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ON GENERALIZED SYMMETRIC MEANS
AND STIRLING NUMBERS OF THE SECOND KIND

1. Introduction and notation. Let t_0, t_1, ..., t_k $(k \in N_0)$ be non-negative real numbers. The *generalized n-th symmetric mean* $h_n(t_0, t_1, ..., t_k)$ is defined in the following way

$$(1.1) \qquad h_n(t_0, t_1, ..., t_k) = \binom{n+k}{k}^{-1} \sum_{i_0+i_1+...+i_k=n} t_0^{i_0} t_1^{i_1} ... t_k^{i_k}$$

$(i_0, i_1, ..., i_k \in \{0, 1, ..., n\}$; $n \in N_0$; $h_0(t_0, t_1, ..., t_k) = 1)$ (cf. [11]). The sum (1.1) involves $\binom{n+k}{k}$ terms. In what follows we assume that k is arbitrary but fixed. Without loss of generality we may assume $t_0 \leqslant t_1 \leqslant ... \leqslant t_k$.

In Section 3 we derive recurrence relations as well as inequalities that hold for the means (1.1). For our aims we recall the definition of logarithmic convexity. A real sequence $\{a_l\}$ is said to be logarithmically convex if

$$(1.2) \qquad a_l^2 \leqslant a_{l-1} a_{l+1} \qquad \text{for all } l$$

(see, e.g., [11]). Strong logarithmic convexity means strict inequality in (1.2). Analogously we define logarithmic concavity. In the monograph [11] the question concerning the proof of logarithmic convexity of the means (1.1) has been stated. Earlier K. V. Menon [9] gave such a proof for $n = 1, 2, 3$ only. Recently DeTemple and Robertson [4] have proved that

$$h_n^2(t_0, t_1) \leqslant h_{n-1}(t_0, t_1) h_{n+1}(t_0, t_1) \qquad (n \in N).$$

We will establish logarithmic convexity of the means (1.1) without any restrictions on the parameters n and k (see Theorem 3.2).

The second object of our study are the Stirling numbers of the second kind usually denoted by $S(n, k)$ $(n, k \in N_0)$. These are defined to be the

number of ways of partitioning a set of n elements into k non-empty subsets (cf. [15]). They are closely related to the means (1.1) (see (4.5)). In Section 4 we prove a number of recurrence relations as well as some inequalities that are valid for them. Among other things, we prove the strong logarithmic concavity for $S(\cdot, k)$, i.e.,

$$(1.3) \qquad S^2(n, k) > S(n-1, k)S(n+1, k) \qquad (2 \leqslant k \leqslant n).$$

Strong logarithmic concavity for the sequence $S(n, \cdot)$ has been established by Harper [6] and Lieb [8].

The useful tool in our considerations are the B-splines of Curry and Schoenberg [3]. The definition and some elementary properties of those important splines are postponed to Section 2.

2. Preliminaries. In this section we give a definition and some facts concerning the B-splines of Curry and Schoenberg [3]. Let $\ldots \leqslant x_{-1} \leqslant x_0 \leqslant x_1 \leqslant \ldots$ be a bi-infinite partition of R with at most k ($k \in N$) values of the x's equal to each other, i.e., $x_i < x_{i+k}$ for all $i \in Z$. The function

$$M_{i,k}(t) = k[x_i, x_{i+1}, \ldots, x_{i+k}](\cdot - t)_+^{k-1} \qquad (i \in Z; k \in N; t \in R)$$

is the *B-spline* of degree $k-1$ (order k). Here $[x_i, x_{i+1}, \ldots, x_{i+k}]f$ is the divided difference of order k for the function f in the points x_l ($l = i$, $i+1, \ldots, i+k$) and as usual $t_+^{k-1} = (\max\{0, t\})^{k-1}$. For the reader's convenience we list below some well-known properties of B-splines.

$1°$ $M_{i,k}(t) > 0$ for $t \in (x_i, x_{i+k})$ and $M_{i,k}(t) = 0$ otherwise. Thus,

$$\operatorname{supp} M_{i,k} = [x_i, x_{i+k}].$$

$2°$ In each interval $[x_{i+j}, x_{i+j+1}]$ ($x_{i+j} < x_{i+j+1}; j = 0, 1, \ldots, k-1$) $M_{i,k}$ coincides with an algebraic polynomial of degree $k-1$ or less.

