

The time transportation problem was posed and solved by Barsov (see [1], p. 90-101) in 1959. The method was based on the simplex method.

Other methods equivalent to Barsov's method in the sense that they produce the same sequence of basic solutions provided we start with the same initial solution were published by Zukhovitski and Avdeyeva (see [11], p. 204-216) in 1964, by Szwarc [9] (see also [8]) in 1966 and by Hammer [6] in 1969.

In all these methods having a basic feasible solution X_B (see Section 2 for necessary definitions) with $t(X_B) = t_{kl}$ one has to consider a set, say G, of (u, v) satisfying

(a) $t_{uv} < t_{kl}$,

(b) $(u, v) \notin B$,

(c) $B \cup (u, v)$ contains a cycle set, say Γ, such that (u, v) and (k, l) belong to different half-cycle subsets of Γ.

If G is non-empty, then by choosing (p, q) such that

$$t_{pq} = \min_{(u,v)\in G} t_{uv}$$

a new basic solution, in which x_{pq} is a basic variable, is constructed.

However, the technique of finding (p, q) is different in all these methods.

An original method based on the labelling method of finding a maximal flow (see [2]) was proposed by Garfinkel and Rao [3] in 1971.

Two methods were given in Nesterov [7] (see p. 72-80) in 1962. One of them is based on Kantorovitch's linear programming method. The other one, due to I. V. Romanovski, reduces Problem T to a sequence of Problems C. In Romanovski's method for a given basic solution X_B with $t(X_B) = t_{kl}$ a new basic solution is found by solving Problem C with the matrix $C = (c_{ij})$ defined as follows:

$$c_{ij} = \begin{cases} 0 & \text{if } t_{ij} < t_{kl}, \\ 1 & \text{if } t_{ij} = t_{kl}, \\ +\infty & \text{if } t_{ij} > t_{kl}. \end{cases}$$

One can notice from Sections 3 and 4 of this paper that $+\infty$ in this formula can be replaced by $h+1$, where h denotes the number of elements of the set $\{(i, j) \mid t_{ij} = t_{kl}\}$.

2. Notation, definitions and properties of Problem C. We need the following known definitions and properties based on [10].

First, we define some subsets of $\Phi = \{1, \ldots, m\} \times \{1, \ldots, n\}$. The set $R_i = \{i\} \times \{1, \ldots, n\}$ is the i-th row set, $C_j = \{1, \ldots, m\} \times \{j\}$ is the j-th column set, and

$$L_k = \begin{cases} R_k & \text{for } k = 1, \ldots, m, \\ C_{k-m} & \text{for } k = m+1, \ldots, m+n \end{cases}$$

is the k-th line set.

A non-empty subset Γ of Φ is a *cycle set* if the intersection of Γ and any line set is either empty or consists of two elements. A set consisting of exactly half elements of Γ and such that its intersection with any line set consists of at most one element is a *half-cycle subset* of Γ. Any cycle set can be divided into exactly two half-cycle subsets which are disjoint.

A subset B of Φ is a *basic set* if it consists of $m+n-1$ elements and contains no cycle set.

A matrix $X = (x_{ij})$ satisfying (1.1) is a *feasible solution*. A feasible solution minimizing (1.2) is an *optimal solution*. A feasible solution $X_B = (x_{ij}^B)$ satisfying $x_{ij}^B = 0$ for $(i,j) \notin B$, where B is a basic set, is called a *basic solution*.

Cost matrices $C = (c_{ij})$ and $C' = (c'_{ij})$ are said to be *equivalent* if

$$c'_{ij} = c_{ij} + u_i + v_j \quad \text{for } (i,j) \in \Phi,$$

where $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ are arbitrary constants. A matrix $C_B = (c_{ij}^B)$ equivalent to $C = (c_{ij})$ and satisfying $c_{ij}^B = 0$ for $(i,j) \in B$, where B is a basic set, is called a *zero matrix*.

To any basic set B there exists exactly one zero matrix C_B and at most one basic solution X_B (to each basic set B there exists exactly one matrix X satisfying equations of (1.1), but it does not have to be non--negative).

If B is a basic set and $(p,q) \notin B$, then $B \cup (p,q)$ contains exactly one cycle set, say Γ. Moreover, Γ contains (p,q). Let Γ_1, Γ_2 be half-cycle subsets of Γ defined in such a way that $(p,q) \in \Gamma_1$. Then

$$c_{pq}^B = \sum_{(i,j)\in\Gamma_1} c_{ij} - \sum_{(i,j)\in\Gamma_2} c_{ij}. \tag{2.1}$$

Assume, in addition, that $X_B = (x_{ij}^B)$ is a basic solution. If (r,s) is an element of Γ_2 satisfying

$$x_{rs}^B = \min_{(i,j)\in\Gamma_2} x_{ij}^B,$$

then $B_1 = B \cup (p,q) - (r,s)$ is a basic set, and $X_{B_1} = (x_{ij}^{B_1})$ is a basic solution. It can be found by the formula

$$x_{ij}^{B_1} = \begin{cases} x_{ij}^B + x_{rs}^B & \text{for } (i,j) \in \Gamma_1, \\ x_{ij}^B - x_{rs}^B & \text{for } (i,j) \in \Gamma_2, \\ x_{ij}^B & \text{for } (i,j) \in \Phi - \Gamma. \end{cases} \tag{2.2}$$

Remark 1. Since $\Gamma \neq \Phi$ and $x_{rs}^B \geqslant 0$, then $X_{B_1} \neq X_B$ if and only if $x_{rs}^B > 0$.

It follows from (2.1) and (2.2) that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}^{B_1} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}^B + c_{pq}^B x_{rs}^B. \tag{2.3}$$

An optimal solution to Problem C can be found by the transportation algorithm. Starting with an arbitrary basic solution we construct a sequence of basic solutions by repeating the same step transforming (if necessary) one basic solution, say X_B, into another one, say X_{B_1}. This step can be described as follows:

1. If $C_B \geqslant 0$, then X_B is an optimal solution. Otherwise proceed to 2.
2. Choose (p, q) such that $c_{pq}^B < 0$.
3. Find $\Gamma, \Gamma_1, \Gamma_2$ and (r, s) specified above.
4. Construct X_{B_1}, where $B_1 = B \cup (p, q) - (r, s)$.

By use of some perturbation technique, if necessary, an optimal solution can be found in finitely many steps.

3. Reducing Problem L to Problem C. Let h_k for $k = 0, \ldots, K$ denote the number of elements of Φ_k and consider the sequence $w_0, w_1, \ldots, w_{k+1}$ defined as follows:

$$(3.1) \qquad w_k = \begin{cases} 0 & \text{for } k = K, \\ 1 & \text{for } k = K-1, \\ (h_{k+1}+1)w_{k+1} & \text{for } k = K-2, \ldots, 0. \end{cases}$$

LEMMA 1. *We have*

$$w_k - \sum_{l=k+1}^{K} h_l w_l = 1 \quad \textit{for } k = K-1, K-2, \ldots, 0.$$

Proof. We prove the lemma by induction.
For $k = K-1$ we have, by (3.1),

$$w_{K-1} - h_K w_K = 1 - h_0 \cdot 0 = 1.$$

Assume

$$w_p - \sum_{l=p+1}^{K} h_l w_l = 1,$$

where p is any integer number satisfying $K-1 \geqslant p \geqslant 1$. Then

$$w_{p-1} - \sum_{l=p}^{K} h_l w_l = (h_p+1)w_p - h_p w_p - \sum_{l=p+1}^{K} h_l w_l = w_p - \sum_{l=p+1}^{K} h_l w_l = 1$$

which completes the proof.

