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SOME APPLICATIONS OF GRAPH THEORY IN THE THEORY OF HYPERGRAPHS

1. Introduction and basic concepts. A number of combinatorial problems can be reduced to the determination of fundamental sets of a suitable hypergraph (set system). The purpose of this paper is to derive some bounds for the numbers describing the fundamental sets of a hypergraph; this will be done by applying the theory of graphs to the theory of hypergraphs. The numbers considered here are the domination number, the independence number, and the transversal number of a hypergraph.

A *hypergraph* $H = (X; E_1, E_2, \ldots, E_m)$ is given by a finite set $X = \{x_1, \ldots, x_n\}$, whose elements are the vertices of H, and by subsets $E_1, \ldots, E_m$ of X called the *edges* of H. If $|E_i| \leqslant 2$ for all i, H is an *undirected graph*. We denote the family $\{E_1, \ldots, E_m\}$, briefly, by $E(H)$, and thus $H = (X, E(H))$.

A *transversal set* of H is a set $T \subset X$ such that $T \cap E_i \neq \emptyset$ for any $E_i \in E(H)$. A set T of smallest cardinality is called a *minimum transversal set* of H, and the number of vertices of such a transversal set, denoted by $\tau(H)$, is the transversal number of H. A set $S \subset X$ is called *independent* in H if $S \cap E_i \not\supseteq E_i$ for any edge E_i of H, $|E_i| \geqslant 2$. The *independence number* $\alpha(H)$ of H equals the greatest cardinality of an independent set S in H.

We assume that the reader is familiar with the basic concepts in the theory of undirected and directed graphs (cf. [1] and [4]).

2. The dominating set of a hypergraph. The definition of a dominating set in a graph implies immediately the following definition for the dominating set of a hypergraph:

Let H be a hypergraph. A set $D \subset X$ is a *dominating set* of H if, for any vertex y of H, $y \notin D$, $\{y\} \cup D \supset E_j$ for at least one edge E_j which contains y in H. The number of vertices in a set D of smallest cardinality is called the *domination number* $\delta(H)$ of H.

Let us associate with H a graph $G_D(H) = (V_D, E_D)$ defined as follows: The vertices of $G_D(H)$ are all the pairs xE_i, where $x \in X$, $E_i \in E(H)$, and

$x \in E_i$. Two distinct vertices xE_i and yE_j are *joined by an undirected edge* in $G_D(H)$ if and only if at least one of the three following conditions holds:

(1) $x = y$,

(2) $E_i = E_j$,

(3) there is a vertex $zE_h \in V_D$ such that $z = x$ and $E_h = E_j$, or $z = y$ and $E_h = E_i$.

The following theorem illuminates the role of the graph $G_D(H)$:

THEOREM 1. *Let H be a given hypergraph and $G_D(H)$ a graph associated with H. For any independent set I of $G_D(H)$, there is a dominating set $D = \{x \mid xE_i \in \bar{I}\}$, $\bar{I}$ is the complement of I in V_D, and, for any dominating set D of H, there is an independent set I of $G_D(H)$. Moreover, any minimum dominating set D_0 of H is determined by a maximum independent set I_0 of $G_D(H)$, and conversely.*

Proof. Let $yE_j \in I$; we show that $E_j - \{y\} \subset \{x \mid xE_i \in \bar{I}\}$. Let $z \in E_j$ in H, $z \neq y$. According to (2), zE_j is joined in $G_D(H)$ by an edge to yE_j and, according to (3), zE_h is joined by an edge to yE_j for any E_h of H, $z \in E_h$. Thus, for no $z \neq y$, $z \in E_j$, z is an element of the set $\{w \mid wE_i \in I\}$, and hence $\{x \mid xE_i \in \bar{I}\}$ is a dominating set of H. Similarly we can see that, for any dominating set D of H, there is at least one set $\{yE_i \mid y \notin D$ and $\{y\} \cup D \supset E_i,\ y \in E_i\}$ which is an independent set of $G_D(H)$.

As each independent set of $G_D(H)$ is generated by a dominating set of H and *vice versa*, and as the relation $yE_j \in I$ implies the relation $E_j - \{y\} \subset \{x \mid xE_i \in \bar{I}\}$, the last part of the theorem is obvious.

