ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIII, 1 (1972)

Ś. ZĄBEK (Lublin)

ON A VARIATIONAL ESTIMATION OF ERROR

1. Let us consider [1] the boundary-value differential problem:

$$L[y] \equiv \sum_{i=0}^{m} (-1)^i [p_i(x) y^{(i)}]^{(i)} = r(x), \tag{1}$$

$$y^{(i)}(a) = y^{(i)}(b) = 0 \quad \text{for } i = 0, 1, \ldots, m-1, \tag{2}$$

where $a < b$, m is a natural number, $y^{(0)} \equiv y$, $y^{(i)} \equiv d^i y / dx^i$, all the $p_i(x)$ are non-negative real functions of the class $C^{(i)}$ in the closed interval $\langle a, b\rangle$ for $i = 0, 1, \ldots, m$, and $p_m(x) \geqslant p > 0$ for $x \in \langle a, b\rangle$.

Let $v(x)$ denote an arbitrary function of the class $C^{(2m)}$ in $\langle a, b\rangle$, satisfying (2), and let $Y[x]$ be the exact solution of problem (1)-(2). We consider $v(x)$ as an approximate solution of the same problem. Thus, the maximal absolute error of this approximation is

$$\hat{e} = \max_{\langle a,b\rangle} |Y(x) - v(x)|.$$

Let us take $z(x) = [Y(x) - v(x)]/\hat{e}$. The function $z(x)$ is evidently one of the class $C^{(2m)}$, satisfies (2) and we have $\sup_{\langle a,b\rangle} |z(x)| = 1$.

Bertram [1] has showed that, for every natural m,

$$\hat{e} \leqslant \int_a^b |L[v] - r(x)|\, dx \Big/ \int_a^b \sum_{i=0}^{m} p_i(x) [z^{(i)}(x)]^2 dx. \tag{3}$$

Thus, if we want to get an over-estimation for $\hat{e}$, it is sufficient to under-estimate the integral

$$I_m = \int_a^b \sum_{i=0}^{m} p_i(x) [z^{(i)}(x)]^2 dx$$

and (3) gives an *a posteriori* estimation of the error of an approximate solution of problem (1)-(2).

---

(1) The main results of this paper were firstly presented without proofs in [4].

Substituting $t = (x-a)/(b-a)$, $w(t) = z[(b-a)t+a]$, $w^{(i)}(t) = z^{(i)}[(b-a)t+a](b-a)^i$ and $q_i(t) = p_i[(b-a)t+a]/(b-a)^{2i-1}$, we obtain

$$I_m = \int_0^b \sum_{i=0}^m q_i(t)[w^{(i)}(t)]^2 dt,$$

where $q_i(t) \in C^{(i)}$ in $\langle 0, 1\rangle$, $q_i(t) \geqslant 0$ and $q_m(t) \geqslant q = p/(b-a)^{2m-1} > 0$ (²). Evidently, $w(t) \in C^{(2m)}$, $\max\limits_{\langle 0,1\rangle} |w(t)| = 1$, and $w^{(i)}(0) = w^{(i)}(1) = 0$ for $i = 0, 1, \ldots, m-1$.

Let $\Phi_m[u, \alpha, \beta]$ denote a functional of the form

$$\Phi_m[u, \alpha, \beta] = \int_\alpha^\beta \sum_{i=0}^m q_i(t)[u^{(i)}(t)]^2 dt, \tag{4}$$

where $0 \leqslant \alpha < \beta \leqslant 1$. If we define Z^m as the class of functions $u(t)$, $u(t) \in C^{(2m)}$ for $t \in \langle 0, 1\rangle$, $\max\limits_{\langle 0,1\rangle} |u(t)| = 1$, and $u^{(i)}(0) = u^{(i)}(1) = 0$ for $i = 0, 1, \ldots$ $\ldots, m-1$, then

$$I_m \geqslant \inf_{Z^m} \Phi_m[u, 0, 1]. \tag{5}$$

We can assume the maximum of $u(t)$ in the interval $\langle 0, 1\rangle$ to be equal to 1, because $\Phi_m[-u, \alpha, \beta] = \Phi_m[u, \alpha, \beta]$.