Let us define an $(m \times n)$-matrix $C = (c_{ij})$ by

$$(3.2) \qquad c_{ij} = w_k \quad \text{for } (i, j) \in \Phi_k \text{ and } k = 0, \ldots, K.$$

In what follows it will be understood that the matrix C of Problem C is defined by (3.2).

Let Γ_1, Γ_2 be the two half-cycle subsets of some cycle set Γ and let

$$g_1 = (g_{10}, \ldots, g_{1K}), \qquad g_2 = (g_{20}, \ldots, g_{2K}),$$

where g_{lk} denotes the number of elements of $\Gamma_l \cap \Phi_k$. Write

$$c(\Gamma_1) = \sum_{(i,j)\epsilon\Gamma_1} c_{ij}, \qquad c(\Gamma_2) = \sum_{(i,j)\epsilon\Gamma_2} c_{ij}. \tag{3.3}$$

PROPOSITION 1. *We have*

$$g_1 - g_2 = 0 \Leftrightarrow c(\Gamma_1) - c(\Gamma_2) = 0,$$
$$g_1 - g_2 \succ 0 \Leftrightarrow c(\Gamma_1) - c(\Gamma_2) > 0,$$
$$g_1 - g_2 \prec 0 \Leftrightarrow c(\Gamma_1) - c(\Gamma_2) < 0.$$

Proof. By (3.2) and the definition of g_1 and g_2 we have

$$c(\Gamma_1) - c(\Gamma_2) = \sum_{k=0}^{K} (g_{1k} - g_{2k}) w_k.$$

Then, clearly, $g_1 - g_2 = 0$ implies $c(\Gamma_1) - c(\Gamma_2) = 0$.

Assume $g_1 - g_2 \succ 0$. Let $g_{1p} - g_{2p}$ be the first non-vanishing component of $g_1 - g_2$. Consequently, $g_{1p} - g_{2p} \geqslant 1$ and, since Γ_1 and Γ_2 have the same number of elements, we have, by (1.4), $p < K$. Obviously, $0 \leqslant g_{lk} \leqslant h_k$ for $l = 1, 2$ and $k = 0, \ldots, K$. Hence

$$c(\Gamma_1) - c(\Gamma_2) = \sum_{k=0}^{K} (g_{1k} - g_{2k}) w_k = (g_{1p} - g_{2p}) w_p + \sum_{k=p+1}^{K} g_{1k} w_k - \sum_{k=p+1}^{K} g_{2k} w_k.$$

Since $g_{1p} - g_{2p} \geqslant 1$, $g_{1k} \geqslant 0$, $g_{2k} \leqslant h_k$ and $w_k \geqslant 0$, we get

$$c(\Gamma_1) - c(\Gamma_2) \geqslant w_p - \sum_{k=p+1}^{K} h_k w_k.$$

Now, by Lemma 1, $c(\Gamma_1) - c(\Gamma_2) \geqslant 1 > 0$ which means that $g_1 - g_2 \succ 0$ implies $c(\Gamma_1) - c(\Gamma_2) > 0$. This, by the symmetry, yields also

$$g_1 - g_2 \prec 0 \Rightarrow c(\Gamma_1) - c(\Gamma_2) < 0$$

which completes the proof.

LEMMA 2. *Let X_{B_1} be a basic solution obtained from X_B by the transportation algorithm. Assume $X_{B_1} \neq X_B$. Then*

$$\sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij}^{B_1} < \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}^{B}, \tag{3.4}$$

$$y(X_{B_1}) \prec y(X_B). \tag{3.5}$$

Proof. By Remark 1, $x_{rs}^B > 0$. Then (3.4) follows from $c_{pq}^B < 0$ and (2.3).

From the definitions of g_1 and g_2, (2.2) and the definition of $y(X)$ we get

$$y(X_{B_1}) = y(X_B)+(g_1-g_2)x_{rs}^B. \tag{3.6}$$

By (2.1), (3.3) and Proposition 1, $c_{pq}^B < 0$ implies $g_1-g_2 \prec 0$ and, since $x_{rs}^B > 0$, we have $(g_1-g_2)x_{rs}^B \prec 0$. This and (3.6) imply (3.5). The proof is completed.

LEMMA 3. *If a basic solution X_{B_0} is not optimal to Problem* C, *then it is not optimal to Problem* L.

Proof. Construct a sequence $X_{B_0}, X_{B_1}, \ldots, X_{B_u}$ by use of the transportation algorithm where X_{B_u} is an optimal solution to Problem C. Obviously, $X_{B_u} \neq X_{B_0}$. Let X_{B_v} be the first element of that sequence which is different from X_{B_0}. Utilizing Lemma 2 we get

$$y(X_{B_0}) = y(X_{B_1}) = \ldots = y(X_{B_{v-1}}) \succ y(X_{B_v})$$

which completes the proof.

LEMMA 4. *If a basic solution X_{B_0} is not optimal to Problem* L, *then it is not optimal to Problem* C.

Proof. By the assumption there exists a feasible solution $\bar{X} = (\bar{x}_{ij})$ such that $y(\bar{X})-y(X_{B_0}) \prec 0$. Let $y_l(\bar{X})-y_l(X_{B_0})$ be the first non-vanishing component of $y(\bar{X})-y(X_{B_0})$. Hence

$$y_k(\bar{X})-y_k(X_{B_0}) = 0 \text{ for } k = 0, \ldots, l-1, \quad y_l(\bar{X})-y_l(X_{B_0}) < 0, \tag{3.7}$$

and since for any feasible solution $X = (x_{ij})$, we have

$$\sum_{k=0}^{K} y_k(X) = \sum_{k=0}^{K} \sum_{(i,j)\in\Phi_k} x_{ij} = \sum_{(i,j)\in\Phi} x_{ij} = \sum_{i=1}^{m} a_i.$$

Then (3.7) implies $l < K$.

Take

$$\Phi'_{l+1} = \bigcup_{k=l+1}^{K} \Phi_k,$$

$$y'_{l+1}(X) = \sum_{(i,j)\in\Phi'_{l+1}} x_{ij} = \sum_{k=l+1}^{K} \sum_{(i,j)\in\Phi_k} x_{ij},$$

$$y'(X) = \big(y_0(X), y_1(X), \ldots, y_l(X), y'_{l+1}(X)\big).$$

By the definition of Problem L, sets $\Phi_0, \Phi_1, \ldots, \Phi_K$ are not empty and, since $l < K$, the sets $\Phi_0, \Phi_1, \ldots, \Phi_l, \Phi'_{l+1}$ are not empty. Moreover,

$$\bigcup_{k=0}^{l} \Phi_k + \Phi'_{l+1} = \bigcup_{k=0}^{K} \Phi_k = \Phi.$$

Hence, the problem, say Problem L′, of minimizing $y'(X)$ subject to (1.1) has the form of Problem L.