As a conclusion, all the minimal dominating sets of a given hypergraph H can be enumerated by applying to $G_D(H)$ a suitable method developed for enumerating all the maximal independent sets of a graph.

For any $x \in X$, there is at most one element xE_i in the set I of the graph $G_D(H)$. Hence, $\delta(H) = |X| - \alpha(G_D(H))$, where $\alpha(G_D(H))$ is the independence number of the graph $G_D(H)$. Thus any bound for $\alpha(G_D(H))$ can be translated into a bound for the domination number of H. In the latter part of this section we shall consider a simple lower bound for the independence number of a graph giving an upper bound for $\delta(H)$; for other bounds for $\alpha(G)$ of a graph G, the reader is referred to the monography of Berge [1], p. 260-274.

Let $G = (V_G, E_G)$ be a given graph and I a maximal independent set of G. For any I of G, one can construct a *spanning subforest* of G as follows:

According to the maximality of I, any vertex $z \in V_G - I$ is joined in G by an edge to at least one vertex $x \in I$. If a vertex z of $V_G - I$ is joined to more than one vertex of I, remove from G those edges incident to z until z is joined to exactly one vertex x of I.

Clearly, the subgraph which arises after carrying out the above operation contains no circuits, and hence must be a spanning forest $F(I)$ of G. According to the operation above, the number of edges in $F(I)$ equals the number of vertices in the set $V_G - I$. In particular, if I_0 is a maximum independent set of G, the forest $F(I_0)$ of G has a minimum number of edges among the spanning forests $F(I)$ of G.

As one can easily prove (see, e.g., [2]), if G_1 is a given graph and $x_1 \in V_{G_1}$, then $I_2 \cup \{x_1\}$ is a maximal independent set of G_1 if and only if I_2 is a maximal independent set of the graph G_2 obtained from G_1 by removing the vertex set $\{x_1\} \cup \Gamma_{G_1} x_1$, i.e. the vertex x_1 and the vertices incident to it in G_1. This fact and the minimum number of edges in the spanning forest $F(I_0)$ implies immediately the following natural way of determining an approximation for I_0 and $\alpha(G)$:

ALGORITHM 1. Let $G = G_1$ be a given graph, and put $i := 1$. Let x_i be a vertex of the smallest degree in G_i; remove from G_i the vertices of the set $\{x_i\} \cup \Gamma_{G_i} x_i$, and denote the graph thus obtained by G_{i+1}.

If $G_{i+1} = \emptyset$, then stop; and if not, then put $i := i+1$ and apply the rule above to this new graph.

If G_{r+1} is the first empty graph of the process, $\{x_1, \ldots, x_r\}$ is an approximation for I_0 and $\alpha(G) \geqslant r$.

As an example, consider the hypergraph $H = (X, E(H))$, where $X = \{a, b, c, d\}$ and $E(H) = \{E_1, E_2, E_3, E_4, E_5\}$ with the expressions $E_1 = \{b, d\}$, $E_2 = \{b, c\}$, $E_3 = \{a, b, c\}$, $E_4 = \{a, d\}$, and $E_5 = \{c, d\}$. Fig. 1 illustrates the graph $G_D(H) = G_D(H)_1$ and the determination pro-

