ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XII, 1 (1971)

N. K. BASU (Calcutta)

POLYNOMIAL APPROXIMATION TO INTEGRAL TRANSFORMS

0. Introduction. The polynomial approximation of the integral transforms of the Laplace or Fourier type in a series of symmetric Jacobi polynomials has been studied by Wimp [5]. He obtained the coefficients in such an approximation as the Hankel transforms containing the given function. In the present note the approximation of the Laplace or Fourier transform $g(x)$ of some function $f(t)$ has been obtained in a series of Jacobi polynomials $P_n^{(a,\beta)}(x)$ in the interval $-1 < x < 1$. The coefficients in such cases are found as integrals containing confluent hypergeometric functions which have ultimately been reduced to a series of Hankel transforms. A similar treatment has been applied to the case of inverse Laplace transforms. Several examples have been included to show the practical applications of the method.

1. Laplace transform. The Jacobi polynomial is defined as

$$P_n^{(a,\beta)}(x) = \binom{n+a}{n} F(-n, n+a+\beta+1; a+1; \tfrac{1}{2}-\tfrac{1}{2}x)$$
$$= (-1)^n \binom{n+\beta}{n} F(-n, n+a+\beta+1; \beta+1; \tfrac{1}{2}+\tfrac{1}{2}x).$$

Let $g(x)$ be the Laplace transform of a function $f(t)$; then

$$(1)\qquad g(x) = L\{f(t)\} = \int_0^\infty e^{-xt} f(t)\, dt = \sum_{n=0}^\infty A_n P_n^{(a,\beta)}(x).$$

To determine the A_n, s, we replace the kernel e^{-xt} in (1) by an expansion obtained by putting $y = it$ in the relation ([2], vol. 2, eq. (4), p. 213)

$$(2)\qquad e^{ixy} = (2iy)^{-\frac{1}{2}(a+\beta)-1} \sum_{n=0}^\infty \frac{\Gamma(n+a+\beta+1)}{\Gamma(2n+a+\beta+1)} M_{k,m}(2iy) P_n^{(a,\beta)}(x),$$
$$-1 < x < 1;\ a, \beta > -1,$$

where $k = (a-\beta)/2$, $m = n+\frac{1}{2}(a+\beta+1)$, and $M_{k,m}(x)$ is the Whittakar function of the first kind.

Thus we obtain

$$(3)\qquad e^{-xt} = (-2t)^{-\frac{1}{2}(a+\beta)-1}\sum_{n=0}^{\infty}\frac{\Gamma(n+a+\beta+1)}{\Gamma(2n+a+\beta+1)}M_{k,m}(-2t)P_n^{(a,\beta)}(x).$$

Now, from [2], vol. 1, eq. (1), p. 264, we have

$$(4)\qquad M_{k,m}(-2t) = e^t(-2t)^{n+\frac{1}{2}(a+\beta)+1}\Phi(n+\beta+1;2n+a+\beta+2;-2t),$$

and applying the Kummer transformation ([2], vol. 1, eq. (7), p. 253)

$$(5)\qquad \Phi(n+\beta+1;2n+a+\beta+2;-2t) = e^{-2t}\Phi(n+a+1;2n+a+\beta+2;2t),$$

we obtain A_n from (1), using (3), (4) and (5), in the form

$$(6)\qquad A_n = (-1)^n 2^n\frac{\Gamma(n+a+\beta+1)}{\Gamma(2n+a+\beta+1)}\int_0^{\infty}e^{-t}t^n\Phi(n+a+1;2n+a+\beta+2;2t)f(t)\,dt.$$

