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NOTE ON THE FIRST ORDER ORTHOGONAL PROJECTIONS

Let

$$W = \{w \in L^2(-1, 1)\colon\ w \geqslant 0,\ \int_{-1}^{1} w(y)\,dy = 1\}$$

and let $\{\varrho_i^w\}_{i=0}^{\infty}$ be the system of the orthonormal polynomials with respect to the weight function $w \in W$. Then for any $f \in C[-1, 1]$ the equality

$$(L_n^w f)(x) = \sum_{i=0}^{n} \varrho_i^w(x) \int_{-1}^{1} w(y) f(y) \varrho_i^w(y)\,dy$$

defines an orthonormal projection of the space $C[-1, 1]$ onto Π_n. The norm of this projection is ([2])

$$\|L_n^w\| = \max_{-1 \leqslant x \leqslant 1} \int_{-1}^{1} w(y) \Big|\sum_{i=0}^{n} \varrho_i^w(x) \varrho_i^w(y)\Big|\,dy. \tag{1}$$

In [1] the numerical values of the norms (1) for the weight functions $w(x) = (1-x^2)^{\lambda-1/2}$ for $\lambda = -.1,\ .0(.2)1.0,\ 3.0,\ 5.0$ and $1 \leqslant n \leqslant 10$ are given. These results are a motivation for the following

PROBLEM. For any natural number n, find the weight function $w^* \in W$ with the property

$$\|L_n^{w^*}\| = \inf_{w \in W} \|L_n^w\|. \quad \blacksquare \tag{2}$$

We consider the simplest case $n = 1$ and we prove that the best weight function does not exist.

Let $\hat{W} \subset W$ denote the subset of all even functions. For any $w \in W$ the orthonormal polynomials have the form

$$\varrho_0^w(x) = 1, \quad \varrho_1^w(x) = (x - I_1)/\sqrt{I_2 - I_1^2}, \quad \ldots$$

where

$$I_k = \int_{-1}^{1} w(y)\, y^k\, dy.$$

We note that $|I_k| \leqslant 1$ for any natural k.

At first we consider the case $w \in \hat{W}$, so that $I_1 = 0$. We have

LEMMA 1. *If* $w \in \hat{W}$ *then*

$$\|L_1^w\| = \max_{-1 \leqslant x \leqslant 1} \int_{-1}^{1} w(y) \max\{1, |xy|/I_2\}\, dy. \quad \blacksquare$$

LEMMA 2. *If* $w \in \hat{W} \cap C[-1, 1]$ *then*

$$\|L_1^w\| = 1 - 2\int_{I_2}^{1} w(y)\, dy + (2/I_2) \int_{I_2}^{1} w(y)\, y\, dy.$$

Proof. Since w is even we may assume that $0 \leqslant x \leqslant 1$. From Lemma 1 we have the equality

$$\|L_1^w\| = \max_{0 \leqslant x \leqslant 1} \Lambda_1^w(x) = 2 \max_{0 \leqslant x \leqslant 1} \int_{0}^{1} w(y) \max\{1, xy/I_2\}\, dy,$$

hence

$$\Lambda_1^w(x) = \begin{cases} 1, & 0 \leqslant x \leqslant I_2, \\ 2\displaystyle\int_{0}^{I_2/x} w(y)\, dy + (2x/I_2) \int_{I_2/x}^{1} y w(y)\, dy, & I_2 < x \leqslant 1. \end{cases}$$

By continuity of w, we have $[\Lambda_1^w(x)]' \geqslant 0$ and $\Lambda_1^w(x) \leqslant \Lambda_1^w(1)$. $\blacksquare$

Let us consider the sequence of the weight functions $w_n \in \hat{W} \cap C[-1, 1]$ of the form

$$w_n(x) = \begin{cases} 0, & |x| \leqslant (n-1)/n, \\ n^2|x| - n(n-1), & (n-1)/n \leqslant |x| \leqslant 1. \end{cases}$$

Using these functions we obtain

LEMMA 3.

$$\inf_{w \in W} \|L_1^w\| \leqslant \inf_{w \in \hat{W}} \|L_1^w\| = 1. \quad \blacksquare$$

Now we may prove

THEOREM. *The best weight function* w^* *does not exist.*

Proof. Let us assume that such a function $w^* \in W$ exists. Of course, it is impossible that $w^*(y) \equiv 0$ a.e. Also it is impossible that $I_2 = 1$, $I_1 = \pm 1$ and $I_2 = I_1^2$. Thus

$$I_1^2 < I_2. \tag{3}$$

The norm of the projection $L_1^{w^*}$ is equal to

$$\|L_1^{w^*}\| = \max_{-1\leqslant x\leqslant 1} \int_{-1}^{1} w^*(y)\left|1+\frac{(x-I_1)(y-I_1)}{I_2-I_1^2}\right|dy.$$

Then using orthogonality for any $-1 \leqslant x \leqslant 1$ we have

$$1 = \int_{-1}^{1} w^*(y)\left[1+\frac{(x-I_1)(y-I_1)}{I_2-I_1^2}\right]dy \leqslant \int_{-1}^{1} w^*(y)\left|1+\frac{(x-I_1)(y-I_1)}{I_2-I_1^2}\right|dy = 1,$$

hence

$$\int_{-1}^{1} w^*(y)\min\left\{0,\ 1+\frac{(x-I_1)(y-I_1)}{I_2-I_1^2}\right\}dy = 0. \tag{4}$$

The function

$$\Phi_x(y) = \min\left\{0,\ 1+\frac{(x-I_1)(y-I_1)}{I_2-I_1^2}\right\}$$

is nonzero for $x,\ y\in[-1,\ 1]$ if

$$-1 \leqslant y < \frac{I_1 x - I_2}{x-I_1} \quad \text{and} \quad \frac{I_2+I_1}{1+I_1} < x \leqslant 1$$

or

$$\frac{I_2-I_1 x}{I_1-x} < y \leqslant 1 \quad \text{and} \quad -1 \leqslant x < -\frac{I_2-I_1}{1-I_2}.$$

The equality (4) is possible only if $w^*(y) \equiv 0$ a.e. for

$$y\in\left[-1,\ -\frac{I_2-I_1}{1-I_2}\right)\cup\left(\frac{I_2+I_1}{1+I_1},\ 1\right].$$

We may assume $I_1 \geqslant 0$, since if $I_1 < 0$ then we may consider the function $\bar{w}(x) = w^*(-x)$ which has the properties

$$\int_{-1}^{1} \bar{w}(y)\,dy > 0 \quad \text{and} \quad \|L_1^{\bar{w}}\| = \|L_1^{w^*}\|.$$

It is easy to verify by the Cauchy–Bunyakovsky inequality that

$$\alpha = \frac{I_2+I_1}{1+I_1} > \frac{I_2-I_1}{1-I_1} = \beta,$$

hence

$$I_2 = \int_{-\beta}^{\alpha} w^*(y)\,y^2\,dy \leqslant \max_{-\beta\leqslant y\leqslant\alpha} y^2 = \left(\frac{I_2+I_1}{1+I_1}\right)^2.$$

From the last inequality we obtain $I_2 \leqslant I_1^2$. In view of (3) this is impossible. ∎

References

[1] W. A. Light, *A comparison between Chebyshev and ultraspherical expansion*, J. Inst. Maths. Applics. 21 (1978), p. 455-460.

[2] S. Paszkowski, *Zastosowania numeryczne wielomianów i szeregów Czebyszewa*, PWN, Warszawa 1975.

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