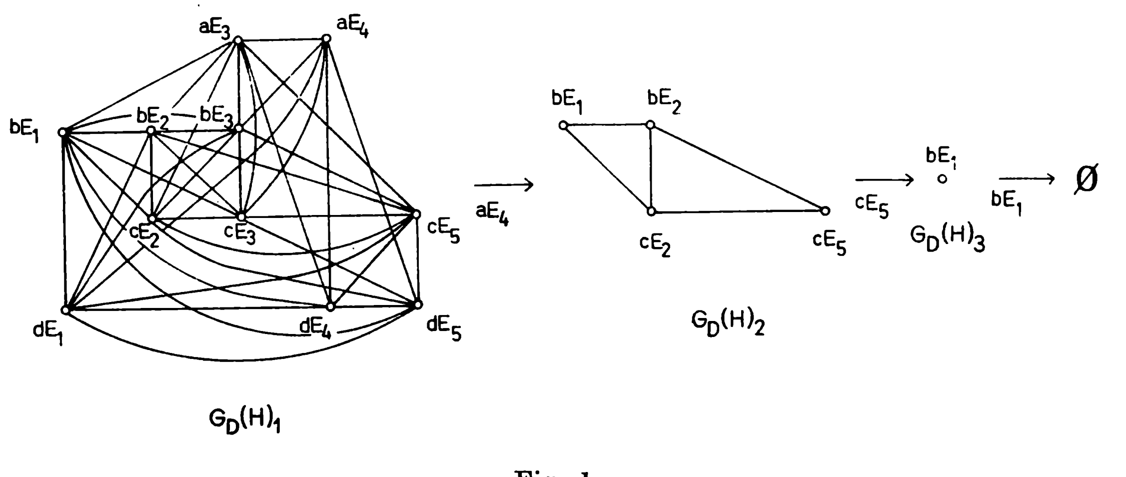

Fig. 1

cess of an approximation for I_0. In $G_D(H)$, aE_4 and dE_4 are the vertices of smallest degree. By removing aE_4 and the vertices adjacent to it from $G_D(H)_1$, we obtain the graph $G_D(H)_2$; Fig. 1 shows the further details of the determination process. The approximation for I_0 of $G_D(H)$, given by the process, is $\{aE_4, cE_5, bE_1\}$, and hence an approximation for the minimum dominating set D_0 of H is $\{b\}$, and $\delta(H) \leqslant 1$.

As a remark, consider briefly the maximum matching of a hypergraph H. A *matching* M of H is a family of edges such that $E_i \cap E_j = \emptyset$ for any two edges $E_i, E_j \in M$, $i \neq j$. This implies immediately that M is an independent set of the graph $G_E = (V_E, E_E)$, where $V_E = E(H)$ and two vertices E_i, E_j are adjacent if $E_i \cap E_j \neq \emptyset$. Thus Algorithm 1 offers a way of finding an approximation for the maximum matching M_0 of H.

3. Transversal and independent sets of H**.** A transversal set of H is a natural generalization of the concept of a covering set of a graph G. As one can expect, the definitions of a transversal and an independent set in H imply that a set T is a transversal set of H if and only if its complement $\overline{T}$ in X is an independent set of H. Hence, a way of solving the determination problem for one of these sets is also a way of solving this problem for the other.

Let us associate with H a *directed graph* $\vec{G}_T(H) = (V_T, E_T)$ defined as follows: The vertices of $\vec{G}_T(H)$ are all the pairs xE_i, where $x \in X$, $E_i \in E(H)$, and $x \in E_i$. Two distinct vertices xE_i and yE_j are joined in $\vec{G}_T(H)$ by an undirected edge if and only if $x = y$ or $E_i = E_j$, and there is a directed edge in $\vec{G}_T(H)$ from xE_i to yE_j if $x \in E_j$ in H (i.e. $xE_j \in V_T$).

THEOREM 2. *For any dominating set C of $\vec{G}_T(H)$, there is a transversal set $\{x \mid xE_i \in C\}$ in H, and, for any transversal set T of H, there is a dominating set*

$$\{yE_j \mid y \in T \text{ and } E_j \text{ is one of the edges containing } y\} \quad \text{in } \vec{G}_T(H).$$

Furthermore, any minimum transversal set T_0 of H is generated by a minimum dominating set C_0 of $\vec{G}_T(H)$, and conversely.

Proof. Let T be a transversal set of H. If $x \in T$, then xE_i dominates any vertex yE_j of $\vec{G}_T(H)$ when $x, y \in E_j$, according to the definition of $\vec{G}_T(H)$. As $T \cap E_i \neq \emptyset$ for any $E_i \in E(H)$, the set

$$\{xE_{i(x)} \mid x \in T \text{ and } E_{i(x)} \text{ is one of the edges containing } x\}$$

is a dominating set of $\vec{G}_T(H)$. The proofs of the other parts of the theorem are similar, and hence we omit them.

The idea of the graph $\vec{G}_T(H)$ has been used, in a generalized form, in Sysło's paper [5] for determining the family of minimal transversal sets of a hypergraph H (the 1-sets of representatives). By means of the graph $\vec{G}_T(H)$ we can translate the way of determining upper and lower bounds for the domination number of a graph introduced in [3] into a method for determining an approximation for T_0 and $\tau(H)$ of H.

