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L. V. K. V. SARMA and D. V. KRISHNA (Madras)

OSCILLATION OF AXISYMMETRIC BODIES IN A STRATIFIED FLUID

0. Summary. The flow due to the oscillation of an axisymmetric body in a rotating stratified fluid is considered. First, taking into account the effect of stratification on the inertial terms, the solution is obtained when the oscillating body is an oblate spheroid. Using the radiation condition to eliminate the incoming waves, the solution is found in terms of oblate spheroidal wave functions. In the limit as the frequency of oscillation σ becomes small, we find that u, w, the radial and axial components of the fluid velocity, tend to a finite limit, whereas v, the swirl component of the velocity is $O(1/\sigma)$. Then, neglecting the effect of stratification on the inertial terms, the solution of this problem is obtained and it is found that, in the absence of rotation and as $\sigma \to 0$, u and w tend to zero everywhere except on the tangential planes where u has a finite limit.

1. Introduction. The problem of axisymmetric bodies moving in a non-homogeneous fluid has been considered by Warren [6]. Stating from the unsteady Oseen-type linearized equations for a slender body, he obtains the limiting form of the solution for large time which shows that as the body moves, waves are produced in the downstream side only. In this paper we consider the flow due to the oscillation of an axisymmetric body in a stratified fluid. The fluid is taken to be inviscid, incompressible and unbounded in all directions. In the case of a rotating fluid this problem was considered by Görtler [2], Morgan [3] and Sarma [4]. They have shown that there are real characteristic cones arising in the fluid on which the velocities become infinite when $\sigma < 2\Omega$. As the frequency of oscillation σ tends to zero, these cones tend to cylinder with its generators parallel to the axis of rotation and circumscribing the body. Here we consider the vertical oscillation of the body in a stratified fluid when the fluid is subjected to a constant rotation Ω about the vertical axis.

First, in Section 2 the solution for the flow due to the oscillation of an oblate spheroid is obtained taking into account the effect of stratification on the inertial terms. Using the radiation condition to eliminate the incoming waves, the solution is obtained in terms of oblate spheroidal wave functions. Here we find that if σ is very small, then no discontinuous or singular surfaces arise and the radial and axial components of the velocity u and w tend to a finite limit whereas the swirl component of the velocity v is $O(1/\sigma)$. Thus, in the presence of stratification and rotation, the linearized equations are inadequate to study the steady flow in an unbounded fluid. However, in the absence of rotation, the swirl velocity is zero and at any general point u and w tend to finite limits as σ approaches zero, thus showing that the steady flow is possible. (Also there are real characteristic cones when $\beta g > \sigma^2$.)

Second, in Section 3 the solution for the above-mentioned problem when β is small (thus neglecting the effect of stratification on the inertial terms) is obtained and it is found that the flow exhibits the same characteristics when both rotation and stratification are taken together except for the wave nature. In the absence of rotation, we now find that, as $\sigma \to 0$, the velocity components u and w tend to zero everywhere except on the tangential planes $|z| = b$, where the radial velocity u has a finite limit $Ua^2/2r|b|$. The case of circular disc is deduced from that of the spheroid in the last section.

2. Governing equations and solution of the problem. The equations of motion of an inviscid fluid with respect of a frame of reference rotating with a constant angular velocity Ω about a fixed axis can be written (see [7]) in the vector form as

$$\varrho\left(\frac{\partial\bar{V}}{\partial t}+(\bar{V}\cdot\nabla\bar{V})\right) = -\nabla p+\varrho\bar{X}-2\varrho\Omega\times\bar{V}-\varrho\Omega\times(\Omega\times\bar{r}),$$

where $\bar{V}$ is the velocity vector, $\bar{r}$ — the coordinate vector, and $\bar{X}$ represents the body force. When we consider stratified incompressible fluids, gravity being the only body force, the equations governing axisymmetric motion in cylindrical coordinates (r, θ, z) are

$$\text{(1)}\qquad \begin{cases} \varrho\left(\dfrac{du}{dt}-\dfrac{v^2}{r}-2\Omega v-\Omega^2 r\right) = -\dfrac{\partial P}{\partial r}, \\ \varrho\left(\dfrac{dv}{dt}+\dfrac{uv}{r}+2\Omega u\right) = 0, \\ \varrho\dfrac{dw}{dt} = -\dfrac{\partial P}{\partial z}-\varrho g, \end{cases}$$

$$\left(\frac{d}{dt} = \frac{\partial}{\partial t}+u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right),$$

where u, v, w denote the velocity components in the direction r, θ, z, respectively, and the axis of rotation is taken in the direction opposing gravity.