$3°$ Let x_{i+j} be a knot of multiplicity r, i.e., let $x_{i+j-1} < x_{i+j} = \cdots$ $\ldots = x_{i+j+r-1} < x_{i+j+r}$, then $M_{i,k}$ is exactly $k-1-r$ times continuously differentiable on (x_{i+j-1}, x_{i+j+r}) (see, e.g., [3], [16]).

The B-splines satisfy the following fundamental recurrence relation

$$(2.1) \qquad \frac{k-1}{k} M_{i,k}(t) = \frac{t - x_i}{x_{i+k} - x_i} M_{i,k-1}(t) + \frac{x_{i+k} - t}{x_{i+k} - x_i} M_{i+1,k-1}(t)$$

$$(i \in Z; k \geqslant 2; t \in R)$$

(cf. [1], [16]).

In the sequel we use the moments of B-splines $M_{i,k}$. Let $i \in Z$. By $\mu_n(i, k)$ we denote the nth moment of $M_{i,k}$, i.e.

$$\mu_n(i, k) = \int\limits_{x_i}^{x_{i+k}} t^n M_{i,k}(t)\, dt \qquad (n \in N_0).$$

The explicit formula for $\mu_n(i, k)$ in terms of the knots x_i has been derived in [13] and [16]. Namely, we have

(2.2) $$\mu_n(i, k) = \binom{n+k}{k}^{-1} \sum_{i \leqslant j_1 \leqslant j_2 \leqslant \ldots \leqslant j_n \leqslant i+k} x_{j_1} x_{j_2} \ldots x_{j_n}.$$

The last sum involves $\binom{n+k}{k}$ terms.

3. The recurrence formulae and inequalities for the generalized symmetric means.

In our further considerations an important role plays the identity (3.1) given below. Let $M_{0,k}$ be the B-spline with knots $0 \leqslant t_0 \leqslant t_1 \leqslant \ldots \leqslant t_k$. Making use of (1.1) and (2.2) it is easy to check that

(3.1) $$h_n(t_0, t_1, \ldots, t_k) = \mu_n(0, k) \qquad (n \in N_0; k \in N).$$

Setting $\mu_n(0, 0) = t_0^n$ we see that (3.1) is also valid for $k = 0$. Thus, in view of (3.1) all results established for the means (1.1) also hold for the moments of $M_{0,k}$ as well. For the sake of notation we write often $h_n(i, j)$ instead of $h_n(t_i, t_{i+1}, \ldots, t_j)$ $(0 \leqslant i \leqslant j \leqslant k)$.

Our first result is given by the following theorem.

THEOREM 3.1. *For any* $n \in N_0$ *and* $k \in N$ *the generalized symmetric means satisfy the following recurrence relations*:

(3.2) $$h_n(0, k) = \frac{1}{n+k} [nt_k h_{n-1}(0, k) + kh_n(0, k-1)],$$

(3.3) $$h_n(0, k) = \frac{k}{(n+1)(t_k - t_0)} [h_{n+1}(1, k) - h_{n+1}(0, k-1)] \qquad (t_0 < t_k),$$

(3.4) $$h_n(0, k) = \frac{k}{(n+k)(t_k - t_0)} [t_k h_n(1, k) - t_0 h_n(0, k-1)] \qquad (t_0 < t_k),$$

(3.5) $$h_n(t_0 + \gamma, \ldots, t_k + \gamma) = \sum_{l=0}^{n} \binom{n}{l} \gamma^l h_{n-l}(t_0, \ldots, t_k) \qquad (k \in N_0; \gamma \geqslant -t_0).$$

Proof. Let

(3.6) $$y_k(t) = \sum_{n=0}^{\infty} \mu_n(0, k) \frac{t^n}{n!}$$

denote the exponential generating function for the moments $\mu_n(0, k)$ of the B-spline $M_{0,k}$. In [13] we have shown that y_k satisfies the following differential-difference equation