By the *degree of a vertex x in a hypergraph H_p*, denoted by $\deg_{H_p}(x)$, we mean the sum $|E_{i_1}|+|E_{i_2}|+\ldots+|E_{i_s}|-1$, $x \in E_{i_t}$ in H_p for any value of t, $t = 1, \ldots, s$.

The operations indicated by the algorithm in [3] are the most simple to carry out by introducing a bipartite graph $B(H) = (V_B^1 \cup V_B^2, E_B)$, where $V_B^1 = X$, $V_B^2 = E(H)$, and an undirected edge $(x, E_i) \in E_B$ if $x \in E_i$ in H. Now we are ready to write a translation of the algorithm in [3]:

ALGORITHM 2. Let $H = H_1$ be a given hypergraph, and put $p := 1$. Let x_p be a vertex of greatest degree in H_p. Remove from $B(H_p)$ the vertex x_p and all the vertices adjacent to it.

Remove from $B(H_p)$ all the isolated vertices, and denote the bipartite graph thus obtained by $B(H_{p+1})$, and the corresponding hypergraph by H_{p+1}.

Is $B(H_{p+1})$ an empty graph? If yes, then stop; and if not, then put $p := p+1$ and apply the algorithm to this new graph.

If $B(H_{q+1})$ is the first empty graph of the process, then $\{x_1, \ldots, x_q\}$ is a transversal set of H, $X - \{x_1, \ldots, x_q\}$ is an independent set of H, $\tau(H) \leqslant q$, and $\alpha(H) \geqslant |X| - q$.

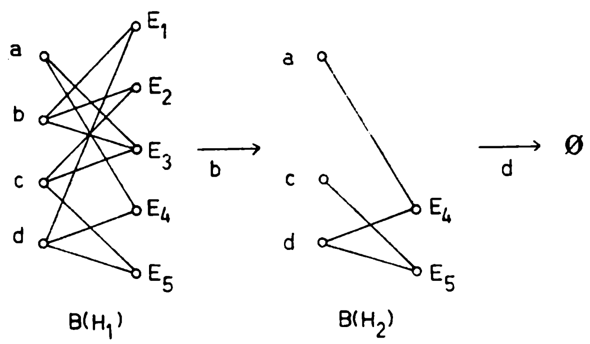

Fig. 2

As an example, consider again the hypergraph $H = H_1$ defined at the end of Section 2. Fig. 2 illustrates the corresponding bipartite graph $B(H_1)$ and the determination process. In $H = H_1$, the vertices have the following degrees: $\deg_{H_1}(a) = 4$, $\deg_{H_1}(b) = 6$, $\deg_{H_1}(c) = 6$, and $\deg_{H_1}(d) = 5$. By removing b and the vertices adjacent to it from $B(H_1)$ we obtain the graph $B(H_2)$ which determines the hypergraph H_2. In H_2, $\deg_{H_2}(a) = 1$, $\deg_{H_2}(c) = 1$, and $\deg_{H_2}(d) = 3$. The removal of d and the vertices adjacent to it implies an empty graph according to Algorithm 2, and hence $\{b, d\}$ is an approximation for T_0, $\{a, c\}$ for I_0, $\tau(H) \leqslant 2$, and $\alpha(H) \geqslant 2$.

A *covering edge family* of a hypergraph H is a family A of edges such that, for any $x \in X$, there is an edge $E_h \in A$, and $x \in E_h$. The minimum number of edges in such a family A is called the *edge covering number* $\varrho(H)$ of H. As mentioned in [1], p. 402, A is a transversal set of the dual hypergraph H^* of H, and hence Algorithm 2 offers a way of finding a rapid approximation for A_0 and an upper bound for $\varrho(H)$.

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ZASTOSOWANIE TEORII GRAFÓW W TEORII HIPERGRAFÓW

STRESZCZENIE

W pracy przedstawione są metody otrzymywania oszacowań charakterystycznych liczb związanych z hipergrafami, polegające na wyznaczaniu oszacowań odpowiednich liczb charakterystycznych związanych z przyporządkowanymi grafami Rozpatrywane są oszacowania dla minimalnej liczby wierzchołków dominujących, maksymalnej liczby wierzchołków niezależnych i minimalnej liczby reprezentantów.