Let L_h^m ($h > 0$) denote the class of functions $c(t)$ with absolutely continuous $(m-1)$-th derivative, $|c(t)| \leqslant 1$ in the interval $\langle 0, h\rangle$, $c(h) = 1$, and $c^{(i)}(0) = 0$ for $i = 0, 1, \ldots, m-1$. Further, let P_h^m ($h < 1$) denote the class of functions $d(t)$ with absolutely continuous $(m-1)$-th derivative, $|d(t)| \leqslant 1$ in the interval $\langle h, 1\rangle$, $d(h) = 1$, $d^{(i)}(1) = 0$ for $i = 0, 1, \ldots, m-1$.

Thus, we define U_h^m ($0 < h < 1$) as the class of functions $u(t)$ ($t \in \langle 0, 1\rangle$) such that for every function $u(t) \in U_h^m$ there exist functions $c(t) \in L_h^m$ and $d(t) \in P_h^m$ satisfying the relation

$$u(t) = \begin{cases} c(t) & \text{for } z \in \langle 0, h\rangle, \\ d(t) & \text{for } t \in \langle h, 1\rangle. \end{cases}$$

We take now

$$U^m = \bigcup_{h \in \langle 0, 1\rangle} U_h^m.$$

Evidently, $Z^m \subset U^m$ and from (5) we have

$$\begin{aligned} I_m &\geqslant \inf_{Z^m} \Phi_m[u, 0, 1] \geqslant \inf_{U^m} \Phi_m[u, 0, 1] = \inf_{h \in \langle 0, 1\rangle} \inf_{U_h^m} \Phi_m[u, 0, 1] \\ &= \inf_{h \in \langle 0, \rangle} \left(\min_{L_h^m} \Phi_m[u, 0, h] + \min_{P_h^m} \Phi_m[u, h, 1]\right). \end{aligned} \tag{6}$$

(²) Similarly, if for some i ($0 \leqslant i \leqslant m-1$) $p_i(x) \geqslant P_i > 0$, then $q_i(t) \geqslant Q_i = P_i/(b-a)^{2i-1} > 0$.

The last identity is a conclusion from the relation

$$\Phi_m[u, 0, 1] = \Phi_m[u, 0, h] + \Phi_m[u, h, 1].$$

The existence of the minima of these component functionals results from the corollary proved in paper [3].

We may calculate or estimate these minimal values by methods of the calculus of variations. Further, we may calculate the lower limit of a function of h by the classical methods of calculus. Finally, we get a constant H^m, $H^m \leqslant I_m$.

Proceeding in this way, Tatarkiewicz [2] has received the estimations

$$I_1 \geqslant 4\Big/\int_a^b (p_1(x))^{-1} dx = H_1^1 \geqslant 4p/(b-a) = H_2^1, \tag{7}$$

and from this, for $p_1(x) \equiv p = 1$, $a = 0$ and $b = 1$,

$$I_1 \geqslant 4 = H_3^1. \tag{8}$$

Moreover, assuming $p_0(x) \geqslant P_0 > 0$, he has received

$$I_1 \geqslant 2p\sqrt{P_0/p}/[(b-a)\tanh(\sqrt{P_0/p}/2)] = H_4^1. \tag{9}$$

It is easy to see that, from the inequality $0 < \tanh t < t$ for $t > 0$, we have $H_4^1 \geqslant H_2^1$.

Bertram [1] has received, for every natural m,

$$I_m \geqslant p(2m-1)[(m-1)!\ 2^m]^2/(b-a)^{2m-1} = H_2^m. \tag{10}$$

This estimation for $m = 1$ is evidently identical to that one with H_2^1 in (7).

We present here an estimation with the constant H_1^m for an arbitrary natural m, analogous to H_1^1.

2. For this purpose we have to prove some lemmas.

LEMMA 1. *Let $f(x)$ be a function defined in the open interval* $(0, 1)$ *by the formula*

$$f(x) = \Big[x^n \int_0^x g(t)\,dt\Big]^{-1} + \Big[(1-x)^n \int_x^1 g(t)\,dt\Big]^{-1},$$

where $g(t)$ is a continuous and positive function defined in the closed interval $\langle 0, 1\rangle$, and n is a non-negative integer number.

Then

$$\min_{(0,1)} f(x) \geqslant 2^{n+2}\Big/\int_0^1 g(t)\,dt$$

(the equality holds for $n = 0$).