(3.7) $$(k - t_k t) y_k(t) + t y_k'(t) = k y_{k-1}(t) \qquad (k \in N).$$

Substituting (3.6) into (3.7) and equating the coefficients of t^n one gets

(3.8) $$(n+k) \mu_n(0, k) = nt_k \mu_{n-1}(0, k) + k\mu_n(0, k-1).$$

Hence and from (3.1) the assertion (3.2) follows. In order to prove (3.3) we observe that the identities (2.5) and (3.3) in [13] together with (3.1) yield

(3.9) $h_n(0, k) = \binom{n+k}{k}^{-1} [t_0, \ldots, t_k] t^{n+k}$ $(k \in N_0)$.

Hence and from the equality

$$[t_0, \ldots, t_k] t^{n+k} = \frac{1}{t_k - t_0} \{[t_1, \ldots, t_k] t^{n+k} - [t_0, \ldots, t_{k-1}] t^{n+k}\}   (k \in N)$$

the desired recurrence (3.3) follows. For the proof of (3.4) we employ the recurrence (2.1) setting $i = 0$ and $x_{i+j} = t_j$ $(j = 0, 1, \ldots, k)$. Further multiplying both sides by t^n and integrating over $[t_0, t_k]$ one obtains, in view of (3.1), the following:

$$\frac{k-1}{k}(t_k - t_0) h_n(0, k)$$

$$= t_k h_n(1, k-1) - t_0 h_n(0, k-1) + h_{n+1}(0, k-1) - h_{n+1}(1, k)   (k > 1).$$

Hence and from (3.3) we get the assertion (3.4) for $k > 1$. Direct calculations show that (3.4) is also valid for $k = 1$. We now prove the last statement of our theorem. For $k = 0$ the assertion is a trivial one. Assume $k > 0$. Let $M_{0,k}(\cdot | t_0, \ldots, t_k) \equiv M_{0,k}(\cdot)$. For any $\gamma \in R$ we have

$$M_{0,k}(\cdot | t_0 + \gamma, \ldots, t_k + \gamma) = M_{0,k}(\cdot - \gamma | t_0, \ldots, t_k).$$

Thus we have for $\gamma \geqslant -t_0$

$$h_n(t_0 + \gamma, \ldots, t_k + \gamma) = \int_{t_0 + \gamma}^{t_k + \gamma} t^n M_{0,k}(t | t_0 + \gamma, \ldots, t_k + \gamma) dt$$

$$= \int_{t_0 + \gamma}^{t_k + \gamma} t^n M_{0,k}(t - \gamma | t_0, \ldots, t_k) dt$$

$$= \int_{t_0}^{t_k} (z + \gamma)^n M_{0,k}(z | t_0, \ldots, t_k) dz,$$

where $z = t - \gamma$. Hence the assertion follows. The proof is completed. ∎

The recurrence (3.2) is useful if one wants to calculate the value of $h_n(0, k)$. Is is easy to see that there is no cancellation in evaluating the right-hand side of (3.2).

Now we derive some inequalities for the means (1.1). For convenience we denote $h_n(0, k)$ by h_n. Among other things we derive the logarithmic convexity of the sequence $\{h_n\}_{n=0}^\infty$.

We are now ready to prove the following

THEOREM 3.2. *Let* $m, n \in N_0$ *and let*

$$\alpha \in \begin{cases} [t_k; \infty) & \text{if} \quad m \text{ odd}, \\ (-\infty, \infty) & \text{if} \quad m \text{ even}. \end{cases}$$

Then

(3.10)
$$\sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} \alpha^l h_{n+m-l} \geq 0.$$