Since the fluid is incompressible,

(2) $$\frac{d\varrho}{dt} = 0.$$

The equation of continuity reduces to

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z} = 0.$$

The stratification ϱ_0 in the undisturbed state is taken to be a function of z alone as $\varrho_0 = \varrho_0' e^{-\beta z}$, where ϱ_0' is the characteristic density and β is the stratification parameter. (In addition, β is taken to be so small that as z varies βz always remains finite.) If P_0 is the corresponding pressure then

$$P_0 = \varrho_0\left(\frac{g}{\beta}+\frac{\Omega^2 r^2}{2}\right).$$

Consider a spheroid whose axis of symmetry coincides with the axis of rotation, oscillating along the axis of rotation with a velocity $Ue^{i\sigma t}$. Choosing the origin to be at the centre, the section of the spheroid in the (r, z)-plane is taken as $r^2/a^2+z^2/b^2 = 1$.

Let u, v, w be the components of the fluid velocity along the directions r, θ, z, respectively; let also P and ϱ be the perturbed presure and density, respectively. Now u, v and w are taken to be so small that their products can be neglected. Hence, substituting u, v, w, P_0+P, $\varrho_0+\varrho$ into equations (1) and (2) and using the Boussinesq approximation, we obtain (see [5]) the linearized equations of motion to be

(3) $$\begin{cases} \varrho_0\left(\dfrac{\partial u}{\partial t}-2\Omega v\right) = -\dfrac{\partial P}{\partial r}, \\ \varrho_0\left(\dfrac{\partial v}{\partial t}+2\Omega u\right) = 0, \\ \varrho_0\dfrac{\partial w}{\partial t} = -\dfrac{\partial P}{\partial z}-\varrho g, \end{cases}$$

(4) $$\frac{\partial \varrho}{\partial t}+w\frac{\partial \varrho_0}{\partial z} = 0,$$

(5) $$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z} = 0.$$

With respect to an instantaneously fixed axis situated at the centre of the body, the boundary conditions are

(i) $$u, v, w \to 0 \quad \text{as } z \to \infty \text{ for fixed } r \text{ and } t,$$

and

(ii) $$ur+\frac{a^2}{b^2}wz = \frac{a^2}{b^2}Uze^{i\sigma t},$$

i.e. on the body the normal velocity should be zero.

Eliminating u, v, w from (3) and (4), using the continuity equation (5), we obtain the governing equation in terms of P to be

$$\left(\frac{\partial^2}{\partial t^2}+\beta g\right)\nabla^2 P+(4\Omega^2-\beta g)\frac{\partial^2 P}{\partial z^2}+\beta\left(\frac{\partial^2}{\partial t^2}+4\Omega^2\right)\frac{\partial P}{\partial z} = 0.$$

Taking oscillations for the variables as $P = P'e^{i\sigma t}$, etc., this equation reduces to (dropping the dashes from the variables)

(6) $$\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\frac{\partial P}{\partial r}+\frac{4\Omega^2-\sigma^2}{\beta g-\sigma^2}\frac{\partial^2 P}{\partial z^2}+\beta\left(\frac{4\Omega^2-\sigma^2}{\beta g-\sigma^2}\right)\frac{\partial P}{\partial z} = 0.$$

The velocities are given by

$$u = \frac{-i\sigma}{\varrho_0(4\Omega^2-\sigma^2)}\frac{\partial P}{\partial r},\quad v = \frac{2\Omega}{\varrho_0(4\Omega^2-\sigma^2)}\frac{\partial P}{\partial r},\quad w = \frac{-i\sigma}{\varrho_0(\beta g-\sigma^2)}\frac{\partial P}{\partial z}.$$

In terms of P, the boundary conditions are $P\to 0$ as $z\to\infty$ and

(7) $$r\frac{\partial P}{\partial r}+\frac{a^2}{b^2}z\left(\frac{4\Omega^2-\sigma^2}{\beta g-\sigma^2}\frac{\partial P}{\partial z}\right) = \frac{i\varrho_0 a^2(4\Omega^2-\sigma^2)Uz}{b^2\sigma}$$

on $r^2/a^2+z^2/b^2 = 1$.