Proof. The function $f(x)$ is positive and a continuous one, and

$$\lim_{x\to 0+} f(x) = \lim_{x\to 1-} f(x) = +\infty.$$

Then, it follows that the minimum of $f(x)$ in $(0,1)$ exists.

Let us consider the function

$$F(x,y) = \Big[y^n \int_0^x g(t)\,dt\Big]^{-1} + \Big[(1-y)^n \int_x^1 g(t)\,dt\Big]^{-1},$$

where n is a natural number (3). The function $F(x,y)$ is defined in the open square S: $0<x<1$, $0<y<1$. It follows from the evident relation $f(x) = F(x,x)$ that

$$\min_{(0,1)} f(x) = \min_{(0,1)} F(x,x) \geqslant \min_S F(x,y).$$

Moreover, $F(x,y)$ is positive and its limit, when (x,y) tends to the boundary line of S, is equal to $+\infty$. Thus, there exists such a point (x_1,y_1) in the interior of S that $F(x_1,y_1)$ is an absolute minimum of $F(x,y)$. It is necessary that $\partial F(x,y)/\partial x$ and $\partial F(x,y)/\partial y$ vanish in this point. These constraints can be written in the form

$$y^n \Big(\int_0^x g(t)\,dt\Big)^2 = (1-y)^n \Big(\int_x^1 g(t)\,dt\Big)^2,$$

$$y^{n+1} \int_0^x g(t)\,dt = (1-y)^{n+1} \int_x^1 g(t)\,dt.$$

Squaring the second equation and substituting into the first one we have

$$[(1-y)/y]^n y^{2n+2} = (1-y)^{2n+2} \quad \text{or} \quad [(1-y)/y]^{n+2} = 1.$$

For $y \in (0,1)$ we have $y>0$ and $1-y>0$, then $1-y=y$ and we obtain $y_1 = 0.5$. Substituting this value into one of the original equations, we obtain

$$\int_0^x g(t)\,dt = \int_x^1 g(t)\,dt$$

and, from this,

$$2\int_0^x g(t)\,dt = \int_0^1 g(t)\,dt.$$

Then

$$\int_0^x g(t)\,dt = \int_x^1 g(t)\,dt = 0.5 \int_0^1 g(t)\,dt.$$

(3) See Tatarkiewicz [2], p. 397, who has proved this lemma for $n = 0$.

The function $g(t)$ is positive and $\int_0^x g(t)\,dt$ increases for increasing x. Thus, there exists an only value of x_1 such that

$$\int_0^{x_1} g(t)\,dt = 0.5 \int_0^1 g(t)\,dt$$

and the function $F(x, y)$ assumes its absolutely minimal value for this x_1 and for $y_1 = 0.5$.

Therefore, we have

$$\min_S F(x, y) = 2^{n+2} \Big/ \int_0^1 g(t)\,dt$$

and our lemma is proved.

LEMMA 2. *If the function $f(x)$, defined in the open interval $(0, 1)$, has a continuous first derivative in this interval and if it is non-negative and strictly convex, then the function $F(x) = f(x)+f(1-x)$ has the absolute minimum in the interval $(0, 1)$ for $x = 0.5$.*

Proof. The lemma is evidently true, because $F(x)$ is strictly convex and symmetric relative to the point $x = 0.5$.

LEMMA 3. *For every natural m and real t, the vector x with components*

$$x_j = (-1)^{m+j} m!/[(j-1)!\, t^{m-j+1}] \quad (j = 1, 2, \ldots m)$$

is the solution of the system of equations $A_m x = e_m$, where A_m is the m-th degree square matrix with elements

$$a_{ij} = \frac{d^{m-i-j+1}}{dt^{m-i-j+1}} \left(\frac{t^m}{m!}\right)$$

and e_m is the m-th column of the identity matrix.

Proof. The k-th row of the matrix A_m has the components

$$a_k' = (t^k/k!,\ t^{k-1}/(k-1)!, \ldots, t,\ 1, 0, \ldots, 0),$$

where the element equal to t lies in the k-th column. In the m-th row, it is the last element.