Introducing

$$w'_k = \begin{cases} 0 & \text{for } k = l+1, \\ 1 & \text{for } k = l, \\ (\bar{h}_{k+1}+1)w_{k+1} & \text{for } k = l-1, \ldots, 0, \end{cases}$$

$$c'_{ij} = \begin{cases} w'_k & \text{for } (i,j)\in\Phi_k \text{ and } k = 0,1,\ldots,l, \\ w'_{l+1} = 0 & \text{for } (i,j)\in\Phi'_{l+1}, \end{cases}$$

we can pose the problem, say Problem C′, of minimizing

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c'_{ij}x_{ij}$$

subject to (1.1). Obviously, $w'_k > 0$ for $k = 0, 1, \ldots, l$. This, the definition of $C' = (c'_{ij})$ and (3.7) imply

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c'_{ij}\bar{x}_{ij} = \sum_{k=0}^{l}\sum_{(i,j)\in\Phi_k} w'_k\bar{x}_{ij} < \sum_{k=0}^{l}\sum_{(i,j)\in\Phi_k} w'_k x_{ij}^{B_0} = \sum_{i=1}^{m}\sum_{j=1}^{n} c'_{ij}x_{ij}^{B_0}$$

which means that X_{B_0} is not an optimal solution to Problem C′.

Construct a sequence $X_{B_0}, X_{B_1}, \ldots, X_{B_u}$ by use of the transportation algorithm where X_{B_u} is an optimal solution to Problem C′. Obviously, $X_{B_u} \neq X_{B_0}$. Let v be the number such that

$$X_{B_0} = X_{B_1} = \ldots = X_{B_{v-1}} \neq X_{B_v}. \tag{3.8}$$

Let $(p, q), \Gamma, \Gamma_1, \Gamma_2$ and (r, s) denote elements or sets found at points 2 and 3 of the step transforming $X_{B_{v-1}}$ into X_{B_v}. Obviously, $x_{rs}^{B_{v-1}} > 0$ and $c_{pq}^{\prime B_{v-1}} < 0$, where $C'_{B_{v-1}} = (c_{ij}^{\prime B_{v-1}})$ denotes a zero matrix equivalent to C' constructed for the basic set B_{v-1}. From $c_{pq}^{\prime B_{v-1}} < 0$ and formulae (2.1), (3.3) rewritten for C' we get $c'(\Gamma_1) - c'(\Gamma_2) < 0$. By Proposition 1, $g'_1 - g'_2 \dashv 0$, where

$$g'_1 = (g_{10}, \ldots, g_{1l}, g'_{1,l+1}) = \Big(g_{10}, \ldots, g_{1l}, \sum_{k=l+1}^{K} g_{1k}\Big),$$

$$g'_2 = (g_{20}, \ldots, g_{2l}, g'_{2,l+1}) = \Big(g_{10}, \ldots, g_{2l}, \sum_{k=l+1}^{K} g_{1k}\Big).$$

Since

$$g'_1 - g'_2 \dashv 0 \quad \text{and} \quad \sum_{k=0}^{K}(g_{1k} - g_{2k}) = 0,$$

there exists a t (where $0 \leqslant t \leqslant l$) such that

$$g_{1k} - g_{2k} = 0 \text{ for } k = 0, \ldots, t-1, \quad g_{1t} - g_{2t} < 0.$$

This implies $g_1 - g_2 \prec 0$ and, by Proposition 1, $c(\Gamma_1) - c(\Gamma_2) < 0$. Applying (3.3) and (2.1) we get $c_{pq}^{B_{v-1}} < 0$, where $C_{B_{v-1}} = (c_{ij}^{B_{v-1}})$ is a zero matrix equivalent to C constructed for the basic set B_{v-1}. From $c_{pq}^{B_{v-1}} < 0$, $x_{rs}^{B_{v-1}} > 0$ and (2.3) rewritten for X_{B_v} and $X_{B_{v-1}}$ we obtain

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}^{B_v} < \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}^{B_{v-1}}.$$

This and $X_{B_0} = X_{B_{v-1}}$ (see (3.8)) means that X_{B_0} is not an optimal solution to Problem C.

COROLLARY 1. *A basic solution is optimal to Problem* L *if and only if it is optimal to Problem* C.

Proof. The corollary follows immediately from Lemmas 3 and 4.

COROLLARY 2. *Problem* L *has a basic optimal solution.*

Proof. The corollary follows from Corollary 1 and the fact that Problem C has a basic optimal solution.

THEOREM 1. *A feasible solution is optimal to Problem* L *if and only if it is optimal to Problem* C.

Proof. It is known that any feasible solution X can be written as

$$X = \sum_{k=1}^{l} \lambda_k X_{B_k},$$

where $X_{B_1}, \ldots, X_{B_l}$ are basic solutions,

$$\sum_{k=1}^{l} \lambda_k = 1, \quad \lambda_k \geqslant 0 \quad \text{for } k = 1, \ldots, l.$$

Clearly, X is an optimal solution to Problem L if and only if $X_{B_1}, \ldots$ $\ldots, X_{B_l}$ are optimal to Problem L. This, by Corollary 1, is equivalent to the statement that $X_{B_1}, \ldots, X_{B_l}$ are optimal to Problem C which, in turn, is equivalent to the statement that X is optimal to Problem C.

THEOREM 2. *Let $X_{B_0}, X_{B_1}, \ldots, X_{B_u}$ be a sequence of basic solutions obtained by the use of the transportation algorithm. Then the sequence $y(X_0), y(X_{B_1}), \ldots, y(X_{B_u})$ is lexicographically non-increasing.*

This theorem is an immediate consequence of Lemma 2.

4. Reducing Problem T into Problem C. In the next sections we deal with Problem T. In this section we construct Problem L such that any optimal solution to Problem L is an optimal solution to Problem T. Thus, by Theorem 1, Problem T can be solved by solving the corresponding classical transportation problem, i.e. Problem C.

Let $t_0, t_1, \ldots, t_K$ be a sequence having the following properties:

(a) $t_0 < t_1 < \ldots < t_K$,

(b) for any t_k of that sequence, there exists $(i,j)\in\Phi$ such that $t_{ij} = t_k$,
(c) for any $(i,j)\in\Phi$ there exists t_k such that $t_k = t_{ij}$.

Write

$$\Phi_k = \{(i,j) \mid t_{ij} = t_{K-k}\} \quad \text{for } k = 0, 1, \ldots, K. \tag{4.1}$$

In what follows it will be understood that sets $\Phi_0, \ldots, \Phi_K$ of Problem L are defined by (4.1) (clearly, $\Phi_0, \ldots, \Phi_K$ are non-empty disjoint sets satisfying (1.4)).