But from [4], eqs. (2.2) and (2.8), taking $\lambda = 1$, $\delta = n+a+\frac{1}{2}$ we have

$$(7)\qquad \Phi(n+a+1;2n+a+\beta+2;2t) = e^t\frac{\Gamma(2n+2a+2)\sqrt{\pi}}{\Gamma(n+a+1)(2t)^{n+a+\frac{1}{2}}}I_{n+a+\frac{1}{2}}(t)+$$
$$+\frac{2e^t\sqrt{\pi}}{\Gamma(n+a+1)(2t)^{n+a+\frac{1}{2}}}\sum_{k=1}^{\infty}(-1)^k\frac{(n+k+a+\frac{1}{2})\Gamma(2n+2a+k+1)}{k!}\times$$
$$\times R_k(n+a+1,2n+a+\beta+2,n+a+\tfrac{1}{2};1)I_{n+k+a+\frac{1}{2}}(t),$$

where

$$R_k(n+a+1,2n+a+\beta+2,n+a+\tfrac{1}{2};1) = R_k(n+a+1,2n+a+\beta+2,n+a+\tfrac{1}{2})$$
$$= (-1)^k\frac{\Gamma(k+a-\beta)}{\Gamma(a-\beta)}\cdot\frac{\Gamma(2n+a+\beta+2)}{\Gamma(2n+a+\beta+k+2)}.$$

Also if $\alpha = \beta$, then

$$R_k(n+\alpha+1, 2(n+\alpha+1), n+\alpha+\tfrac{1}{2}) = \begin{cases} 0 & \text{if } k \neq 0, \\ 1 & \text{if } k = 0. \end{cases} \tag{8}$$

Substituting (7) in (6) we obtain

$$A_n = (-1)^n 2^{-(\frac{1}{2}+\alpha)}\sqrt{\pi}\,\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}\cdot\frac{1}{\Gamma(n+\alpha+1)}\times \tag{9}$$
$$\times\sum_{k=0}^{\infty}\Bigg[(-1)^k\frac{(2n+2\alpha+2k+1)\Gamma(2n+2\alpha+k+1)}{k!}\times$$
$$\times R_k(n+\alpha+1, 2n+\alpha+\beta+2, n+\alpha+\tfrac{1}{2})\int_0^{\infty} I_{n+k+\alpha+\frac{1}{2}}(t)\,t^{-(\alpha+\frac{1}{2})}f(t)\,dt\Bigg].$$

Now, since ([2], vol. 2, eq. (12), p. 5)

$$I_\nu(z) = J_\nu(ze^{i\pi/2})\exp\left\{-i\frac{\pi}{2}\nu\right\},$$

we have

$$I_{n+k+\alpha+\frac{1}{2}}(t) = J_{n+k+\alpha+\frac{1}{2}}(it)\exp\left\{-i\frac{\pi}{2}(n+k+\alpha+\tfrac{1}{2})\right\}.$$

Taking now

$$2^{-(\alpha+\frac{1}{2})}\sqrt{\pi}\,\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}\cdot\frac{1}{\Gamma(n+\alpha+1)} = \Omega_n$$

and

$$(-1)^k\frac{(2n+2\alpha+2k+1)\Gamma(2n+2\alpha+k+1)}{k!}\times \tag{10}$$
$$\times R_k(n+\alpha+1, 2n+\alpha+\beta+2, n+\alpha+\tfrac{1}{2}) = G_{n,k},$$

equation (9) becomes

$$A_n = (-1)^n\,\Omega_n\sum_{k=0}^{\infty}G_{n,k}\,e^{-i\pi/2(n+k+\alpha+1)}H\left\{\frac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=i\\ \nu=n+k+\alpha+\frac{1}{2}}}, \tag{11}$$

where $H\{F(t)\} = \int_0^{\infty} F(t)J_\nu(yt)(yt)^{1/2}dt$ denotes the Hankel transform of $F(t)$.

In the symmetric case $\alpha = \beta$, from (8) and (10) $G_{n,k} = 0$ for $k \neq 0$, and so only for $k = 0$ (11) gives the coefficients for the symmetric Jacobi expansion as

$$A_n' = e^{i\pi/2(n-\alpha-1)}\Omega_n G_{n,0}H\left\{\frac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=i\\ \nu=n+\alpha+\frac{1}{2}}}, \tag{12}$$

where

$$\Omega_n G_{n,0} = 2^{\frac{1}{2}-\alpha}\sqrt{\pi}\,\frac{(n+\alpha+\frac{1}{2})\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)}. \tag{13}$$

The coefficients C_n for the Chebyshev expansion of $g(x)$ follow from (12) by putting $a = -1/2$, and all these results are identical with those of Wimp [5].