Moreover, if $n \in N$, *then*

(3.11)
$$\frac{n}{n+k} h_{n-1} h_{n+1} \leq h_n^2 \leq h_{n-1} h_{n+1}.$$

Proof. If $k = 0$ or $t_0 = \ldots = t_k$, then the thesis is obvious. Thus, we assume $k > 0$ and $t_0 < t_k$. Now the inequality (3.10) follows immediately from the following one

$$\int_{t_0}^{t_k} t^n (\alpha - t)^m M_{0,k}(t) \, dt \geq 0.$$

In order to prove the first inequality in (3.11) we use the following one

(3.12)
$$\frac{n}{n+k} t_k h_{n-1} \leq h_n \quad (n \in N).$$

This is an obvious consequence of (3.2). Further, setting $m = 1$ and $\alpha = t_k$ into (3.10), we arrive at

$$\frac{1}{t_k} h_{n+1} \leq h_n \quad (n \in N_0).$$

Hence and from (3.12) the assertion follows. For the proof of the second inequality in (3.11) we employ (3.10) setting $m = 2$. In such a case we get

$$h_n \alpha^2 - 2h_{n+1} \alpha + h_{n+2} \geq 0 \quad (n \in N_0).$$

Since the leading coefficient of the above quadratic is positive, the assertion follows. This completes the proof. ∎

From the proofs of (3.10) and the second of the inequalities (3.11) we see that these results are true not only for the means h_n. Particularly it is known that any totally monotonic sequence is logarithmically convex (cf. [2]). We recall that a real sequence $\{c_n\}_{n=0}^{\infty}$ is said to be totally (or completely) monotonic (TM) if the inequalities $(-1)^m \Delta^m c_n \geq 0$ are valid for all $m, n \in N_0$ (see, e.g., [2], [17]). Generally the means h_n are not in TM. This confirms the following

COROLLARY 3.1. *If* $1 \leqslant \dfrac{1}{k+1} \sum\limits_{l=0}^{k} t_l$, *then* $h_0 \leqslant h_1 \leqslant \ldots$ *On the other hand if*

$t_k \leqslant 1$, *then* $\{h_n\}_{n=0}^{\infty} \in TM$.

Proof. In order to prove the first part of the thesis we observe that if for some $l \in N$ is $h_{l-1} \leqslant h_l$, then also $h_l \leqslant h_{l+1}$. This fact follows directly from the second inequality (3.11). According to the definition of (1.1) we have h_0 $= 1$, $h_1 = \dfrac{1}{k+1} \sum\limits_{l=0}^{k} t_l$. The assertion follows. The second statement follows from the classical results due to Hausdorff ([7]) or from (3.10) with $\alpha = 1$. ∎

Theorem 2.52 of [2] and the identity (3.1) give the following

THEOREM 3.3. *Let* $p, q > 1$, $1/p + 1/q = 1/r \leqslant 1$. *Then*

$$(3.13) \qquad\qquad h_{r(l+m)}^{1/r} \leqslant h_{pl}^{1/p} h_{qm}^{1/q} \qquad (l, m \in N_0)$$

provided $r(l+m)$, pl *and* qm *are non-negative integers.*

Other inequalities for the means h_n can be easily derived making use of the well-known inequalities that hold for TM sequences and the so-called Stieltjes moment sequences. For details and results see [2], [17].

4. The recurrence formulae and inequalities for Stirling numbers of the second kind. In Section 1 we have recalled the definition of Stirling numbers of the second kind. For further purposes we summarize below some more or less elementary facts concerning these numbers. The $S(n, k)$ is positive for $1 \leqslant k \leqslant n$ and equal to zero for other values of k. As usual we set $S(0, 0)$ $= 1$. The explicit formula for $S(n, k)$ is the following one

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

The $S(\cdot, \cdot)$ are combinatorially distributed by the following difference equation

$$(4.1) \qquad S(n+1, k) = S(n, k-1) + kS(n, k) \qquad (k \in N; n \in N_0).$$

These and other properties of these numbers are given in Riordan [15].

Rennie and Dobson [14] proved by elementary arguments that

$$(4.2) \quad (n-k)S(n, k) > (k+1)S(n, k+1) \qquad (n \geqslant 4; \tfrac{1}{2}(n+1) \leqslant k \leqslant n-1),$$

$$(4.3) \qquad\qquad S(2n, n) > S(2n, n+1) \qquad (n \geqslant 2).$$

It is known (cf. [5], [14]) that, for fixed n, $S(n, k)$ has a single maximum, i.e., that there is k_n such that

$$S(n, 1) < S(n, 2) < \ldots < S(n, k_n)$$

and

$$S(n, k_n) \geqslant S(n, k_n+1) > \ldots > S(n, n).$$

In the above mentioned paper [14] the authors proved that

(4.4) $$k_n \leqslant \tfrac{1}{2}(n+1).$$

The asymptotic behaviour of k_n has been studied in [10], [14] (see also the references therein).