LEMMA 5. *Given two feasible solutions $X' = (x'_{ij})$ and $X'' = (x''_{ij})$. If $y(X')$ is not lexicographically greater than $y(X'')$, then $t(X') \leqslant t(X'')$.*

Proof. Let $(p,q)\in\Phi$ and $K-r\in\{0,\ldots,K\}$ satisfy

$$t(X') = \max_{(i,j)\in\Phi} t_{ij}\operatorname{sgn} x'_{ij} = t_{pq} = t_{K-r}.$$

This, $t_0 < t_1 < \ldots < t_K$ and (4.1) imply

$$\begin{aligned} &\sum_{(i,j)\in\Phi_k} x'_{ij} = 0 \quad \text{for } k = 0, 1, \ldots, r-1, \\ &\sum_{(i,j)\in\Phi_r} x'_{ij} > 0. \end{aligned} \tag{4.2}$$

Suppose $t(X'') < t(X') = t_{K-r}$. Then

$$\sum_{(i,j)\in\Phi_k} x''_{ij} = 0 \quad \text{for } k = 0, 1, \ldots, r. \tag{4.3}$$

From (4.2) and (4.3), utilizing (1.5) and (1.6), we get $y(X') \succ y(X'')$ which is contradictory to the assumption. This contradiction proves the lemma.

Lemma 5 enables us to state the following theorems:

THEOREM 3. *An optimal solution to Problem L is optimal to Problem T.*

THEOREM 4. *An optimal solution to Problem C is optimal to Problem T.*

THEOREM 5. *Let $X_{B_0}, X_{B_1}, \ldots, X_{B_u}$ be a sequence of basic solutions obtained by use of the transportation algorithm. Then $t(X_{B_0}) \geqslant t(X_{B_1}) \geqslant \ldots \ldots \geqslant t(X_{B_u})$.*

Theorem 3 follows from Lemma 5, Theorem 4 — from Theorems 1 and 3, and Theorem 5 — from Lemma 5 and Theorem 2.

According to Theorem 4, Problem T can be solved by solving Problem C which is a classical transportation problem. As can be easily seen the matrix $C = (c_{ij})$ can be constructed by letting

$$c_{ij} = w_k \quad \text{for } (i,j)\in D_k \text{ and } k = 0, \ldots, K,$$

where

$$D_k = \{(i, j) \mid t_{ij} = t_k\} \quad \text{for } k = 0, \ldots, K,$$

$$w_k = \begin{cases} 0 & \text{for } k = 0, \\ 1 & \text{for } k = 1, \\ (h_{k-1}+1)w_{k-1} & \text{for } k = 2, \ldots, K, \end{cases}$$

and h_k denotes the number of elements of D_k.

It should be stressed that Problems C and T are not equivalent. Although Problems C and L are equivalent, Problems L and T are not, since $y(X') \neq y(X'')$ and $t(X') = t(X'')$ very often. Hence a set of optimal solutions to Problems C is contained in the set of optimal solutions to Problem T, but these two sets are not necessarily equal.

At this stage it should be pointed out that some elements of the matrix C might be of great magnitude unless some devices are applied. If, for instance, all elements of the matrix T are different, which is the worst possible case, then the set of elements of C is equal to $\{0, 1, 2, 4, \ldots, 2^{mn-2}\}$. This is an obvious disadvantage of the method. However, it can be avoided by finding lower and upper bounds for the minimum value of $t(X)$ in the way proposed in the next two sections.

Now we describe a general idea of use of these bounds.

Let t^* be a minimum value of (1.3) in Problem T and let t', t'' be numbers such that $t' \leqslant t^* \leqslant t''$. Write

$$(4.4) \qquad \begin{aligned} t_{ij}^1 &= \max(t', t_{ij}), \qquad t_{ij}^2 = \min(t_{ij}, t''+\varepsilon), \\ t_{ij}^3 &= \min[\max(t', t_{ij}), t''+\varepsilon], \end{aligned}$$

where ε is an arbitrary positive number.

Let $t^l(X)$ for $l = 1, 2, 3$ denote a function obtained from

$$t(X) = \max_{(i,j)\in\Phi} t_{ij} \operatorname{sgn} x_{ij}$$

by replacing t_{ij} by t_{ij}^l. By *Problem* T^l we understand the problem of minimizing $t^l(X)$ subject to (1.1).

PROPOSITION 2. *A feasible solution is optimal to Problem* T *if and only if it is optimal to Problems* T^1, T^2 *and* T^3.

Proof. Let X be any feasible solution. It follows from $t^* \geqslant t'$ that $t(X) \geqslant t'$. Suppose (p, q) satisfies $t_{pq} = t(X) \geqslant t'$. This implies $x_{pq} > 0$ and $x_{ij} = 0$ whenever $t_{ij} > t_{pq}$. Since $t_{pq} \geqslant t'$, by the definition of t_{ij}^1 we get

$$t_{pq}^1 = \max(t', t_{pq}) = t_{pq} \qquad \text{and} \qquad t_{ij}^1 = \max(t', t_{ij}) = t_{ij}$$

whenever $t_{ij} > t_{pq}$. Hence we have $t_{pq}^1 = t_{pq}$, $x_{pq} > 0$ and $x_{ij} = 0$ whenever $t_{ij}^1 > t_{pq}$. This means that $t^1(X) = t_{pq} = t(X)$. Thus $t^1(X) = t(X)$ holds for any feasible solution X, and Problems T and T^1 are equivalent.

Since $t^* \leqslant t''$, Problem T is equivalent to the problem of minimizing $t(X)$ subject to (1.1) and $t(X) \leqslant t''$. Let X be any feasible solution satisfying $t(X) \leqslant t''$ and let $t_{pq} = t(X)$ for some $(p, q) \in \Phi$. Then $x_{ij} = 0$ whenever $t_{ij} > t_{pq}$. In particular, $x_{ij} = 0$ whenever $t_{ij} > t''$ (since $t'' \geqslant t(X) = t_{pq}$).

Let (i, j) be any element of Φ. If $t_{ij} > t''$, then $x_{ij} = 0$ and $t_{ij}\operatorname{sgn} x_{ij} = t_{ij}^2 \operatorname{sgn} x_{ij} = 0$. If $t_{ij} \leqslant t''$, then $t_{ij}^2 = t_{ij}$ and, obviously, $t_{ij}\operatorname{sgn} x_{ij} = t_{ij}^2 \operatorname{sgn} x_{ij}$. Hence $t(X) = t^2(X)$ which proves the equivalency of Problems T and T^2.

Finally, since Problems T and T^1 are equivalent, t_{ij} can be replaced by $t_{ij}^1 = \max(t', t_{ij})$ and we get $t_{ij}^3 = \min(t_{ij}^1, t+\varepsilon)$. As we have proved, for any feasible solution X we have $t^1(X) = t(X)$ and, consequently, $t' \leqslant t^{1*} \leqslant t''$, where t^{1*} denotes the minimum value of $t^1(X)$. Applying the second part of the proof, we can prove the equivalency of Problems T and T^3.

The proof is completed.

Clearly, bounds t' and t'' can be always found in such a way that there exist two elements of T which are equal to t' and t'', respectively. Utilizing the equivalency of Problems T and T^3 we can construct the matrix C as follows:

PROCEDURE 1.