2. Inverse Laplace transform. We consider now the inverse problem, i.e. $g(x)$ being given, to find $f(t)$ as a polynomial in $P_n^{(a,\beta)}(t)$, where t lies between $-1 < t < 1$. Now, referring to $g(x)$ as given in (1)

$$f(t) = \frac{1}{2\pi i} \lim_{\nu\to\infty} \int_{c-i\nu}^{c+i\nu} e^{xt} g(x)\,dx,$$

where the real number c is so chosen that all singularities of $g(x)$ are to the left of the line of integration. The inversion is valid for $t > 0$. To take into account negative values of t two functions $f_1(t)$ and $f_2(t)$ are defined as follows:

$$f_1(t) = \begin{cases} f(t), & t > 0, \\ f(0)/2, & t = 0, \\ 0, & t < 0, \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} f(t), & t < 0, \\ f(0)/2, & t = 0, \\ 0, & t > 0. \end{cases}$$

Hence for all real t we may write

$$f(t) = f_1(t) + f_2(t).$$

We now define

$$g_1(x) = \int_0^\infty e^{-xt} f(t)\,dt = L\{f(t)\},$$

$$g_2(-x) = \int_0^\infty e^{-xt} f(-t)\,dt = L\{f(-t)\},$$

so that $g_1(x)$ and $g_2(x)$ are the Laplace transforms of $f_1(t)$ and $f_2(-t)$. Hence, following an analysis by Elliott [1], we have for all t

$$(14)\qquad f(t) = \frac{1}{2\pi i} \lim_{\nu\to\infty} \left\{ \int_{c_1-i\nu}^{c_1+i\nu} e^{xt} g_1(x)\,dx + \int_{-c_2-i\nu}^{-c_2+i\nu} e^{xt} g_2(-x)\,dx \right\}.$$

If t lies in $-1 < t < 1$, then interchanging x and t and replacing x by $-x$ we obtain from (3)

$$(15)\qquad e^{xt} = (2x)^{-\frac{1}{2}(a+\beta)-1} \sum_{n=0}^{\infty} \frac{\Gamma(n+a+\beta+1)}{\Gamma(2n+a+\beta+1)} M_{k,m}(2x) P_n^{(a,\beta)}(t).$$

Now, if

$$f(t) = \sum_{n=0}^{\infty} A_n P_n^{(a,\beta)}(t),$$

then using (4), (5), (14) and (15), we infer that

$$(16)\quad A_n = 2^n \frac{\Gamma(n+a+\beta+1)}{\Gamma(2n+a+\beta+1)}\cdot\frac{1}{2\pi i}\times$$
$$\times \lim_{\nu\to\infty}\Big\{\int_{c_1-i\nu}^{c_1+i\nu} e^x x^n \Phi(n+a+1; 2n+a+\beta+2; -2x) g_1(x)\,dx +$$
$$+\int_{-c_2-i\nu}^{-c_2+i\nu} e^x x^n \Phi(n+a+1; 2n+a+\beta+2; -2x) g_2(-x)\,dx\Big\}.$$

Applying (7), relation (16) can be reduced to the form

$$(17)\quad A_n = \frac{1}{2\pi i}\,\Omega_n \sum_{k=0}^{\infty} e^{i\pi k} G_{n,k}\Big[\lim_{\nu\to\infty}\Big\{\int_{c_1-i\nu}^{c_1+i\nu} x^{-(a+\frac{1}{2})} I_{n+k+a+\frac{1}{2}}(x) g_1(x)\,dx +$$
$$+\int_{-c_2-i\nu}^{-c_2+i\nu} x^{-(a+\frac{1}{2})} I_{n+k+a+\frac{1}{2}}(x) g_2(-x)\,dx\Big\}\Big].$$

If $a = \beta$, then as before

$$(18)\quad A_n' = \frac{1}{2\pi i}\,\Omega_n G_{n,0} \lim_{\nu\to\infty}\Big\{\int_{c_1-i\nu}^{c_1+i\nu} x^{-(a+\frac{1}{2})} I_{n+a+\frac{1}{2}}(x) g_1(x)\,dx +$$
$$+\int_{-c_2-i\nu}^{-c_2+i\nu} x^{-(a+\frac{1}{2})} I_{n+a+\frac{1}{2}}(x) g_2(-x)\,dx\Big\},$$

where $\Omega_n G_{n,0}$ is given by (13).