Introducing the oblate spheroidal coordinate transformation defined by

$$z = \left(\frac{4\Omega^2-\sigma^2}{\beta g-\sigma^2}\right)^{1/2}c\xi\eta \qquad\text{and}\qquad r = c(1+\xi^2)^{1/2}(1-\eta^2)^{1/2},$$

the governing equation (6) transforms to

(8)
$$\left[(1+\xi^2)\frac{\partial^2}{\partial\xi^2}+(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+2\xi\frac{\partial}{\partial\xi}-2\eta\frac{\partial}{\partial\eta}-\left(\frac{\beta' c}{2}\right)^2(\xi^2+\eta^2)\right]Pe^{\beta' c\xi\eta/2} = 0,$$

where

$$\beta' = \beta\left(\frac{4\Omega^2-\sigma^2}{\beta g-\sigma^2}\right)^{1/2}.$$

The spheroid $r^2/a^2+z^2/b^2 = 1$ is given by

$$\xi = \xi_0 = \left(\frac{(\beta g-\sigma^2)b^2}{\sigma^2(b^2-a^2)+(4\Omega^2a^2-\beta g b^2)}\right)^{1/2}$$

and

$$c^2 = \frac{\sigma^2(b^2-a^2)+4\Omega^2a^2-\beta g b^2}{4\Omega^2-\sigma^2}.$$

The boundary conditions (7) reduce to $P \to 0$ as $\xi \to \infty$ and

$$\left(\frac{\partial P}{\partial \xi}\right)_{\xi=\xi_0} = \mu\eta e^{-\beta b\eta}, \tag{9}$$

where

$$\mu = \frac{i\varrho_0' U}{\sigma}[\sigma^2(b^2-a^2)+4\Omega^2 a^2-\beta g b^2]^{1/2}(\beta g-\sigma^2)^{1/2}.$$

Equation (8) separates into the two equations

$$\frac{d}{d\xi}\left[(\xi^2+1)\frac{df}{d\xi}\right]+(m^2-\lambda^2\xi^2)f = 0, \tag{10}$$

$$\frac{d}{d\eta}\left[(\eta^2-1)\frac{dg}{d\eta}\right]+(m^2+\lambda^2\eta^2)g = 0, \tag{11}$$

where m^2 is the separation constant and $\lambda = \beta' c/2$. These equations have oblate spheroidal wave functions as their solutions (see [1]). The angular solution of equation (11), which is finite in the range $-1 \leqslant \eta \leqslant 1$, can be expressed as

$$S_n'(-i\lambda, \eta) = \sum_{r=0}^{\infty}{}' d_r^{0n}(-i\lambda)P_r(\eta),$$

where $P_r(\eta)$ is the Legendre polynomial of r-th order and summation ($'$) is over even or odd values of r according to as n is even or odd.

The coefficients $d_r^{0n}(-i\lambda)$ are given by

$$\sum_{r=0}^{\infty}\frac{\underline{|2r}}{2^{2r}(\underline{|r})^2}d_{2r}^{02n} = \frac{\underline{|2n}}{2^{2n}(\underline{|n})^2} \qquad (n = 0, 1, \ldots),$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r\underline{|2(r+1)}}{2^{2r+1}\underline{|r}\,\underline{|r+1}}d_{2r+1}^{02n+1} = \frac{(-1)^n\underline{|2(n+1)}}{2^{2n+1}\underline{|n}\,\underline{|n+1}} \qquad (n = 0, 1, \ldots).$$

(The other angular solutions S_n^2 cannot be considered because of their singular nature at $\eta = \pm 1$.)

The radial solutions $R_n(-i\lambda, i\xi)$ of equation (10) can be chosen as $R_n^3(-i\lambda, i\xi)$ and $R_n^4(-i\lambda, i\xi)$ which behave at infinity as

$$\frac{1}{\lambda\xi}\exp\left\{\pm i\left(\lambda\xi-\frac{n+1}{2}\pi\right)\right\},$$

respectively, and hence satisfy condition (9) at infinity. Now, applying the radiation condition, the ξ-dependence is confined to the radial function

$R_n^4(-i\lambda, i\xi)$. Thus

$$R_n^4(-i\lambda, i\xi) = \frac{\sum_{r=0}^{\infty}{}' i^{r-n} d_r^{0n} h_r^2(\lambda\xi)}{\sum_r{}' d_r^{0n}}.$$