For $k < m$ we have

$$a_k' x = \sum_{j=1}^{k+1} (-1)^{k+j} m!\, t^{k-j+1}/[(k-j+1)!\,(j-1)!\, t^{m-j+1}]$$
$$= (-1)^{m+1} \frac{m!\, t^{k-m}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} = 0.$$

Similarly, for $k = m$,

$$a'_m x = (-1)^{m+1} \frac{m!\, t^{m-m}}{m!} \sum_{j=0}^{m-1} (-1)^j \binom{m}{j}$$

$$= (-1)^{m+1} \left[\sum_{j=0}^{m} (-1)^j \binom{m}{j} - (-1)^m \binom{m}{m} \right] = 1.$$

Thus, $A_m x = e_m$.

Before the formulation of the next lemma, we introduce a new symbol

$$^{(k)}\int_a^x f(x)\, dx^k$$

defined for integer non-negative k as follows:

$$^{(0)}\int_a^x f(x)\, dx^0 = f(x), \qquad ^{(k)}\int_a^x f(x)\, dx^k = \int_a^x \left[^{(k-1)}\int_a^u f(u)\, du^{k-1} \right] du.$$

Similarly, we introduce the symbol $^{(k)}\int_x^a f(x)\, dx^k$.

Evidently,

$$^{(k)}\int_x^a f(x)\, dx^k = (-1)^k\, ^{(k)}\int_a^x f(x)\, dx^k,$$

$$^{(r)}\int_a^x \left[^{(k)}\int_a^x f(x)\, dx^k \right] dx^r = {}^{(r+k)}\int_a^x f(x)\, dx^{r+k},$$

$$^{(r)}\int_x^a \left[^{(k)}\int_x^a f(x)\, dx^k \right] dx^r = {}^{(r+k)}\int_x^a f(x)\, dx^{r+k}.$$

LEMMA 4. *If* 1° $f(x)$ *is a positive, continuous function, defined in the interval* $\langle 0, 1\rangle$; 2° $a \in \operatorname{Int}\langle 0, 1\rangle$; 3° n *is a non-negative integer*; *then*

$$\int_0^a f(t)(a^{-1}t-1)^n\, dt = (-1)^n a^{-n} n! \int_0^a \left[^{(n)}\int_0^t f(t)\, dt^n \right] dt$$

and

$$\int_a^1 f(t)(a^{-1}t-1)^n\, dt = a^{-n} n! \int_a^1 \left[^{(n)}\int_t^1 f(t)\, dt^n \right] dt.$$

Proof. Both formulae are identities when $n = 0$. We may obtain them for positive n integrating n times by parts their left-hand sides.

LEMMA 5. *Supposing as in lemma* 4, *we have*

$$\int_0^a \left[^{(n)}\int_0^t f(t)(a^{-1}t-1)^n\, dt^n \right] dt = (-1)^n a^{-n} n! \binom{2n}{n} \int_0^a \left[^{(2n)}\int_0^t f(t)\, dt^{2n} \right] dt,$$

$$\int_a^1 \left[{}^{(n)}\int_t^1 f(t)(a^{-1}t-1)^n dt^n\right] dt = a^{-n} n! \binom{2n}{n} \int_a^1 \left[{}^{(2n)}\int_t^1 f(t)\, dt^{2n}\right] dt.$$

Proof ([4]). Multiplying the first formula by $(-1)^n n!\, a^n$, we obtain

$$n! \int_0^a \left[{}^{(n)}\int_0^t f(t)(a-t)^n dt^n\right] dt = (2n)! \int_0^a \left[{}^{(2n)}\int_0^t f(t)\, dt^{2n}\right] dt.$$

Similarly, the first formula proved in Lemma 4 can be transformed into

$$\int_0^a \left[{}^{(r)}\int_0^t g(t)\, dt^r\right] dt = \int_0^a g(t)(a-t)^r dt/r!.$$

Assuming successively: 1° $m = n$, $g(t) = f(t)(a-t)^n$ and 2° $m = 2n$, $g(t) = f(t)$, we may transform the left-hand side of the equality to be proved into the same form as the right-hand side, namely into

$$\int_0^a f(t)(a-t)^{2n} dt.$$

Analogically, using the second formula of Lemma 4, we may prove the second equality of this lemma.