In this section we derive new recurrence formulae as well as inequalities for the numbers $S(\cdot, \cdot)$. Among other things we establish the inequality (1.3) and the lower bound for k_n. Some of our results form a counterpart for the inequalities (4.2) and (4.3) (see Theorem 4.3 and Corollary 4.2).

The key identity that is used repeatedly is

(4.5) $$S(n+k, k) = \binom{n+k}{k} h_n(t_0, t_1, \ldots, t_k) \qquad (n, k \in N_0),$$

where $t_l = l$ for all $l = 0, 1, \ldots, k$.

This follows immediately from (3.3) in [13] and from (3.1).

In the sequel we write $M_k(\cdot)$ instead of $M_{0,k}(\cdot|0, 1, \ldots, k)$. The B-splines M_k are referred to as forward B-splines. For further use we recall a differentiation formula

(4.6) $$M'_k(t) = M_{k-1}(t) - M_{k-1}(t-1) \qquad (k = 2, 3, \ldots).$$

This holds for any $t \in R$ except the case $k = 2$ where we must assume $t \in R \backslash \{1\}$ (see, e.g., [16]). By $\mu_n(k)$ we denote the nth moment of the M_k.

We are ready to state and prove the following

THEOREM 4.1. *Let* $n, l \in N_0$; $k \in N$. *Then*

(4.7) $$\sum_{i=p-n}^{p} (-1)^{p-i} \binom{i-k+1}{l} \binom{p}{i} S(i+1, k) = \sum_{i=p-l}^{p} \binom{i-k+1}{n} \binom{p}{i} S(i, k-1),$$

where $p = n+k+l-1$.

Proof. To verify (4.7) for $k = 1$ we take into account the identities $S(m, 0) = \delta_{m0}$ $(m \geqslant 0)$ and $S(m, 1) = 1$ $(m \geqslant 1)$. Let $k > 1$. We define

(4.8) $$u(t) = (t-1)^n t^l \qquad (n, l \in N_0),$$

and

(4.9) $$I = \int_0^k u(t) M'_k(t) dt.$$

Performing integration by parts we obtain

(4.10) $$I = -\int_0^k u'(t) M_k(t) dt.$$

Substituting (4.8) into (4.10) and further applying the binomial formula we get

(4.11) $$I = \sum_{j=0}^{n} (-1)^{n-j+1} \binom{n}{j} (j+l) \mu_{j+l-1}(k).$$

On the other hand inserting (4.6) into (4.9), we obtain easily

$$I = \int_0^{k-1} [u(t) - u(t+1)] M_{k-1}(t)\,dt.$$

Using (4.8) one obtains

$$I = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \mu_{j+l}(k-1) - \sum_{j=0}^{l} \binom{l}{j} \mu_{j+n}(k-1)$$

after little algebra. Comparing this with (4.11) we get

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [(j+l)\mu_{j+l-1}(k) + \mu_{j+l}(k-1)] = \sum_{j=0}^{l} \binom{l}{j} \mu_{j+n}(k-1).$$

Applying to the above the identity

$$(4.12) \qquad S(n+k, k) = \binom{n+k}{k} \mu_n(k) \qquad (n, k \in N_0)$$

(see [13], (3.3)) we obtain with the help of (4.1) the assertion. The proof i completed. ∎

From the above theorem many particular recurrences follo\ immediately. We recor̃d only two of them.

COROLLARY 4.1. *Let* $k, p \in N$. *Then*

$$(4.13) \qquad kS(p, k) = \sum_{i=k-1}^{p-1} \binom{p}{i} S(i, k-1),$$

and

$$(4.14) \qquad S(p, k-1) = \sum_{i=k-1}^{p} (-1)^{p-i} \binom{p}{i} S(i+1, k).$$

Proof. For the proof of (4.13) we insert $n = 0$ into (4.7). Furthe employing the recurrence formula (4.1) we obtain the desired assertion. Th recurrence (4.14) follows directly from (4.7) letting $l = 0$. ∎

Our next result reads as follows.