Hence, the most general solution of euqation (8) can be taken as

$$Pe^{\lambda\xi\eta} = \sum_{r=0}^{\infty} A_n S_n^1(-i\lambda, \eta) R_n^4(-i\lambda, i\xi),$$

where the constants A_n should be determined by the condition on the body. Now

$$P = e^{-\lambda\xi\eta} \sum_{n=0}^{\infty} A_n S_n^1(\eta) R_n^4(i\xi).$$

Applying the boundary condition on the spheroid

$$\left(\frac{\partial P}{\partial \xi}\right)_{\xi=\xi_0} = \mu\eta e^{-\beta b\eta},$$

we get

$$\text{(12)} \quad \sum_{n=0}^{\infty} A_n \left\{ R_n'^4(i\xi_0) \sum_{r=0}^{\infty}{}' d_r^{0n} P_r(\eta) - \lambda R_n^4(i\xi_0) \sum_{r=0}^{\infty}{}' d_r^{0n} \eta P_r(\eta) = \mu\eta e^{\nu\eta}, \right.$$

where

$$R_n'^4(i\xi_0) = \left[\frac{d}{d\xi} R_n^4(i\xi)\right]_{\xi=\xi_0}.$$

Using the orthogonal properties of the Legendre polynomials, we have

$$\int_{-1}^{1} P_r P_l d\eta = \begin{cases} 0 & \text{for } r \neq l, \\ 2/(2l+1) & \text{for } r = l, \end{cases}$$

$$\int_{-1}^{1} P_r P_l \eta d\eta = \begin{cases} 0 & \text{for } r \neq l+1, \\ 2r/(4r^2-1) & \text{for } r = l+1, \end{cases}$$

and

$$\int_{-1}^{1} \eta^r P_l d\eta = \begin{cases} \dfrac{2r(r-1)\ldots(r-l+2)}{(r+l+1)(r+l-1)\ldots(r-l+3)} & \text{for } r-1 \text{ even}, \\ 0 & \text{for } r < l \text{ and } r-l \text{ odd}. \end{cases}$$

From (12) we obtain

$$\sum_n B_{2n} R'^4_{2n}(i\xi_0)\, d_0^{02n} - \frac{\lambda}{3} \sum_n B_{2n+1} R^4_{2n+1}(i\xi_0)\, d_1^{02n+1} = \Psi_{1,0}(\nu),$$

$$\lambda \sum_n B_{2n} R^4_{2n}(i\xi_0) \left\{ d_2^{02n} + \frac{2}{5}\, d_2^{02n} \right\} - \sum_n B_{2n+1} R'^4_{2n+1}(i\xi_0)\, d_1^{02n+1} = -\Psi_{1,1}(\nu),$$

. . . .

$$\sum_n B_{2n} R'^4_{2n}(i\xi_0) \frac{d_{2l}^{02n}}{4l+1} -$$

$$-\lambda \sum_n B_{2n+1} R^4_{2n+1}(i\xi_0) \left\{ \frac{2l d_{2l-1}^{02n+1}}{16l^2-1} + \frac{(2l+1)\, d_{2l+1}^{02n+1}}{4(2l+1)^2-1} \right\} = \frac{\Psi_{1,2l}(\nu)}{3},$$

$$\lambda \sum_n B_{2n} R^4_{2n}(i\xi_0) \left\{ \frac{(2l+1)\, d_{2l}^{02n}}{4(2l+1)^2-1} + \frac{2(l+1)\, d_{2l+2}^{02n}}{16(l+1)^2-1} \right\} -$$

$$- \sum_n B_{2n+1} R'^4_{2n+1}(i\xi_0) \frac{d_{2l+1}^{02n+1}}{4l+3} = \frac{-\Psi_{1,2l+1}(\nu)}{3},$$

where

$$\mu B_n = A_n,$$

$$\Psi_{1,l} = \frac{1}{l}\left[(2l-1)\Psi'_{1,l-1} - (l-1)\Psi_{1,l-2}\right],$$

$$\Psi_{1,1} = \sqrt{\frac{\pi}{2\nu}}\, \left[I_{1/2}(\nu) + 2I_{5/2}(\nu)\right],$$

$$\Psi_{1,0} = 3\sqrt{\frac{\pi}{2\nu}}\, I_{3/2}(\nu).$$

These infinite simultaneous equations, for the values $l = 0, 1, 2, \ldots$, determine the infinite set of constants B_n.

Thus P is uniquely determined. Hence the velocities at any general point are given by

$$u = \frac{-U\sqrt{\varrho_0'/\varrho}\,(\beta g - \sigma^2)^{1/2} r(\Phi_{\mathrm{Re}}\cos\sigma t - \Phi_{\mathrm{Im}}\sin\sigma t)}{[\sigma^2(b^2-a^2) + 4\Omega^2 a^2 - \beta g b^2]^{1/2}(\xi^2+\eta^2)},$$