3. Introducing the notation

$$\Psi_m[u, a, \beta] = \int_a^\beta q_m(t)[u^{(m)}(t)]^2 dt,$$

we evidently have

$$\Phi_m[u, a, \beta] \geqslant \Psi_m[u, a, \beta]$$

and, in virtue of (6),

$$I_m \geqslant \inf_{h\in\langle 0,1\rangle} \left(\min_{L_h^m} \Psi_m[u, 0, h] + \min_{P_h^m} \Psi_m[u, h, 1]\right). \tag{11}$$

Let us denote by u_L (analogically, u_P) the function realizing the minimum of $\Psi_m[u, 0, h]$ on L_h^m (of $\Psi_m[u, h, 1]$ on P_h^m), They exist in virtue of Corollary 1 from [3]. Both functions u_L and u_P must satisfy the Euler-Poisson equation

$$\left(q_m(t)\, u^{(m)}\right)^{(m)} = 0. \tag{12}$$

Moreover, the function u_L must satisfy the boundary constraints of the class L_h^m (i.e. $u_L^{(i)}(0) = 0$ for $i = 0, 1, \ldots, m-1$, $u_L(h) = 1$), and

---

([4]) The general idea of this proof was proposed to me by A. Bielecki.

the function u_P must satisfy the boundary constraints of the class P_h^m (i.e. $u_P^{(i)}(1) = 0$ for $i = 0, 1, \ldots, m-1$, $u_P(h) = 1$). Both these functions must satisfy $m-1$ "natural boundary conditions" for $t = h$:

$$(13) \qquad [q_m(t)u^{(m)}(t)]^{(k)}|_{t=h} = 0 \quad \text{for } k = 0, 1, \ldots, m-2.$$

Then, to calculate u_L and u_P, we solve equation (12) with constraints (13) and boundary constraints of the corresponding class of functions L_h^m or P_h^m. There are exactly $2m$ boundary conditions and equation (12) is of order $2m$.

Integrating this equation, we see that there exist m constants C_1, $C_2, \ldots, C_m$, satisfying with $u(t)$ the equations

$$(14) \quad [q_m(t)u^{(m)}]^{(k)} = \sum_{j-1}^{m-k} C_j \frac{t^{m-j-k}}{(m-j-k)!} \quad \text{for } k = 0, 1, \ldots, m-1.$$

But, in virtue of (13), these constants both for u_L and for u_P must satisfy the equations

$$(15) \quad \begin{aligned} \sum_{j=1}^{m-k} C_j h^{m-j-k}/(m-j-k)! &= 0 \quad \text{for } k = 1, 2, \ldots, m-2, \\ \sum_{j=1}^{m-1} C_j h^{m-j}/(m-j)! &= -C_m. \end{aligned}$$

The matrix of this system is A_{m-1} as in Lemma 3, with $t = h$, and the vector of the right-hand sides is equal to $-C_m e_{m-1}$. Thus, $C_j = -C_m x_j$ for $j = 1, 2, \ldots, m-1$, i.e.

$$(16) \quad C_j = (-1)^{m+j}(m-1)!\, C_m/[(j-1)!\, h^{m-j}] \quad \text{for } j = 1, 2, \ldots, m,$$

because for $j = m$ we have an identity.

Thus, for any function u_L (respectively, u_P), there exists such a constant C that

$$q_m(t)u^{(m)} = C \sum_{j=1}^{m} (-1)^{m+j} \frac{(m-1)!\,t^{m-j}}{(j-1)!\,(m-j)!\,h^{m-j}}.$$

Otherwise,

$$\begin{aligned} u^{(m)} &= C[q_m(t)]^{-1} \sum_{j=1}^{m} (-1)^{m+j} \binom{m-1}{j-1} (t/h)^{m-j} \\ &= (-1)^{m+1} C[q_m(t)]^{-1} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (t/h)^{m-j-1} \\ &= (-1)^{m+1} C[q_m(t)]^{-1} (t/h-1)^{m-1}. \end{aligned}$$