1. Define the sequence $t_0, t_1, \ldots, t_l$ satisfying

(a) $t' = t_0 < t_1 < \ldots < t_l = t''$;

(b) for each $(i, j) \in \Phi$ satisfying $t' \leqslant t_{ij} \leqslant t''$ there exists $t_k \in \{t_0, \ldots, t_l\}$ such that $t_k = t_{ij}$;

(c) for each $t_k \in \{t_0, \ldots, t_l\}$ there exists $(i, j) \in \Phi$ such that $t_{ij} = t_k$.

2. Take

$$E_0 = \{(i, j) \mid t_{ij} \leqslant t_0\},$$
$$E_k = \{(i, j) \mid t_{ij} = t_k\} \quad \text{for } k = 1, \ldots, l,$$
$$E_{l+1} = \{(i, j) \mid t_{ij} > t_l\}.$$

3. Define the sequence $w_0, w_1, \ldots, w_{l+1}$ taking

$$w_0 = 0, \quad w_1 = 1, \quad w_k = (h_{k-1}+1)w_{k-1} \text{ for } k = 2, \ldots, l+1,$$

where $h_1, \ldots, h_l$ denote the numbers of elements of $E_1, \ldots, E_l$, respectively.

4. Construct the matrix $C = (c_{ij})$ taking

$$c_{ij} = w_k \quad \text{for } (i, j) \in E_k \text{ and } k = 0, 1, \ldots, l+1.$$

5. A lower bound for minimum time. Now we give the method of obtaining a lower bound t' for the minimum value t^* of $t(X)$ in Problem T. This method was first proposed in [5].

Consider the following problem:

PROBLEM RT. Minimize

$$t(X) = \max_{(i,j)\in\Phi} t_{ij}\operatorname{sgn} x_{ij}$$

subject to

$$(5.1)\qquad \begin{aligned} &\sum_{j=1}^{n} x_{ij} = a_i \quad \text{for } i = 1, \ldots, m,\\ &0 \leqslant x_{ij} \leqslant b_j \quad \text{for } (i,j)\in\Phi. \end{aligned}$$

Let t^R denote the minimum value of $t(X)$ in Problem RT.

LEMMA 6. $t^R \leqslant t^*$.

Proof. The lemma is obvious, since any matrix X satisfying constraints (1.1) of Problem T satisfies constraints (5.1) of Problem RT.

Since

$$\max_{(i,j)\in\Phi} t_{ij}\operatorname{sgn} x_{ij} = \max_{i=1,\ldots,m} (\max_{i=1,\ldots,n} t_{ij}\operatorname{sgn} x_{ij}),$$

Problem RT decomposes into m independent Problems R_iT (where $i = 1, \ldots, m$) of the following form:

PROBLEM R_iT. Minimize

$$t(X) = \max_{j=1,\ldots,n} t_{ij}\operatorname{sgn} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad 0 \leqslant x_{ij} \leqslant b_j \quad \text{for } j = 1, \ldots, n.$$

Let t_i^R denote the minimum value of $t(X)$ in Problem R_iT. The following lemma is obvious.

LEMMA 7. $t^R = \max(t_1^R, t_2^R, \ldots, t_m^R)$.

An optimal solution X_i^R to Problem R_iT and an optimal solution X^R can be found in a very elementary way (see Procedure 2).

In an analogous way Problem CT and Problems C_jT for $j = 1, \ldots, n$ can be posed and solved; without going into details we pose Problem CT.

PROBLEM CT. Minimize $t(X)$ subject to

$$\begin{aligned} &\sum_{i=1}^{m} x_{ij} = b_j \quad \text{for } j = 1, \ldots, n,\\ &0 \leqslant x_{ij} \leqslant a_i \quad \text{for } (i,j)\in\Phi. \end{aligned}$$

A lower bound t' for t^* can be found by the following procedure:

PROCEDURE 2.

1. Construct an optimal solution $X^R = (x_{ij}^R)$ to Problem RT in the following way:

(a) for any $i = 1, \ldots, m$ find a permutation $\{j_1, j_2, \ldots, j_n\}$ of the set $\{1, \ldots, n\}$ such that $t_{ij_1} \leqslant t_{ij_2} \leqslant \ldots \leqslant t_{ij_n}$;

(b) take

$$x_{ij_k}^R = \max\left[0, \min\left(a_i - \sum_{l=0}^{k-1} b_{j_l}, b_{j_k}\right)\right] \quad \text{for } k = 1, \ldots, n,$$

where it is understood that $b_{j_0} = 0$.

2. Construct an optimal solution $X^C = (x_{ij}^C)$ to Problem CT using a procedure analogous to Step 1.

3. Take $t' = \max[t(X^R), t(X^C)]$.

This procedure is justified by the following proposition:

PROPOSITION 3. *The minimum value t^* of $t(X)$ in Problem* T *satisfies the inequality $t^* \geqslant t' = \max[t(X^R), t(X^C)]$.*

Proof. If $t' = 0$, the proposition is obvious, since $T \geqslant 0$, $X \geqslant 0$ and, consequently, $t(X) \geqslant 0$ for any feasible solution X.

Assume $t' > 0$. Without loss of generality we can also assume $t' = t(X^R)$. By rearranging rows and columns of T, if necessary, we get $t' = t(X^R) = t_{1p}\operatorname{sgn} x_{1p}^R$ and $t_{11} \leqslant t_{12} \leqslant \ldots \leqslant t_{1n}$. Since $t' > 0$, we have $x_{1p}^R > 0$ and $t' = t_{1p}$.

Suppose, to the contrary, that there exists a feasible solution $\bar{X} = (\bar{x}_{ij})$ such that $t(\bar{X}) < t_{1p}$. This and $t_{11} \leqslant t_{12} \leqslant \ldots \leqslant t_{1n}$ imply $\bar{x}_{1j} = 0$ for $j = p, \ldots, n$. Since, obviously, $\bar{x}_{1j} \leqslant b_j$ for $j = 1, \ldots, p-1$, we have

$$\sum_{j=1}^{p-1} \bar{x}_{1j} \leqslant \sum_{j=1}^{p-1} b_j$$

and, consequently,

$$\sum_{j=1}^{n} \bar{x}_{1j} \leqslant \sum_{j=1}^{p-1} b_j. \tag{5.2}$$

On the other hand, $x_{1p}^R > 0$ and $t_{11} \leqslant t_{12} \leqslant \ldots \leqslant t_{1n}$ yield (see Step 1 of Procedure 2)

$$a_1 - \sum_{j=1}^{p-1} b_j > 0.$$

This and (5.2) imply

$$\sum_{j=1}^{n} \bar{x}_{1j} < a_1$$

which is contradictory to (1.1). This contradiction proves Proposition 3.

6. Method of constructing an initial basic solution. Clearly, any method of constructing an initial basic solution to Problem C can be used to obtain an initial solution to Problem T. This very often requires, however, that the corresponding matrix C has been constructed before. Even if the matrix $T^1 = (t^1_{ij})$ defined by (4.4) is used instead of the original matrix $T = (t_{ij})$, it may cause a great magnitude of some elements of the matrix C. Therefore, it is recommended that an initial solution has been found before the matrix C was constructed. According to Proposition 4 and Procedure 1, the value $t(X_0)$ of the function $t(X)$ for the initial solution X_0 can be used as an upper bound for the minimum value t^* of $t(X)$ to construct the matrix C.