$$v = \frac{-2U\Omega\sqrt{\overset{*}{\varrho}_0/\varrho}\,(\beta g - \sigma^2)^{1/2} r(\Phi_{\mathrm{Re}}\sin\sigma t + \Phi_{\mathrm{Im}}\cos\sigma t)}{\sigma[\sigma^2(b^2-a^2) + 4\Omega^2 a^2 - \beta g b^2]^{1/2}(\xi^2+\eta^2)},$$

$$w = \frac{U\sqrt{\varrho_0'/\varrho}\,(\chi_{\mathrm{Re}}\cos\sigma t - \Psi_{\mathrm{Im}}\sin\sigma t)}{(\xi^2+\eta^2)},$$

where Re denotes the real part. Im denotes the imaginary part, and

$$\Phi = \sum_n B_n\{\xi S_n^1(\eta)R_n'^4(i\xi) - \eta S_n'^1(\eta)R_n^4(i\xi)\},$$

$$\chi = \sum_n B_n\{\eta(1+\xi^2)S_n^1(\eta)R_n'^4(i\xi) + \xi(1-\eta^2)S_n'^1(\eta)R_n^4(i\xi) - \\ -\lambda(\xi^2+\eta^2)S_n^1(\eta)R_n^4(i\xi)\},$$

$$\xi = \{2[\sigma^2(b^2-a^2)+4\Omega^2a^2-\beta g b^2]\}^{-1/2}\{(4\Omega^2-\sigma^2)r^2+ \\ +(\beta g-\sigma^2)z^2-\sigma^2(b^2-a^2)-(4\Omega^2a^2-\beta g b^2)+ \\ +[((4\Omega^2-\sigma^2)r^2+(\beta g-\sigma^2)z^2+\sigma^2(b^2-a^2)+4\Omega^2a^2-\beta g b^2)^2- \\ -4(4\Omega^2-\sigma^2)(\sigma^2(b^2-a^2)+4\Omega^2a^2-\beta g b^2)r^2]^{1/2}\}^{1/2},$$

$$\eta = \{2[\sigma^2(b^2-a^2)+4\Omega^2a^2-\beta g b^2]\}^{-1/2}\{[((4\Omega^2-\sigma^2)r^2+ \\ +(\beta g-\sigma^2)z^2+\sigma^2(b^2-a^2)+4\Omega^2a^2-\beta g b^2)^2- \\ -4(4\Omega^2-\sigma^2)(\sigma^2(b^2-a^2)+4\Omega^2a^2-\beta g b^2)r^2]^{1/2}- \\ -[(4\Omega^2-\sigma^2)r^2+(\beta g-\sigma^2)z^2-\sigma^2(b^2-a^2)-4\Omega^2a^2-\beta g b^2]\}^{1/2}.$$

Particular case. Sphere as a limiting case of a spheroid. In the case of a sphere of radius a, we have

$$\xi_0 = \left(\frac{\beta g-\sigma^2}{4\Omega^2-\beta g}\right)^{1/2} \quad \text{and} \quad c = a\left(\frac{4\Omega^2-\beta g}{4\Omega^2-\sigma^2}\right)^{1/2}.$$

Thus, this analysis fails when $4\Omega^2 = \beta g$. In this case the original equation (6) simplifies to

$$(\nabla^2-\beta^2/4)(Pe^{\beta z/2}) = 0.$$

This is to be solved using the boundary condition on the sphere $r^2+z^2 = a^2$, i.e.

$$r\frac{\partial P}{\partial r}+z\frac{\partial P}{\partial z} = \frac{i\varrho_0 U(4\Omega^2-\sigma^2)z}{\sigma}.$$

In terms of spherical polar coordinates (R, θ), the problem reduces to solving the equation

$$\left(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot\theta}{R^2}\frac{\partial}{\partial\theta}-\frac{\beta^2}{4}\right)(Pe^{\beta R\cos\theta/2}) = 0, \tag{13}$$

satisfying the conditions

$$\left(\frac{\partial P}{\partial R}\right)_{R=a} = \nu e^{-\beta a\mu}\mu \quad \text{and} \quad P\to 0 \text{ as } R\to\infty, \tag{14}$$

where

$$\mu = \cos\theta \quad \text{and} \quad \nu = i\varrho_0' U(4\Omega^2 - \sigma^2)/\sigma.$$

The general solution of (13), satifying the condition at ∞, can be taken as

$$P = \frac{1}{\sqrt{R}} e^{-\beta R\mu/2} \sum_n D_n H^1_{n+1/2}(i\beta R/2) p_n(\mu),$$

where $H^1_{n+1/2}$ is the Hankel function and the $p_n(\mu)$ are Legendre polynomials.