In particular, for $a = -1/2$, the coefficients for the Chebyshev expansion follow from (18) as

$$C_n = \frac{1}{\pi i}\lim_{\nu\to\infty}\Big\{\int_{c_1-i\nu}^{c_1+i\nu} I_n(x) g_1(x)\,dx + \int_{-c_2-i\nu}^{-c_2+i\nu} I_n(x) g_2(-x)\,dx\Big\},$$

which corresponds to the result obtained by Elliott [1].

3. Fourier transform. In the case of the Fourier transform

$$g(x) = \int_0^\infty e^{ixt} f(t)\,dt.$$

The kernel e^{ixt} is replaced by (2) taking $y = t$. So if

$$g(x) = \sum_{n=0}^{\infty} A_n P_n^{(a,\beta)}(x),$$

then as before

(19) $$A_n = 2^n e^{in\pi/2}\times$$
$$\times\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}\int_0^\infty t^n e^{it}\Phi(n+\alpha+1; 2n+\alpha+\beta+2; -2it)f(t)\,dt.$$

Applying (7), relation (19) can be reduced to the form

(20) $$A_n = \exp\left\{\frac{(2n+\alpha+\frac{1}{2})i\pi}{2}\right\}\Omega_n\times$$
$$\times\sum_{k=0}^{\infty}\exp\left\{-\frac{i\pi}{2(n+k+\alpha+\frac{1}{2})}\right\}G_{n,k}H\left\{\frac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=1\\ \nu=n+\alpha+k+\frac{1}{2}}}.$$

Taking $\alpha=\beta$ we have

(21) $$A_n' = e^{in\pi/2}\Omega_n G_{n,0}H\left\{\frac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=1\\ \nu=n+\alpha+\frac{1}{2}}}.$$

It may be observed that Wimp's results for the Chebyshev expansion of $g_1(x)$ and $g_2(x)$, i.e. for the Fourier cosine and sine transforms, follow directly from (21).

4. Illustrative examples. In this section several examples have been included for the Laplace transform, inverse Laplace transform, and Fourier transform to show the practical application of the method. It may be remarked in this connection that the integrals for the coefficients A_n occurring in the polynomial approximation of the transform problems ultimately depend on the Hankel transforms which are similar to those obtained by Elliott [1] and Wimp [5]. Hence for the completeness of the discussion most of the examples considered by Wimp have been chosen to illustrate our method.

4.1. *Laplace transform.* If $f(t)=t^{s-1}e^{-at}$, then $g(x)=\Gamma(s)/(x+a)^s$, $\mathrm{Re}(s)>0$, $\mathrm{Re}(x)>-\mathrm{Re}(a)$. Now, employing the relations ([3], vol. 2, p. 332)

$$I_\nu(z)=\frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu}e^{-z}\Phi\left(\nu+\frac{1}{2}; 2\nu+1; 2z\right)$$

and ([3], vol. 1, eq. (11), p. 215)

(22) $$\int_0^\infty e^{-(p+1)t}t^{b-1}\Phi(a; c; 2t)\,dt$$
$$=\frac{\Gamma(b)}{(p+1)^b}\,{}_2F_1\left(a, b; c; \frac{2}{p+1}\right),\qquad \mathrm{Re}(b)>0,\ \mathrm{Re}(p)>1,$$

we obtain

$$(23)\qquad \int_0^\infty e^{-pt}t^{b-1}I_\nu(t)\,dt$$
$$= \frac{1}{2^\nu\Gamma(\nu+1)}\,\frac{\Gamma(\nu+b)}{(p+1)^{\nu+b}}\,{}_2F_1\left(\frac{1}{2}+\nu,\, b+\nu;\, 2\nu+1;\, \frac{2}{p+1}\right),$$
$$\operatorname{Re}(b+\nu)>0,\ \operatorname{Re}(p)>1.$$