We begin with an obvious proposition.

PROPOSITION 4. *The minimum value t^* of $t(X)$ in Problem* T *satisfies the inequality $t^* \leqslant t'' = t(X_0)$, where $X_0 = x^0_{ij}$ is any feasible solution.*

The procedure of constructing a "good" initial solution can be outlined as follows:

PROCEDURE 3.

1. Replace the matrix T by the matrix $T^1 = (t^1_{ij})$ defined by (4.4). To simplify the writing we omit the index 1 in t^1_{ij}.

2. For any $i = 1, \ldots, m$ find a permutation $\{j_1, j_2, \ldots, j_n\}$ of the set $\{1, \ldots, n\}$ such that $t_{ij_1} \leqslant \ldots \leqslant t_{ij_n}$, and for any $j = 1, \ldots, n$ find a permutation $\{i_1, i_2, \ldots, i_m\}$ of the set $\{1, \ldots, m\}$ such that $t_{i_1 j} \leqslant t_{i_2 j} \leqslant \ldots \ldots \leqslant t_{i_m j}$. If $m > n$, proceed to 3; if $m < n$, proceed to 4; and if $m = n$, proceed to 5.

3. Take $p = m$. For each $i = 1, \ldots, m$ and $k = n+1, \ldots, p$ take $t_{ij_k} = t_{ij_n}$. Proceed to 6.

4. Take $p = n$. For each $j = 1, \ldots, n$ and $k = m+1, \ldots, p$ take $t_{i_k j} = t_{i_m j}$. Proceed to 6.

5. Take $p = m = n$.

6. For each $i = 1, \ldots, m$ form the vector $r_i = (r_{i1}, \ldots, r_{i,p+2})$ taking $r_{ik} = t_{ij_k}$ for $k = 1, \ldots, p$, $r_{i,p+1} = a_i$, $r_{i,p+2} = i$. For each $j = 1, \ldots, n$ form the vector $c_j = (c_{j1}, \ldots, c_{j,p+2})$ taking $c_{jk} = t_{i_k j}$ for $k = 1, \ldots, p$, $c_{j,p+1} = b_j$, $c_{j,p+2} = m+j$.

7. Define vectors $l_1, \ldots, l_{m+n}$ by

$$l_k = \begin{cases} r_i & \text{for } k = 1, \ldots, m, \\ c_{k-m} & \text{for } k = m+1, \ldots, m+n. \end{cases}$$

8. Select the lexicographically greatest vector from the set $l_1, \ldots, l_{m+n}$. Suppose it is l_u, where $u \leqslant m$ (the procedure in the case $u > m$ is analogous).

9. If $r_{uj_1} < r_{uj_k}$ for $k = 2, \ldots, n$, take $v = j_1$ and proceed to 11, otherwise, i.e. if $r_{uj_1} = r_{uj_2} = \ldots = r_{uj_l} < r_{uj_{l+1}} \leqslant \ldots \leqslant r_{uj_n}$, where $2 \leqslant l \leqslant n$, proceed to 10.

10. Select the lexicographically greatest vector l_{m+v} from the set $\{l_{m+j_1}, l_{m+j_2}, \ldots, l_{m+j_l}\} = \{c_{j_1}, \ldots, c_{j_n}\}$.

11. Take $x^0_{uv} = \min(a_u, b_v)$. If $a_u > b_v$, exclude the v-th column; if $a_u < b_v$ exclude the u-th row. In the case $a_u = b_v$ exclude the v-th column if $l_{m+v} \prec l_u$ and the u-th row if $l_u \succ l_{m+v}$. Replace a_u by $a_u - b_v$ if the v-th column is excluded and b_v by $b_v - a_u$ otherwise. Return to 2 with the new problem.

It is worthwhile to make some remarks.

Remark 2. Since the last component $l_{k,p+2}$ of l_k is equal to k, the vector which is to be selected at Steps 8 and 10 is unique.

Remark 3. In the case $a_u = b_v$ both the u-th row and the v-th column can be excluded. However, by following the procedure of Step 11 we obtain a basic set even in the case of degeneracy.

Remark 4. The vectors l'_k of the new transportation problem obtained in Step 11 do not have to be formed from the very beginning as it is suggested in Steps 2-7. An obvious procedure of obtaining these vectors from the initial vectors $l_1, \ldots, l_{m+n}$ will be clear from the numerical example considered in Section 7.

The following procedure describes a suggested method of reducing Problem T into Problem C:

PROCEDURE 4.

1. Using Procedure 2 construct the matrices X^R and X^C. Take $t' = \max[t(X^R), t(X^C)]$.

2. Using Procedure 3 construct an initial feasible solution X_0. Take $t'' = t(X_0)$.

3. Using Procedure 1 construct the matrix $C = (c_{ij})$.

4. Taking $X_0 = (x^0_{ij})$ as an initial basic solution, solve Problem C

7. Numerical example. We illustrate the method by the following Problem T taken from [8]:

$$T = (t_{ij}) = \begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & 6 & 21 & 19 & 12 & 7 & 8 \\ 2 & 9 & 13 & 10 & 14 & 15 & 5 \\ 3 & 14 & 11 & 12 & 9 & 12 & 4 \\ 4 & 12 & 16 & 8 & 20 & 19 & 5 \\ \hline & 2 & 6 & 4 & 7 & 3 & \end{array} \tag{7.1}$$

Remark 5. Numbers i, j, a_i and b_j are on the left, above, on the right and below the matrix, respectively.

In order to construct $X^R = (x^R_{ij})$ we are to consider Problem RT or Problems R_1T, R_2T, R_3T and R_4T.

Problem $\mathrm{R_1T}$ can be illustrated as follows:

	1	2	3	4	5	
1	6	21	19	12	7	8
	2	6	4	7	3	

According to Step 3 of Procedure 2 we obtain the inequalities $t_{11} < t_{15} < t_{14} < t_{13} < t_{12}$. Hence $x_{11}^R = \min(8, 2) = 2$ and we obtain

	2	3	4	5	
1	21	19	12	7	6
	6	4	7	3	

Now $x_{15}^R = \min(6, 3) = 3$ and we get

	2	3	4	
1	21	19	12	3
	6	4	7	

Then $x_{14}^R = \min(3, 7) = 3$ and we have

	2	3	
1	21	19	0
	6	4	

Finally, $x_{13}^R = x_{12}^R = 0$.

Problem $\mathrm{R_2T}$ can be illustrated by the following tableau:

	1	2	3	4	5	
2	9	13	10	14	15	5
	2	6	4	7	3	

Remark 6. Observe that the numbers b_j remain unchanged.

In the same way as in Problem $\mathrm{R_2T}$ we can find $x_{21}^R = 2$, $x_{23}^R = 3$, $x_{22}^R = x_{24}^R = x_{25}^R = 0$.