Applying condition (14) and using the orthogonality relation of Legendre polynimoals, we obtain

$$\sum_n D_n \left[H^{1\prime}_{n+1/2}(i\beta a/2) - \frac{1}{2a} H^1_{n+1/2}(i\beta a/2)\right] \frac{1}{\sqrt{a}} \frac{2}{2m+1} \Psi_{n,m}(-\beta a/2) -$$

$$- \frac{\beta}{2\sqrt{a}} H^1_{n+1/2}(i\beta a/2) \left[\frac{2m}{4m^2-1} \Psi_{n,m-1}(-\beta a/2) + \right.$$

$$\left. + \frac{2(m+1)}{4(m+1)^2-1} \Psi_{n,m+1}(-\beta a/2)\right] = \begin{cases} \nu/3 & \text{if } m = 1, \\ 0 & \text{if } m = 0, 2, 3, \ldots, \end{cases}$$

where

$$\Psi_{n,m} = \frac{1}{m} [(2m-1)\Psi_{n,m-1} - (m-1)\Psi_{n,m-2}],$$

$$\Psi_{m,1} = \frac{n}{2n-1} \Psi_{m-1,0} + \frac{n+1}{2n+3} \Psi_{n+1,0},$$

$$\Psi_{n,0} = (2n+1)\sqrt{\frac{\pi}{2\xi}} I_{n+1/2}(\xi).$$

The above-mentioned set of infinite simultaneous equations determines the constants D_n.

Let us discuss the results when $\Omega \neq 0$ and $\beta \neq 0$. From the expressions for the velocity components u, v and w, in general, we find that they become infinite when $\xi^2 + \eta^2 = 0$. They are real if $4\Omega^2 \geqslant \sigma^2 \geqslant \beta g$ or if $4\Omega^2 \leqslant \sigma^2 \leqslant \beta g$. Thus, the flow is continuous everywhere if the frequency of oscillation does not lie between the frequency of rotation and the Brunt-Viassälä frequency. When σ is small, so that its powers may be neglected in comparison with the other terms, these discontinuous cones become imaginary. On the axis of rotation ($r = 0$) we have $u = v = 0$ and

$$w = \mathrm{Re}\left(U\sqrt{\varrho_0'/\varrho} \sum B_n \frac{(1+\xi^2)S^1_n(1)R'^4_n(i\xi) - \lambda(1+\xi^2)S^1_n(\eta)R^4_n(i\xi)}{(1+\xi^2)} e^{i\sigma t}\right).$$

At any general point, as $\sigma \to 0$, we have

$$\begin{aligned} u &\simeq -U\sqrt{\varrho_0'/\varrho}\sqrt{\beta g \Delta}\, r(\Phi_{\mathrm{Re}})_{\sigma=0}\varkappa, \\ v &\simeq -2U\sqrt{\varrho_0'/\varrho}\sqrt{\beta g \Delta}\, r(\Phi_{\mathrm{Im}})_{\sigma=0}\varkappa/\sigma \sim O(1/\sigma), \\ w &\simeq U\sqrt{\varrho_0'/\varrho}\,\Delta(\chi_{\mathrm{Re}})_{\sigma=0}\varkappa, \end{aligned}$$

where

$$\Delta = 4\Omega^2 a^2 - \beta g b \quad \text{and} \quad \varkappa = \left([4\Omega^2(r^2+a^2)+\beta g(z^2-b^2)]^2 - 16\Omega^2 r^2 \Delta\right)^{-1/2}.$$

Thus, as σ approaches zero, the radial and axial velocities will be finite, whereas the swirl component will increase indefinitely.

When $\Omega = 0$ and $\beta \neq 0$, the swirl component of velocity is zero. The discontinuous cones are given by

$$r\sigma \pm \sqrt{\beta g - \sigma^2}\, z \pm \sqrt{(a^2-b^2)\sigma^2 + \beta g b^2} = 0.$$

If $\beta g > \sigma^2$, these cones are real. At any general point, as $\sigma \to 0$, we have, for $|z| \neq b$,

$$u \simeq \frac{-U\sqrt{\varrho_0'/\varrho}\,\beta g b r(\Phi_{\mathrm{Im}})_{\sigma=0}}{\beta g\,|z^2-b^2|}, \qquad w \simeq \frac{-Ub^2\sqrt{\varrho_0'/\varrho}\,(\chi_{\mathrm{Re}})_{\sigma=0}}{|z^2-b^2|}.$$

When $\beta = 0$ and $\Omega \neq 0$, the discontinuous cones are

$$\sqrt{4\Omega^2-\sigma^2}\, r \pm \sigma z \pm \sqrt{a^2(4\Omega^2-\sigma^2)+\sigma^2 b^2} = 0.$$