THEOREM 4.2. *For any* $k \in N$ *and any* $l \in N_0$, *the following recurrence*

$$(4.15) \qquad (k-1)kS(k+l, k) = (l+1)S(k+l, k-1) +$$

$$+ \sum_{j=0}^{l} \binom{k+l}{k+j} [(k-1)(k+j)S(k+j-1, k-1) - (j+1)S(k+j, k-1)]$$

holds.

Proof. For $k = 1$ there is nothing to prove. Assume $k > 1$. For the progressive B-splines the recurrence (2.1) may be rewritten as follows

$$(k-1) M_k(t) = t M_{k-1}(t) + (k-t) M_{k-1}(t-1) \quad (t \in R).$$

Multiplying both sides by t^l ($l \in N_0$), further using the binomial formula and integrating over the interval $[0, k]$ one gets the following recurrence

(4.16) $\quad (k-1) \mu_l(k) = \mu_{l+1}(k-1) + \sum_{j=0}^{l} \binom{l}{j} [(k-1) \mu_j(k-1) - \mu_{j+1}(k-1)].$

Applying (4.12) to (4.16) one obtains the assertion (4.15). This completes the proof. ∎

We give a number of inequalities involving the numbers $S(\cdot, \cdot)$. Some of them follow simply from the results of Section 3 and of this section.

THEOREM 4.3. For the numbers $S(\cdot, \cdot)$ the following inequalities

(4.17) $\quad (n+1) k S(n, k) > (n+1-k) S(n+1, k) \quad (1 \leqslant k \leqslant n),$

(4.18) $\quad (n-k+2) S(n+1, k) > (n+1) S(n, k) \quad (2 \leqslant k \leqslant n),$

(4.19) $\quad k S(n+1, k) > (n+1) S(n, k-1) \quad (1 \leqslant k \leqslant n),$

(4.20) $\quad k(k+1) S(n, k+1) > (n-k) S(n, k) \quad (1 \leqslant k < n)$

hold true.

Proof. The inequality (4.17) is obvious for $k = 1$. Let $k > 1$. The function $t^j (k-1-t) M_{k-1}(t)$ ($j \in N_0$) is positive for $t \in (0, k-1)$. Hence also

$$\int_0^{k-1} t^j (k-1-t) M_{k-1}(t) dt > 0,$$

and further $(k-1) \mu_j(k-1) > \mu_{j+1}(k-1)$. According to (4.12) one gets

(4.21) $\quad (k-1)(k+j) S(k+j-1, k-1) > (j+1) S(k+j, k-1)$

$$(k = 2, 3, \ldots; j \in N_0).$$

Setting above $k+j-1 := n$, $k := k+1$, we get (4.17). In order to prove (4.18) we use the identity (4.7) for $n = 1$. Simple calculations lead to the following recurrence

$$(p-k+1) k S(p, k)$$

$$= p k S(p-1, k) + p(p-k+1) S(p-1, k-1) + \sum_{i=k}^{p-2} (i-k+1) \binom{p}{i} S(i, k-1).$$

Hence one gets $(p-k+1) S(p, k) > p S(p-1, k)$ and the assertion (4.18)

follows directly. Making use of (4.15) and (4.21), we obtain the inequality

$$kS(k+l, k) > (k+l)S(k+l-1, k-1).$$

Setting above $k+l-1:=n$ one gets (4.19). For the proof of (4.20) we employ the identity (4.15), and the inequality (4.21). Hence we get

$$(k-1)kS(k+l, k) > (l+1)S(k+l, k-1).$$

Now setting $k+l:=n$, $k:=k+1$ we obtain the desired inequality (4.20). The proof is completed. ∎

COROLLARY 4.2. *For any* $n \in N$, *the inequalities*

$$(2n+1)nS(2n, n) > (n+1)S(2n+1, n),$$

$$(n+1)S(2n, n+1) > S(2n, n)$$

are valid.

Proof. The first inequality follows directly from (4.17), and the second one follows from (4.20). ∎

With the help of (4.20), we obtain immediately the following

COROLLARY 4.3. $(n+1)^{1/2} \leqslant k_n$.