Since $u_L^{(i)}(0) = 0$ for $i = 0, 1, \ldots, m-1$, we obtain from this

$$u_L(t) = (-1)^{m+1} C \, {}^{(m)}\!\int_0^t (t/h-1)^{m-1}/q_m(t)\, dt^m,$$

where C may be computed from the condition $u_L(h) = 1$; namely,

$$C = (-1)^{m+1} \Big/ \int_0^h \Big[{}^{(m-1)}\!\int_0^t (t/h-1)^{m-1}/q_m(t)\, dt^{m-1}\Big] dt$$

and, finally,

$$u_L(t) = \frac{{}^{(m)}\!\int_0^t (t/h-1)^{m-1}/q_m(t)\, dt^m}{\int_0^h \Big[{}^{(m-1)}\!\int_0^t (t/h-1)^{m-1}/q_m(t)\, dt^{m-1}\Big] dt}. \tag{17}$$

Both the denominator and the numerator of this expression have the same sign independently of the parity of m, and the absolute value of the numerator increases with t, because the integrand does not change the sign inside the interval $\langle 0, h\rangle$. Thus, the function $u_L(t)$ is non-negative in this interval, $u_L(0) = 0$, $u_L(h) = 1$, and it increases with t. Consequently, $0 \leqslant u_L(t) \leqslant 1$ in the interval $\langle 0, h\rangle$ and $u_L \in L_h^m$.

Analogically, using relations $u_P^{(i)}(1) = 0$ for $i = 0, 1, \ldots, m-1$, $u_P(h) = 1$, we obtain

$$u_P(t) = \frac{{}^{(m)}\!\int_t^1 (t/h-1)^{m-1}/q_m(t)\, dt^m}{\int_h^1 \Big[{}^{(m-1)}\!\int_t^1 (t/h-1)^{m-1}/q_m(t)\, dt^{m-1}\Big] dt} \tag{18}$$

and we may prove that $u_P \in P_h^m$.

In virtue of the mentioned Corollary from paper [3], both functionals $\Psi[u, 0, h]$ (in the class L_h^m) and $\Psi[u, h, 1]$ (in the class P_h^m) must attain their minimal values. Since u_L and u_P are unique, satisfying, respectively, the necessary conditions (12) and (13); thus

$$\min_{L_h^m} \Psi[u, 0, h] = \Psi[u_L, 0, h], \qquad \min_{P_h^m} \Psi[u, h, 1] = \Psi[u_P, h, 1]. \tag{19}$$

However, substituting the calculated value of C (for u_L and u_P, respectively) to the obtained above-mentioned expression for $u^{(m)}$, and using the definition of $\Psi[u, \alpha, \beta]$, we have

$$\Psi[u_L, 0, h] = \frac{\int_0^h (t/h-1)^{2m-2}/q_m(t)\, dt}{\Big\{\int_0^h \Big[{}^{(m-1)}\!\int_0^t (t/h-1)^{m-1}/q_m(t)\, dt^{m-1}\Big] dt\Big\}^2}$$

and

$$\Psi[u_p, h, 1] = \frac{\int\limits_h^1 (t/h-1)^{2m-2}/q_m(t)\,dt}{\{\int\limits_h^1 [{}^{(m-1)}\int\limits_t^1 (t/h-1)^{m-1}/q_m(t)\,dt^{m-1}]\,dt\}^2}.$$

Now, in virtue of Lemmas 4 and 5, we may obtain, finally,

$$\Psi[u_L, 0, h] = \left\{\binom{2m-2}{m-1}\int\limits_0^h \left[{}^{(2m-2)}\int\limits_0^t dt^{2m-2}/q_m(t)\right] dt\right\}^{-1},$$

$$\Psi[u_p, h, 1] = \left\{\binom{2m-2}{m-1}\int\limits_h^1 \left[{}^{(2m-2)}\int\limits_t^1 dt^{2m-2}/q_m(t)\right] dt\right\}^{-1}.$$

From this and from (11) and (19) we have

$$I_m \geqslant \inf_{(0,1)} \varphi(h)\Big/\binom{2m-2}{m-1}, \tag{20}$$

where

$$\varphi(h) = \left\{\int\limits_0^h \left[{}^{(2m-2)}\int\limits_0^t dt^{2m-2}/q_m(t)\right] dt\right\}^{-1} + \left\{\int\limits_h^1 \left[{}^{(2m-2)}\int\limits_t^1 dt^{2m-2}/q_m(t)\right] dt\right\}^{-1}.$$

4. The function $\varphi(h)$ is continuous and positive in the open interval $(0,1)$. Moreover,

$$\lim_{h\to 0^+}\varphi(h) = \lim_{h\to 1^-}\varphi(h) = +\infty.$$

Then, there exists a point $h_{\min}$ such that $\varphi(h_{\min}) = \min\limits_{(0,1)}\varphi(h)$. If we can compute directly the integrals in the formula for $\varphi(h)$, this minimum may be calculated by the usual method. In the opposite case, one can estimate it from below as follows.

