

DING REN (Shijiazhuang) and J. R. REAY (Bellingham, Washington)

## AREAS OF LATTICE POLYGONS, APPLIED TO COMPUTER GRAPHICS

*Abstract.* An 1899 theorem of Pick asserts that if a polygon  $P$  in the plane has vertices only at integer lattice points, then the area of  $P$  is  $A = b/2 + i - 1$ , where  $b$  is the number of lattice points on the boundary of  $P$ , and  $i$  the number of lattice points in the interior of  $P$ . This result is extended to polygons whose vertices lie in the regular triangular or the regular hexagonal lattices. The results may be applied to the following problem in computer graphics:

When a straight edge is drawn on a television screen, how do you remove the characteristic jagged, "stair-step" appearance of the line?

**1. Introduction.** Pick's theorem asserts that if the simple planar polygon  $P$  has vertices at lattice points (that is, points where both coordinates are integers) and if  $P$  has  $b$  lattice points on its boundary and  $i$  lattice points in its interior, then the area of  $P$  is given by

$$A(P) = b/2 + i - 1.$$

An example of Pick's theorem is shown in Fig. 1, where the vertices of the polygon  $P$  lie in the set of lattice points formed by the edge-to-edge tiling of the plane with unit squares. There are 7 of these lattice points on the

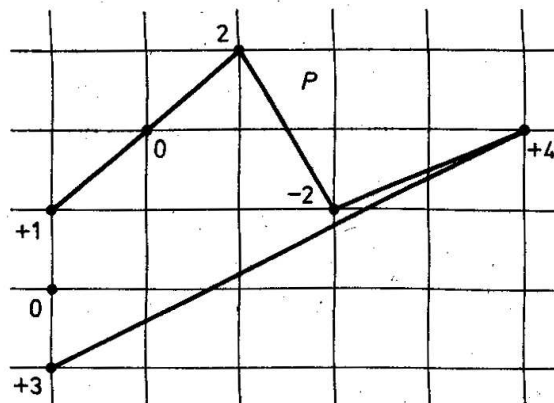


Fig. 1. Example of Pick's theorem

boundary of  $P$  and 4 lattice points in the interior of  $P$ . Hence the area of  $P$  is

$$A(P) = b/2 + i - 1 = 7/2 + 4 - 1 = 6.5,$$

which may be independently verified by direct inspection.

In Section 2 we review some of the notation found in [3], including the concept of the boundary characteristic  $c$  of a lattice polygon  $P$ . This parameter and Pick's number will be the key, in Section 3, to theorems concerning the area of certain polygons in the regular hexagonal tiling of the plane. Theorems from [3] are reviewed and extended. Section 4 reviews other generalizations of Pick's theorem and gives an application to computer graphics.

**2. Notation.** The only face-to-face tilings of the plane using congruent regular polygons are the familiar tilings by squares, triangles, or hexagons shown in Fig. 2. They are denoted by (4)4, (6)3, and (3)6, respectively, which show the number of each type of polygon at each vertex. For example, in the tiling of the plane by equilateral triangles, there are 6 triangles at each vertex.

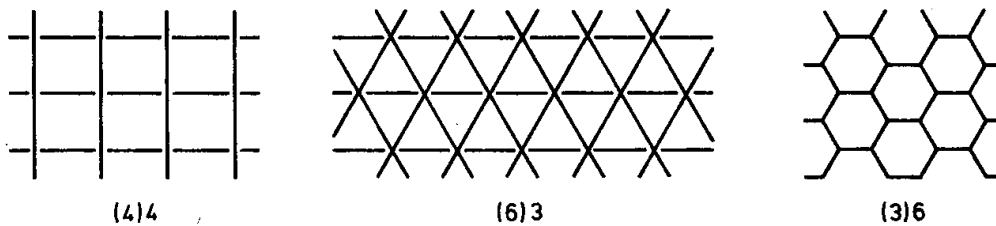


Fig. 2. The planar tilings with congruent regular polygons

In each of these tilings let  $V$  and  $E$  denote the set of all vertices of the tiling and the family of all edges of the tiling, respectively.  $(V, E)$  is a  $d$ -regular planar graph, where the *degree*  $d$  of the tiling is the degree at each vertex. The set  $V$  is the *lattice* determined by the tiling, and will be denoted by  $V[\text{type}]$  whenever it is necessary to distinguish the particular tiling. For each tiling,  $\mathcal{P}$  or  $\mathcal{P}[\text{type}]$  will denote the set of all *lattice polygons*, that is, planar polygons whose boundary is a simple closed curve and whose vertices lie in  $V$ . Let  $\mathcal{C}$  denote the subset of all polygons in  $\mathcal{P}$  whose boundary lies in the union of  $E$ . Thus a polygon  $P$  in  $\mathcal{C}$  must be the finite union of regular polygons which make up the specified tiling, and the boundary of a polygon  $P$  in  $\mathcal{C}$  is actually a simple closed path using the edges in  $E$ . Let  $A(P)$  denote the area of a polygon  $P$  in  $\mathcal{P}$ . Let  $b$  (respectively,  $i$ ) denote the number of points of  $V$  that lie in the boundary (respectively, the interior) of  $P$ . Another useful characteristic of a lattice polygon is  $n = b + 2i - 2$ , which will be called *Pick's number*.