With the help of (23) the coefficients A_n in the Jacobi expansion of $(x+a)^{-s}$ can be evaluated from (11) as

$$(24)\qquad A_n = (-1)^n\frac{\Omega_n}{\Gamma(s)}\sum_{k=0}^{\infty}G_{n,k}\frac{1}{2^{n+k+a+1/2}}\,\frac{\Gamma(n+k+s)}{\Gamma(n+k+a+\frac{3}{2})}(a+1)^{-(n+k+s)}\times$$
$$\times{}_2F_1\left(n+k+a+1;\, n+s+k;\, 2(n+k+a+1);\, \frac{2}{a+1}\right),$$
$$\operatorname{Re}(n+s+k)>0,\ \operatorname{Re}(a)>1.$$

Now, combining the result of (23) with ([3], vol. 1, p. 196)

$$\int_0^\infty e^{-pt}t^{b-1}I_\nu(t)\,dt = \frac{\Gamma(\nu+b)}{\Gamma(\nu+1)}\,\frac{(p-q)^\nu}{q^b}\,{}_2F_1\left(1-b,\, b;\, \nu+1;\, \frac{q-p}{2q}\right),$$

where $q=\sqrt{p^2-1}$, $\operatorname{Re}(b)>0$, $\operatorname{Re}(p)>1$, and using the formula ([4], eq. (3.3))

$${}_2F_1(a,b;c;z) = \frac{w^{\delta-b}2^{2\delta}}{(1+w)^{2\delta}}\sum_{k=0}^{\infty}(-1)^k\frac{(b)_k(2\delta)_k}{k!(\delta)_k}R_k(a,c,\delta)\left(\frac{1-w}{1+w}\right)^k\times$$
$$\times{}_2F_1\left(1-b+\delta,\, b-\delta;\, k+\delta+1;\, -\frac{(1-w)^2}{4w}\right),$$

where $w=\sqrt{1-z}$, $|z|<1$, relation (24) can be reduced after certain manipulations to

$$(25)\qquad A_n = (-1)^n2^n\frac{\Gamma(n+a+\beta+1)}{\Gamma(2n+a+\beta+1)}\cdot\frac{\Gamma(n+s)}{\Gamma(s)}(a+1)^{-(n+s)}\times$$
$$\times{}_2F_1\big(n+a+1,\, n+s;\, 2n+a+\beta+2;\, 2/(a+1)\big),$$
$$\operatorname{Re}(n+s)>0,\ \operatorname{Re}(a)>1.$$

For this particular example the coefficients given in (25) could have been directly obtained from (6) and (22) but for the simplicity of the integral in (11), the latter formula has been used for the evaluation. Moreover, since the Hankel transforms are widely tabulated, it is easier to handle formula (11) than that of (6).

In particular, if $\alpha = \beta = -1/2$, the coefficients of (25) are

$$A'_n = (-1)^n 2^n \frac{\Gamma(n)}{\Gamma(2n)} \frac{\Gamma(n+s)}{\Gamma(s)} (a+1)^{-(n+s)} {}_2F_1\left(n+\tfrac{1}{2}, n+s; 2n+1; 2/(a+1)\right),$$

and so the coefficients of the Chebyshev expansion

$$C_n = (-1)^n \varepsilon_n \frac{\Gamma(n+s)}{\Gamma(s)} (a^2-1)^{-s/2} P_{s-1}^{-n}\left(\frac{a}{\sqrt{a^2-1}}\right),$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{for } n = 0, \\ 2 & \text{for } n > 0. \end{cases}$$

4.2. *The Psi and Log Gamma functions.* Now $f(t) = (-1)^{p+1} e^{-at} t^p \times \times (1-e^{-t})^{-1}$, $\operatorname{Re}(a) > 1$. Then $g(x) = \psi^p(x+a)$, where $\psi(x) = D \log \Gamma(x)$. The coefficients in the Jacobi expansion of $g(x)$ are given by

$$A_n = (-1)^{n+p+1} 2^n \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)} \Gamma(n+p+1) \times$$
$$\times \sum_{r=0}^{\infty} (a+r+1)^{-(n+p+1)} {}_2F_1\left(n+p+1, n+\alpha+1; 2n+\alpha+\beta+2; 2/(a+r+1)\right).$$