It is easy to compute for a natural m

$$\int\limits_0^h \left[{}^{(2m-4)}\int\limits_0^t t\,dt^{2m-4}\right] dt = h^{2m-2}/(2m-2)!$$

and

$$\int\limits_h^1 \left[{}^{(2m-4)}\int\limits_t^1 (1-t)\,dt^{2m-4}\right] dt = (1-h)^{2m-2}/(2m-2)!.$$

Moreover, we see that

$${}^{(2)}\int\limits_0^t dt^2/q_m(t) = \int\limits_0^t \left[\int\limits_0^r ds/q_m(s)\right] dr \leqslant t \max_{r\in\langle 0,h\rangle}\int\limits_0^r ds/q_m(s) = t\int\limits_0^h ds/q_m(s).$$

Similarly,

$$ {}^{(2)}\!\int_t^1 dt^2/q_m(t) \leqslant (1-t)\int_h^1 ds/q_m(s). $$

Thus, for $m \geqslant 2$,

$$ \int_0^h \Big[{}^{(2m-2)}\!\int_0^t dt^{2m-2}/q_m(t)\Big]\,dt = \int_0^h \Big\{{}^{(2m-4)}\!\int_0^t \Big[{}^{(2)}\!\int_0^t dt^2/q_m(t)\Big]\,dt^{2m-4}\Big\}\,dt $$

$$ \leqslant \Big[\int_0^h ds/q_m(s)\Big]\int_0^h \Big[{}^{(2m-4)}\!\int_0^t t\,dt^{2m-4}\Big]\,dt $$

that is

$$ \int_0^h \Big[{}^{(2m-2)}\!\int_0^t dt^{2m-2}q_m(t)\Big]\,dt \leqslant \frac{h^{2m-2}}{(2m-2)!}\int_0^h dt/q_m(t). $$

This inequality is evidently true also for $m=1$. In the same way we obtain, for every natural m, the inequality

$$ \int_h^1 \Big[{}^{(2m-2)}\!\int_t^1 dt^{2m-2}/q_m(t)\Big]\,dt \leqslant \frac{(1-h)^{2m-2}}{(2m-2)!}\int_h^1 dt/q_m(t). $$

Hence

$$ \varphi(h) \geqslant (2m-2)!\,f(h), $$

where

$$ f(h) = \Big[h^{2m-2}\int_0^h dt/q_m(t)\Big]^{-1} + \Big[(1-h)^{2m-2}\int_h^1 dt/q_m(t)\Big]^{-1}. $$

The calculation of these integrals may be more easy than in the formula for $\varphi(h)$ and it may be possible to compute the minimum of the function $f(h)$ directly. If it is also impracticable, then we obtain from Lemma 1 the estimation

$$ \min_{(0,1)} f(h) \geqslant 2^{2m}\Big/\int_0^1 dt/q_m(t), $$

where the integral may be computed approximately.

5. Finally, we obtain the estimations

$$ (21)\qquad I_m \geqslant \min_{(0,1)}\Big\{\Big[\binom{2m-2}{m-1}\int_0^h\Big({}^{(2m-2)}\!\int_0^t dt^{2m-2}/q_m(t)\Big)\,dt\Big]^{-1} + $$

$$ +\Big[\binom{2m-1}{m-1}\int_h^1\Big({}^{(2m-2)}\!\int_t^1 dt^{2m-2}/q_m(t)\Big)\,dt\Big]^{-1}\Big\} = H_1^m, $$

where $q_m(t) = p_m[(b-a)t+a]/(b-a)^{2m-1}$ and

$$(22) \qquad H_1^m \geqslant [2^m(m-1)!]^2 / \int_0^1 dt/q_m(t)$$

$$= [2^m(m-1)!]^2 / \left[(b-a)^{2m-2} \int_a^b dx/p_m(x)\right] = G_1^m,$$

where the integral may be calculated approximately, but with a positive remainder.