Before presenting our next result we introduce more notation. Following K. V. Menon [9] let

$$H_n = \sum_{i_0 + i_1 + \ldots + i_k = n} a_0^{i_0} a_1^{i_1} \ldots a_k^{i_k}$$

$$(n \in N_0; i_0, i_1, \ldots, i_k \in \{0, 1, \ldots, n\}; H_0 = 1)$$

denote a complete symmetric function of the nth order for the non-negative variables $a_0, a_1, \ldots, a_k$. In that paper it is shown that

$$H_{p-m}H_{q+m} \geqslant H_{p-m-1}H_{q+m+1} \qquad (p \leqslant q; 0 \leqslant m < p),$$

$$H_1 \geqslant H_2^{1/2} \geqslant H_3^{1/3} \geqslant \ldots$$

These inequalities are strict unless all but one of the variables are zero. Making use of (1.1), (4.5) and the definition of H_n we get

$$S(n+k, k) = H_n \qquad (n \in N_0; k \in N),$$

provided $a_l = l$ for all $l = 0, 1, \ldots, k$. This identity and the above inequalities lead to the following

THEOREM 4.4. *For any* m, $k \in N$ *the inequalities*

(4.22) $S(m, k)S(n, k) > S(m-1, k)S(n+1, k)$ $(m \leqslant n)$,

(4.23) $S(k+1, k) > S(k+2, k)^{1/2} > S(k+3, k)^{1/3} > \ldots$

hold true.

Inserting $m = n$ into (4.22) one obtains the inequality (1.3).
We have also the following

THEOREM 4.5. *Assume m, $n \in N_0$; $k \in N$. If*

$$\alpha \in \begin{cases} [k, \infty) & \text{if} \quad m \text{ odd}, \\ (-\infty, \infty) & \text{if} \quad m \text{ even}, \end{cases}$$

then

$$\sum_{l=0}^{m} (-1)^l \binom{m}{l} \binom{j+l}{k}^{-1} \alpha^{m-l} S(j+l, k) \geqslant 0,$$

where $j = n+k$. Also the following inequalities

$$(4.24) \qquad S(l-1, k) S(l+1, k) \leqslant \frac{l+1}{l+1-k} S^2(l, k) \leqslant \frac{l}{l-k} S(l-1, k) S(l+1, k)$$

$$(k < l),$$

$$\left[\binom{k+r(l+m)}{k}^{-1} S(k+r(l+m), k) \right]^{1/r}$$

$$\leqslant \left[\binom{k+pl}{k}^{-1} S(k+pl, k) \right]^{1/p} \left[\binom{k+qm}{k}^{-1} S(k+qm, k) \right]^{1/q}$$

$$(p, q > 1; 1/p + 1/q = 1/r \leqslant 1; l, r(l+m), pl, qm \in N_0),$$

$$(4.25) \qquad \binom{k+1}{k}^{-1} S(k+1, k) \leqslant \left[\binom{k+2}{k}^{-1} S(k+2, k) \right]^{1/2}$$

$$\leqslant \left[\binom{k+3}{k}^{-1} S(k+3, k) \right]^{1/3} \leqslant \dots$$

are valid.

Proof. The first three of the above inequalities follow immediately from (3.10), (3.11) and (3.13), respectively, making use of (4.5). For the proof of (4.25) we employ the inequalities $h_1 \leqslant h_2^{1/2} \leqslant h_3^{1/3} \leqslant \dots$ that follow simply from the logarithmic convexity of the means h_n. Hence and with the help of (4.5) we obtain the desired result. This completes the proof. $\blacksquare$

The second inequality of (4.24) has been established in [12].
We conclude this section with the following

PROPOSITION 4.1. *If $k, p \in N$ and $k \leqslant p$, then*

$$S(p, k) \geqslant \sum_{i=k-1}^{p-1} S(i, k-1).$$

Proof. First we observe that $\binom{p}{i} \geqslant k$ for all $i = k-1, \ k, \ ..., \ p-1$

provided $k \leqslant p$. Hence and from (4.13) the assertion follows. ∎

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