They are real if $2\Omega > \sigma$. At any general point, as $\sigma \to 0$, we have, for $r \neq a$,

$$u \simeq 0, \qquad v \simeq \frac{-Uar(\Phi_{\mathrm{Re}})_{\sigma=0}}{|r^2-a^2|}, \qquad w \simeq \frac{Ua^2(\chi_{\mathrm{Re}})_{\sigma=0}}{|r^2-a^2|}.$$

3. Solution neglecting the stratification effect on inertial terms. Starting from equation (6), if we neglect the term containing β, the governing differential equation is of the form

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{4\Omega^2-\sigma^2}{\beta g - \sigma^2}\frac{\partial^2 P}{\partial z^2} = 0,$$

and

$$u = \frac{-i\sigma}{\varrho_0'(4\Omega^2-\sigma^2)}\frac{\partial P}{\partial r}, \qquad v = \frac{2\Omega}{\varrho_0'(4\Omega^2-\sigma^2)}\frac{\partial P}{\partial r}, \qquad w = \frac{-i\sigma}{\varrho_0'(\beta g-\sigma^2)}\frac{\partial P}{\partial z}.$$

This equation can also be derived by neglecting the effect of stratification on inertial terms, i.e. by taking ϱ_0 as a constant (ϱ_0') in the left-hand sides of equations (3).

Using oblate spheroidal transformations as in the previous section, the problem reduces to solving the equation

$$\frac{\partial}{\partial \xi}\left((1+\xi^2)\frac{\partial P}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left((1-\eta^2)\frac{\partial P}{\partial \eta}\right)=0,$$

where P has to satisfy the boundary conditions

$$P \to 0 \text{ as } \xi \to \infty \quad \text{and} \quad \left(\frac{\partial P}{\partial \xi}\right)_{\xi=\xi_0} = \alpha\eta,$$

where $\alpha = i\varrho_0' U\sqrt{\sigma^2(b^2-a^2)+4\Omega^2a^2-\beta g b^2}\sqrt{\beta g-\sigma^2}/\sigma$.

The solution for P satisfying these conditions can be written as

$$P = \frac{\alpha}{\log\dfrac{\xi_0-i}{\xi_0+i}+\dfrac{2i\xi_0}{\xi_0^2+1}}\left(\xi\log\frac{\xi-i}{\xi+i}+2i\right)\eta.$$

Hence, the velocity components are

$$u = \mathrm{Re}\left(\frac{-2i\,Ur\sqrt{\beta g-\sigma^2}\,\eta e^{i\sigma t}}{\Delta(\xi,\eta)}\right),$$

$$v = \mathrm{Re}\left(\frac{4\Omega Ur\sqrt{\beta g-\sigma^2}\,\eta e^{i\sigma t}}{\sigma\Delta(\xi,\eta,)}\right),$$

$$w = \mathrm{Re}\left(\frac{U\left(\log\dfrac{\xi-i}{\xi+i}+\dfrac{2i\xi}{\xi^2+\eta^2}\right)e^{i\sigma t}}{\log\dfrac{\xi_0-i}{\xi_0+i}+\dfrac{2i\xi_0}{\xi_0^2+1}}\right),$$

where

$$\Delta(\xi,\eta) = \sqrt{\sigma^2(b^2-a^2)+4\Omega^2a^2-\beta g b^2}\left(\log\frac{\xi_0-i}{\xi_0+i}+\frac{2i\xi_0}{\xi_0^2+1}\right)(\xi^2+\eta^2)(1+\xi^2).$$

Thus we see that the discontinuous surfaces are the same as in the previous section. In general, also, as σ approaches zero, u and w will remain finite and v will be of the order $O(1/\sigma)$.

When $\Omega = 0$, i.e. in the case of stratification alone, we have

$$u = \mathrm{Re}\left(\frac{-2i\sqrt{\sigma^2-\beta g}\,Ur\eta e^{i\sigma t}}{\sqrt{\sigma^2(a^2-b^2)+\beta g b^2}\left(\log\dfrac{\xi_0-i}{\xi_0+i}+\dfrac{2i\xi_0}{\xi_0^2+1}\right)(\xi^2+\eta^2)(1+\xi^2)}\right),$$

$$w = \mathrm{Re}\left(\frac{U\left(\log\dfrac{\xi-i}{\xi+i}+\dfrac{2i\xi}{\xi^2+\eta^2}\right)e^{i\sigma t}}{\log\dfrac{\xi_0-i}{\xi_0+i}+\dfrac{2i\xi_0}{\xi_0^2+1}}\right),$$