If $p = 0$, then the coefficients in the expansion of $g(x)$ are

$$A_n = (-1)^{n+1} 2^n \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)} \Gamma(n+1) \times \tag{26}$$
$$\times \sum_{r=0}^{\infty} (a+r+1)^{-(n+1)} {}_2F_1\left(n+1, n+\alpha+1; 2n+\alpha+\beta+2; 2/(a+r+1)\right).$$

Taking $\alpha = \beta = -1/2$, for the Chebyshev expansion we have

$$C_n = (-1)^{n+1} 2^{-n+1} \sum_{r=0}^{\infty} (a+r+1)^{-(n+1)} \times \tag{27}$$
$$\times {}_2F_1\left(n+1, n+\tfrac{1}{2}; 2n+1; 2/(a+r+1)\right).$$

Now for $n = 0$ and $n = 1$ it follows from (27) that

$$C_0 = -2 \sum_{r=0}^{\infty} (a+r+1)^{-1} {}_2F_1\left(1, \tfrac{1}{2}; 1; 2/(a+r+1)\right) = -2 \sum_{r=0}^{\infty} \frac{1}{\sqrt{(a+r)^2-1}} \tag{28}$$

and

$$C_1 = \sum_{r=0}^{\infty} (a+r+1)^{-2} {}_2F_1\left(2, \tfrac{3}{2}; 3; 2/(a+r+1)\right) \tag{29}$$
$$= -2 \sum_{r=0}^{\infty} \left[1 - \frac{a+r}{\sqrt{(a+r)^2-1}}\right].$$

The series in (28) diverges, and that in (29) slowly converges. Hence C_0 and C_1 are to be determined in terms of higher computable coefficients, as shown in [5]. Having determined the Jacobi expansion or the Chebyshev expansion of $\psi(x+a)$ from (26) or (27) respectively, the corresponding expansions of $\log\Gamma(x+a)$ can be obtained by integrating the series for $\psi(x+a)$.

4.3. *Inverse Laplace transform.* Let $g(x) = \frac{1}{2}\log\left(1+(a/x)^2\right)$, $\mathrm{Re}(x) > |\mathrm{Im}(a)|$. Then $f(t) = (1-\cos at)/t$. Here $g_1(x) = \frac{1}{2}\log\left(1+(a/x)^2\right)$ and $g_2(x) = -g_1(x)$. With these $g_i(x)$, $i = 1, 2$, the integrant in (17) tends to zero as $v \to \infty$ in $x = c+iv$. Hence (17) gives

$$A_n = \frac{1}{4\pi i}\Omega_n \sum_{k=0}^{\infty}(-1)^n G_{n,k}\int_C x^{-\left(a+\frac{1}{2}\right)} I_{n+k+a+\frac{1}{2}}(x)\log\left(1+\left(\frac{a}{x}\right)^2\right)dx,$$

where C is the contour enclosing the slit from $-ia$ to ia. By displacing the contour to the slit we obtain

$$A_n = \frac{1}{2}\Omega_n \sum_{k=0}^{\infty}[(-1)^k+(-1)^{n+1}]\exp\left\{\frac{i(n+k-1)\pi}{2}\right\}\times$$
$$\times G_{n,k}\int_0^a y^{-\left(a+\frac{1}{2}\right)} J_{n+k+a+\frac{1}{2}}(y)\,dy.$$

If $a = -1/2$, then from [3], vol. 2, eq. (1), p. 333, it follows that

$$A_n = \Omega_n \sum_{r=0}^{\infty}\sum_{k=0}^{\infty}[(-1)^k+(-1)^{n+1}]\exp\left\{\frac{i(n+k-1)\pi}{2}\right\}\times$$
$$\times (G_{n,k})_{a=-\frac{1}{2}} J_{n+k+2r+1}(a),$$

Moreover, taking $\beta = -1/2$, the Chebyshev coefficients are

$$C_n = \begin{cases} 0 & \text{for } n \text{ even,} \\ 4\exp\left\{\dfrac{i(n-1)\pi}{2}\right\}\displaystyle\sum_{r=0}^{\infty} J_{n+2r+1}(a) & \text{for } n \text{ odd.}\end{cases}$$