By the same procedure we find elements of the third and the fourth row of X^R. The whole matrix $X^R = (x_{ij}^R)$ and a matrix $(t_{ij}\,\mathrm{sgn}\,x_{ij}^R)$ are the following:

$$X^R = (x_{ij}^R) = \begin{pmatrix} 2 & 0 & 0 & 3 & 3 \\ 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 1 & 0 & 4 & 0 & 0 \end{pmatrix}, \quad (t_{ij}\,\mathrm{sgn}\,x_{ij}^R) = \begin{pmatrix} 6 & 0 & 0 & 12 & 7 \\ 9 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 12 & 0 & 8 & 0 & 0 \end{pmatrix}.$$

Clearly,

$$t(X^R) = \max_{i,j} t_{ij} \operatorname{sgn} x_{ij}^R = t_{14} \operatorname{sgn} x_{14}^R = t_{41} \operatorname{sgn} x_{41}^R = 12.$$

In a similar way, the following $X^C = (x_{ij}^C)$ can be found:

$$X^C = (x_{ij}^C) = \begin{pmatrix} 2 & 0 & 0 & 3 & 3 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{pmatrix}.$$

Thus, $t(X^C) = t_{22} \operatorname{sgn} x_{22}^C = 13$ and $t' = \max[t(x^R), t(x^C)] = \max(12, 13.$ $= 13$ is a lower bound for the minimum value t^* of $t(X)$ in Problem T)

Replacing the matrix T by the matrix $T^1 = (t_{ij}^1)$ given by $t_{ij}^1 = \max(13, t_{ij})$ we obtain Problem T^1 equivalent to Problem T. Problem T^1 is illustrated by the following tableau:

$$T^1 = (t_{ij}^1) = \begin{array}{c|c|c|c|c|c|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & 13 & 21 & 19 & 13 & 13 & 8 \\ 2 & 13 & 13 & 13 & 14 & 15 & 5 \\ 3 & 14 & 13 & 13 & 13 & 13 & 4 \\ 4 & 13 & 16 & 13 & 20 & 19 & 5 \\ \hline & 2 & 6 & 4 & 7 & 3 & \end{array} \qquad (7.2)$$

In order to obtain an initial feasible solution $X^0 = (x_{ij}^0)$ we use Procedure 3.

Steps 2-7 are illustrated for $i = 1$ and $j = 1$. The index 1 in t_{ij}^1 is omitted.

According to Step 1 for $i = 1$ we get the sequence

$$\{t_{11}, t_{14}, t_{15}, t_{13}, t_{12}\} = \{13, 13, 13, 19, 21\},$$

and for $j = 1$ — the sequence

$$\{t_{11}, t_{21}, t_{41}, t_{31}\} = \{13, 13, 13, 14\}.$$

The latter sequence is in Step 3 supplemented to $\{13, 13, 13, 14, 14\}$.

In Step 6 we form

$$r_1 = (13, 13, 13, 19, 21, 8, 1), \quad \text{where } r_{16} = a_1 = 8, r_{17} = 1,$$

and

$$c_1 = (13, 13, 13, 14, 14, 2, 5), \quad \text{where } c_{16} = b_1 = 2, c_{17} = m+1 = 5.$$

In Step 7 we take $l_1 = r_1$ and $l_5 = c_1$. In this way we can obtain the following vectors $l_1, l_2, \ldots, l_9$:

$$(7.3)\quad \begin{cases} l_1 = r_1 = (13,13,13,19,21,8,1), & l_2 = r_2 = (13,13,13,14,15,5,2), \\ l_3 = r_3 = (13,13,13,13,14,4,3), & l_4 = r_4 = (13,13,16,19,20,5,4), \\ l_5 = c_1 = (13,13,13,14,14,2,5), & l_6 = c_2 = (13,13,16,21,21,6,6), \\ l_7 = c_3 = (13,13,13,19,19,4,7), & l_8 = c_4 = (13,13,14,20,20,7,8), \\ l_9 = c_5 = (13,13,15,19,19,3,9). & \end{cases}$$

The vector $l_6 = c_2$ is the lexicographically greatest and some elements of the second column of $X_0 = (x_{ij}^0)$ are defined. Since the smallest element of the second column of matrix (7.2) is not unique or, in other words, $c_{22} = c_{23} < c_{24} \leqslant c_{21}$, where $c_{22} = t_{22} = 13$, $c_{23} = t_{23} = 13$, $c_{24} = t_{24} = 16$, $c_{21} = t_{21} = 21$, we have to perform Step 10. We have to consider $r_2 = l_2$ and $r_3 = l_3$. Since $l_2 \succ l_3$, we have $(u, v) = (2, 2)$ and the value $x_{22}^0 = \min(a_2, b_2) = \min(5, 6)$ is found.

After excluding the second row and replacing $b_2 = 6$ by $b_2 - a_2 = 6 - -5 = 1$ we obtain the new problem which is illustrated by

$$(7.4)\quad \begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & 13 & 21 & 19 & 13 & 13 & 8 \\ 3 & 14 & 13 & 13 & 13 & 13 & 4 \\ 4 & 13 & 16 & 13 & 20 & 19 & 5 \\ \hline & 2 & 1 & 4 & 7 & 3 & \end{array}$$

Obviously, the vectors $l_1 = r_1$, $l_3 = r_3$, $l_4 = r_4$ remain unchanged. One can obtain new vectors $l_5 = c_1, l_6 = c_2, l_7 = c_3, l_8 = c_4, l_9 = c_5$ without defining them from the very beginning. For instance, in order to obtain the new $l_5 = c_1$ it is enough to exclude 13 which is the element of the first column of matrix (7.2) lying in the excluded second row and repeat $c_{i5} = 14$ once again. In this way the new vectors l_5, l_7, l_8, l_9 can be found. The same has to be done with the new $l_6 = c_2$ but, in addition, $c_{26} = 6$ should be replaced by 1.

The new vectors $l_1, l_3, l_4, \ldots, l_9$ are the following:

$$\begin{array}{ll} l_1 = r_1 = (13, 13, 13, 19, 21, 8, 1), & l_3 = r_3 = (13, 13, 13, 13, 14, 4, 3), \\ l_4 = r_4 = (13, 13, 16, 19, 20, 5, 4), & l_5 = c_1 = (13, 13, 14, 14, 14, 2, 5), \\ l_6 = c_2 = (13, 16, 21, 21, 21, 1, 6), & l_7 = c_3 = (13, 13, 19, 19, 19, 4, 7), \\ l_8 = c_4 = (13, 13, 20, 20, 20, 7, 8), & l_9 = c_5 = (13, 13, 19, 19, 19, 3, 9). \end{array}$$

Again, the vector $l_6 = c_2$ is selected, and since $t_{32} = 13$ is the unique smallest element of the second column of matrix (7.4), we have $x_{32}^0 = \min(4, 1) = 1$.