where

$$\xi_0 = \sqrt{\frac{b^2(\sigma^2-\beta g)}{\sigma^2(a^2-b^2)+\beta g b^2}}, \qquad \xi^2 = \alpha(\gamma+\mu), \qquad \eta^2 = \alpha(\mu-\gamma),$$

with

$$\alpha = 1/(2[\sigma^2(a^2-b^2)+\beta g b^2]),$$
$$\gamma = r^2\sigma^2+(\sigma^2-\beta g)z^2-\sigma^2(a^2-b^2)-\beta g b^2,$$
$$\mu = \sqrt{[r^2\sigma^2+(\sigma^2-\beta g)z^2+\sigma^2(a^2-b^2)+\beta g b^2]^2-4\sigma^2 r^2[\sigma^2(a^2-b^2)+\beta g b^2]}\,.$$

The discontinuous surfaces are $r\sigma \pm \sqrt{\beta g-\sigma^2}z \pm \sqrt{\sigma^2(a^2-b^2)+\beta g b^2} = 0$.

Since $\xi_0 \to i$, from the expression for the velocity we see that $u \to 0$ and $w \to 0$ as $\sigma \to 0$ at any general point in the fluid. Also, on the planes $|z| = b$ we can show that as $\sigma \to 0$ we have $u \simeq Ua^2/2rb$ and $w \simeq 0$.

In the degenerate case when the spheroid becomes a circular disk $r = a$, we have

$$u = \mathrm{Re}\left(\frac{-2\sqrt{\sigma^2-\beta g}\,Ur\eta e^{i\sigma t}}{\sigma a\pi(\xi^2+\eta^2)(1+\xi^2)}\right),$$
$$w = \mathrm{Re}\left(\frac{U}{i\pi}\left(\log\frac{\xi-i}{\xi+i}+\frac{2i\xi}{\xi^2+\eta^2}\right)e^{i\sigma t}\right),$$

if $z \neq 0$ and $u \simeq 0$, $w \simeq 0$ as $\sigma \to 0$. On the plane $z = 0$, as $\sigma \to 0$, we obtain

$$u \simeq 0 \quad \text{and} \quad w \simeq \frac{U}{\pi}\left(2\tan^{-1}\frac{\sqrt{r^2-a^2}}{a}+\frac{2a}{\sqrt{r^2-a^2}}+\pi\right) \quad \text{for } r > a,$$

and

$$u \to \infty \quad \text{and} \quad w \simeq U \quad \text{for } r < a.$$

Thus we see that as the frequency of oscillation approaches to zero, the radial component of the velocity will not remain small on the disk, although, in general, it tends to zero everywhere in the flow field. The vertical component of velocity will remain finite on the plane $z = 0$ outside the disk, whereas it is zero at any general point.

References

[1] C. Flammer, *Spheroidal wave functions.*

[2] H. Görtler, *Einige Bemerkungen über Strömungen in rotierenden Flüssigkeiten,* Z. angew. Math. Mech. 24 (1944), p. 210-214.

[3] G. W. Morgan, *A study of motions in a rotating liquid,* Proc. Roy. Soc., London A206 (1951), p. 108-130.

[4] L. V. K. V. Sarma, *Rotational flows,* Thesis submitted to Indian Institute of Technology, Kharagpur, India 1958.

[5] S. P. Talwar, *Stability of a conducting rotating fluid of variable density*, J. Fluid Mech. 9 (1960), p. 581-592.
[6] F. W. G. Warren, *Wave resistance to vertical motion in a stratified fluid*, ibidem 7 (1960), p. 209-229.
[7] C. S. Yih, *Dynamics of non-homogeneous fluids*, Macmillan, New York 1965.

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY
MADRAS, INDIA

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L. V. K. V. SARMA i D. V. KRISHNA (Madras)

DRGANIA OSIOWO SYMETRYCZNYCH CIAŁ W ROZWARSTWIONEJ CIECZY

STRESZCZENIE

W pracy rozpatruje się przepływ powstały wskutek drgań osiowo symetrycznego ciała w wirującej rozwarstwionej cieczy. Warunek promieniowania został użyty dla wyeliminowania fal przychodzących, a rozwiązanie uzyskano za pomocą zmodyfikowanych sferycznych funkcji falowych. Zbadana jest różnica w charakterystykach dwu przepływów: w jednym z nich wzięto pod uwagę wpływ rozwarstwienia na wyrazy inercjalne, w drugim zaś założono, że wpływ ten jest znikomy.