4.4. *Fourier transform.* If

$$f(t) = \begin{cases} 1, & 0 < t \leqslant a \\ 0, & a < t < \infty,\end{cases}$$

then

$$g_1(x) = \frac{1-\cos ax}{x} = \int_0^{\infty} f(t)\sin tx\,dt$$

and

$$g_2(x) = \frac{\sin ax}{x} = \int_0^\infty f(t)\cos tx\,dt.$$

Using [3], vol. 2, eq. (1), p. 333, we evaluate (20) for $a = -1/2$ to obtain the coefficients in the particular Jacobi expansion of the Fourier transform of $f(t)$ as

$$(30) \qquad A_n = 2e^{in\pi}\Omega_n \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \exp\left\{-\frac{i(n+k)\pi}{2}\right\}(G_{n,k})_{a=-1/2} J_{n+k+2r+1}(a).$$

If, in addition, $\beta = -1/2$, then (30) gives

$$(31) \qquad A_n' = 2e^{in\pi/2}\Omega_n (G_{n,0})_{a=-1/2} \sum_{r=0}^{\infty} J_{n+2r+1}(a),$$

and so the coefficients for the Chebyshev expansion for the sine and cosine integrals follow from (31); this is in agreement with Wimp [5].

If $f(t) = J_0(t)/t$, then the Fourier sine transform equals

$$g_1(x) = \begin{cases} \sin^{-1}x, & 0 < x \leqslant 1, \\ \pi/2, & 1 < x < \infty. \end{cases}$$

Using [2], vol. 2, eq. (32), p. 92, we evaluate (20) for $a = -1/2$ so that

$$(32) \qquad A_n = \frac{2}{\pi}\Omega_n \sum_{k=0}^{\infty} \exp\left\{\frac{i(n-k)\pi}{2}\right\}(G_{n,k})_{a=-1/2} \frac{\sin(n+k)\pi/2}{(n+k)^2}.$$

For $\beta = -1/2$ eq. (32) becomes

$$(33) \qquad A_n' = \begin{cases} 0 & \text{for } n \text{ even}, \\ \dfrac{2}{\pi n^2}\Omega_n (G_{n,0})_{a=-1/2} e^{in\pi/2} \sin\dfrac{n\pi}{2} & \text{for } n \text{ odd}. \end{cases}$$

The coefficients in the symmetric Jacobi expansion ($a = \beta = -1/2$), and also for the Chebyshev expansion in the case of the Fourier sine transform, follow directly from (33).

Remark. It may be observed that the Hankel transforms occurring in all expressions for A_n can be easily evaluated if we take $a = -1/2$, but this does not imply that $\beta = -1/2$ also. It is only for the Chebyshev case that we can take $a = \beta = -1/2$, and that case is specially interesting.

Acknowledgement. I offer my grateful thanks to Mr. P. K. Ghosh, Reader, Dept. of Applied Mathematics, University of Calcutta, for helpful guidance in the preparation of the manuscript. I also thank Prof. Dr. S. K. Chakravorty, Head of the Dept. of Applied Mathematics, University of Calcutta, for giving me facility to work in the department.

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DEPT. OF APPLIED MATHEMATICS
UNIVERSITY COLLEGE OF SCIENCE
CALCUTTA, INDIA

Received on 18. 4. 1969;
revised version on 28. 3. 1970

N. K. BASU (Calcutta)

APROKSYMACJA TRANSFORMAT CAŁKOWYCH PRZY POMOCY WIELOMIANÓW

STRESZCZENIE

W pracy znaleziono aproksymację transformat Laplace'a i Fouriera $g(x)$ pewnej funkcji $f(t)$ przy pomocy wielomianów Jacobiego $P_n^{(\alpha,\beta)}(x)$ w przedziale $-1 < x < 1$. Współczynnikami są całki funkcji hipergeometrycznych, przekształcone ostatecznie w szeregi transformat Hankela. Podobnie zostały potraktowane odwrotne transformaty Laplace'a. Praktyczna przydatność metod jest pokazana na kilku przykładach numerycznych.