We may observe that, taking into consideration the relation $q_m(t) \geqslant q$, we can estimate the function $\varphi(h)$ by

$$\varphi(h) \geqslant q\left\{\left[\int_0^h \left({}^{(2m-2)}\int_0^t dt^{2m-2}\right) dt\right]^{-1} + \left[\int_h^1 \left({}^{(2m-2)}\int_t^1 dt^{2m-2}\right) dt\right]^{-1}\right\}$$

$$= q(2m-1)!\,[1/h^{2m-1} + 1/(1-h)^{2m-1}].$$

In virtue of Lemma 2, the function $1/h^{2m-1}+1/(1-h)^{2m-1}$ assumes its minimal value in the interval $(0,1)$ for $h = 0.5$, and we have

$$(23) \qquad H_1^m \geqslant q(2m-1)[2^m(m-1)!]^2$$

$$= p(2m-1)[2^m(m-1)!]^2/(b-a)^{2m-1} = H_2^m,$$

i.e. estimation (10) obtained by Bertram in another way.

In the same manner we may underestimate G_1^m:

$$(24) \qquad G_1^m \geqslant p[2^m(m-1)!]^2/(b-a)^{2m-1} = G_2^m.$$

It is easy to see that $H_2^m \geqslant G_2^m$. The equality holds for $m = 1$ only.

6. The estimation of I_m by G_1^m is better than that one with H_2^m, when $p_m(x)$ deviates from its lower bound p so much that

$$\int_a^b dx/p_m(x) \leqslant (b-a)/[p(2m-1)].$$

In the opposite case, particularly if $p_m(x) \equiv p = \text{const}$, we have $G_1^m \leqslant H_2^m$ (in this special case $H_1^m = H_2^m$).

We consider as an example the problem

$$\frac{d^2}{dt^2}\left[(1+at)^3\frac{d^2y}{dt^2}\right] + k(1+at)y - lt = 0,$$

$$y(0) = y'(0) = y(1) = y'(1) = 0,$$

where a, k and l are constants, and $a > 0$.

This equation concerns an elastic radial deformation of a filled vertical cylindric cistern for liquids; a is the linear increase coefficient of its wall thickness (the greater one is at the bottom). Here $p_2(t) = (1+at)^3 \geqslant 1 = p$.

For calculation of H_1^2 we should solve a logarithmic equation, but $G_1^2 = 32(1+a)^2/(2+a)$, $H_2^2 = 48$.

When $a > (\sqrt{33}-1)/4 \simeq 5/4$, i.e. when the wall is 5/4 times thicker at the bottom than at the top, the constant G_1^2 is better than H_2^2.

References

[1] G. Bertram, *Eine Fehlerabschätzung für gewisse selbstadjungierte gewöhnliche Randwertaufgaben*, Num. Math. 1 (1959), p. 151-185.

[2] K. Tatarkiewicz, *Une méthode d'éstimation de l'erreur dans le procédé de Ritz*, Ann. Pol. Math. 1 (1955), p. 346-359.

[3] Ś. Ząbek, *Sur le minimum absolu de certaines fonctionnelles*, Ann. Univ. Mariae Curie-Skłodowska, sectio A, 17 (1963), p. 75-79.

[4] — *Kilka oszacowań błędu dla rozwiązań przybliżonych równań różniczkowych liniowych rzędu parzystego*, Algorytmy 1 (1963), p. 37-43.

MATHEMATICAL INSTITUTE
M. CURIE-SKŁODOWSKA UNIVERSITY
LUBLIN

Received on 28. 5. 1971

Ś. ZĄBEK (Lublin)

O PEWNYM WARIACYJNYM OSZACOWANIU BŁĘDU

STRESZCZENIE

W pracy uogólnia się wyniki Tatarkiewicza [2] (poprawiając równocześnie wyniki Bertrama [1]), odnoszące się do oszacowań *a posteriori* błędu rozwiązań przybliżonych problemów (1)-(2). Otrzymuje się mianowicie oszacowania od dołu wartości całki z mianownika wzoru (3) stałymi H_1^m, G_1^m, H_2^m i G_2^m (patrz wzory (21)-(24)). W paragrafie 6 podano przykład zastosowania otrzymanych oszacowań.