After excluding the second column we obtain the new problem and the following vectors:

$$l_1 = r_1 = (13, 13, 13, 19, 8, 1), \quad l_3 = r_3 = (13, 13, 13, 14, 3, 3),$$
$$l_4 = r_4 = (13, 13, 19, 20, 5, 4), \quad l_5 = c_1 = (13, 13, 14, 14, 2, 5),$$
$$l_7 = c_3 = (13, 13, 19, 19, 4, 7), \quad l_8 = c_4 = (13, 13, 20, 20, 7, 8),$$
$$l_9 = c_5 = (13, 13, 19, 19, 3, 9).$$

Following Procedure 3 we find successively $x_{14}^0 = 7$, $x_{43}^0 = 4$, $x_{41}^0 = 1$. At this stage we obtain the problem

$$\begin{array}{c|cc|c} & 1 & 5 & \\ \hline 1 & 13 & 13 & 1 \\ 3 & 14 & 13 & 3 \\ \hline & 1 & 3 & \end{array}$$

with the following vectors:

$$l_1 = r_1 = (13, 13, 1, 1), \quad l_3 = r_3 = (13, 14, 3, 3),$$
$$l_5 = c_1 = (13, 14, 1, 5), \quad l_9 = c_5 = (13, 13, 3, 9).$$

Now $x_{35}^0 = \min(a_3, b_5) = \min(3, 3) = 3$. We exclude the third row since $r_3 \in c_5$. Finally, we get $x_{11}^0 = 1$ and $x_{15}^0 = 0$.

In this way we have obtained the following initial basic solution:

$$X_0 = (x_{ij}^0) = \begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 4 & 0 & 0 \end{pmatrix}.$$

Elements (i, j) corresponding to circled x_{ij} form a basic set. The value of $t(X)$ for X_0 is $t(X_0) = t^1(X_0) = 13$. Since it is equal to the lower bound for the minimum value t^* of $t(X)$ in Problem T, X_0 is an optimal solution to Problem T.

Clearly, in this case the matrix $C = (c_{ij})$ can be constructed by taking $c_{ij} = 0$ if $t_{ij} \leqslant 13$ and by taking $c_{ij} = 1$ if $t_{ij} > 13$. However, the matrix C need not have to be constructed.

Since due to X_0 with $t(X_0) = t'$ Procedure 1 can be hardly illustrated, we take the following solution $\bar{X} = (\bar{x}_{ij})$ obtained by the minimum row method as an initial solution (the basic set is shown by circling the corresponding x_{ij}):

$$\bar{X} = (\bar{x}_{ij}) = \begin{pmatrix} 2 & 0 & 0 & 3 & 3 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 5 & 0 & 0 & 0 \end{pmatrix}.$$

We take $t'' = t(\bar{X}) = 16$ as an upper bound for the minimum value t^* of $t(X)$.

Having $t' = 13$, $t'' = 16$ and matrix (7.1), we define the sequence $\{t_0, t_1, t_2, t_3\} = \{13, 14, 15, 16\}$ in Step 1 of Procedure 1.

Thus

$$E_0 = \{(i,j) \mid t_{ij} \leqslant 13\} = \{(1,1),(1,4),(1,5),(2,1),(2,2),(2,3),(3,2),(3,3),(3,4),(3,5),(4,1),(4,3)\},$$

$$E_1 = \{(i,j) \mid t_{ij} = 14\} = \{(2,4),(3,1)\}, \quad E_2 = \{(i,j) \mid t_{ij} = 15\} = \{(2,5)\},$$

$$E_3 = \{(i,j) \mid t_{ij} = 16\} = \{(4,2)\},$$

$$E_4 = \{(i,j) \mid t_{ij} > 16\} = \{(1,2),(1,3),(4,4),(4,5)\}.$$

Since $h_1 = 2, h_2 = 1, h_3 = 1$ are the numbers of elements of E_1, E_2, E_3, respectively, we have

$$w_0 = 0, \quad w_1 = 1, \quad w_2 = (h_1+1)w_1 = 3, \quad w_3 = (h_2+1)w_2 = 6,$$
$$w_4 = (h_3+1)w_3 = 12.$$

Thus by assigning $c_{ij} = w_k$ for $(i,j) \in E_k$ we obtain Problem C as follows:

$$C = c_{ij} = \begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & 0 & 12 & 12 & 0 & 0 & 8 \\ 2 & 0 & 0 & 0 & 1 & 3 & 5 \\ 3 & 1 & 0 & 0 & 0 & 0 & 4 \\ 4 & 0 & 6 & 0 & 12 & 12 & 5 \\ \hline & 2 & 6 & 4 & 7 & 3 & \end{array}$$

Any optimal solution to that problem is also optimal to Problem T.

8. Some remarks. The proposed method seems to have an obvious advantage, since the procedure of improving a solution (this procedure is always the most troublesome) can be handled by use of the classical transportation algorithm. Before using this algorithm one has to find the lower bound t', the upper bound t'' and the matrix C which can be done by use of Procedures 2, 3 and 1, respectively. Although setting up computer programs for these procedures does not seem to be difficult, one can easily handle them manually even in "large" problems. If bounds t' and t'' are found by Procedures 2 and 3, then (even in "large" problems) elements of C are not of great magnitude.

Clearly, one can find other bounds by use of simpler techniques. For instance, one can take

$$t' = \max(\max_i \min_j t_{ij}, \max_j \min_i t_{ij})$$

(in our example it yields $t' = t_{32} = 11$) as a lower bound. Also an initial solution yielding an upper bound does not have to be constructed by Procedure 3.

One can stress that the method reduces Problem T, in which function (1.3) is not continuous, to a classical transportation problem.

Hammer [6] seeks a solution which lexicographically minimizes the vector

$$z(X) = \Big(t(X), \sum_{t_{ij}=t(X)} x_{ij}\Big).$$

Such a solution can be found by taking

$$\theta = \max\{t_{ij} \mid t_{ij} < t'\},$$

where t' is any lower bound for the minimum value of $t(X)$, and by defining the sequence $t_0, t_1, \ldots, t_l$ satisfying conditions 1 (b) and 1 (c) of Procedure 1 and

$$\theta = t_0 < t_1 < \ldots < t_l = t''.$$

One can easily observe that θ is also a lower bound for the minimum value of $t(X)$. Now an optimal solution to Problem C lexicographically minimizes (see Step 2 of Procedure 1 for definitions of $E_0, E_1, \ldots, E_{l+1}$)

$$\sum_{(i,j)\in E_{l+1}} x_{ij}, \sum_{(i,j)\in E_l} x_{ij}, \ldots, \sum_{(i,j)\in E_1} x_{ij}, \sum_{(i,j)\in E_0} x_{ij},$$

since $\theta < \min t(X)$ and, consequently,

$$\sum_{k=1}^{l+1} \sum_{(i,j)\in E_k} x_{ij} > 0,$$

therefore an optimal solution to Problem C minimizes also $z(X)$.

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MINIMALIZACJA LEKSYKOGRAFICZNA I MINIMALIZACJA CZASU W ZAGADNIENIU TRANSPORTOWYM

STRESZCZENIE

W pracy omawia się zagadnienie transportowe z kryterium czasu. Zagadnienie to jest potraktowane jako szczególny przypadek innego zagadnienia transportowego, w którym pewien wektor jest leksykograficznie minimalizowany. Oba te zagadnienia sprowadzone są do klasycznego zagadnienia transportowego.