With this notation the theorem of Pick [9] may be stated as

**THEOREM (Pick).** *The area of any polygon  $P$  in  $\mathcal{P}[(4)4]$  is  $A(P) = n/2$ .*

Each lattice point  $x$  on the boundary  $\partial P$  of a lattice polygon  $P$  is incident with exactly  $d$  edges of  $E$ . Each of these edges either (1) lies in the boundary of  $P$  or (2) extends locally into the exterior of  $P$  near  $x$  or (3) extends locally into the interior of  $P$  near  $x$ . We will denote these three disjoint sets of edges by  $B(x)$ ,  $F(x)$ , and  $G(x)$ , respectively. We define the *boundary characteristic*  $c(P)$  of a polygon  $P$  in  $\mathcal{P}$  as

$$c(P) = f - g,$$

where

$$f = \sum_{x \in (V \cap \partial P)} (|F(x)| + (1/2)|B(x)|)$$

and

$$g = \sum_{x \in (V \cap \partial P)} (|G(x)| + (1/2)|B(x)|).$$

(Throughout the paper,  $|S|$  denotes the cardinality of the set  $S$ .) Clearly,  $f + g = bd$  since there are  $d$  edges incident at each of the  $b$  lattice points on the boundary of  $P$ , and  $f + g$  also counts these incidences. Intuitively, to compute the boundary characteristic  $c = c(P)$  of the lattice polygon  $P$ , travel once around the boundary of  $P$ , and add  $+1$  each time we find a lattice edge that starts on the boundary and sticks out locally into the exterior of  $P$ , and add  $-1$  each time we find such an edge that locally pokes into the interior of  $P$ . For the example in Fig. 1,  $f = 18$  and  $g = 10$ , so  $f + g = 28 = 7 \cdot 4 = bd$ . The contribution to  $c$  at each of the 7 vertices on the boundary of  $P$  is shown, and these numbers sum to  $c = 8 = f - g$ . The examples in Fig. 3 (and similar variants) show that in  $\mathcal{P}[(3)6]$  the parameter  $c$  may

- (a) assume any non-negative integer,
- (b) assume any non-positive integer,
- (c) vary even though  $b$  and  $i$  remain the same (triangles  $XYZ$ ,  $XZW$ , and  $XZU$  have  $b = 3$ ,  $i = 0$ , and  $n = 1$ , but  $c(XYZ) = 5$ ,  $c(XZW) = 7$ , and  $c(XZU) = 9$ ).

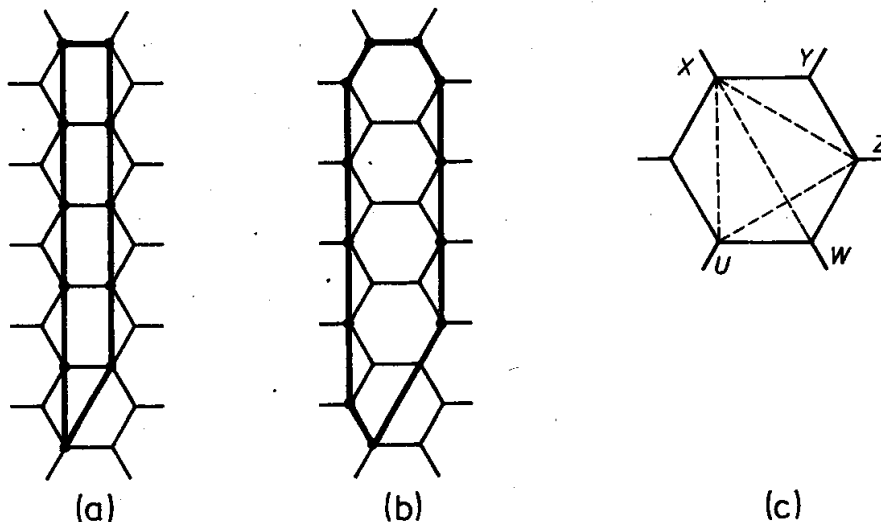


Fig. 3. The boundary characteristic in (3)6

(a)  $c = 15$ , (b)  $c = -1$ , (c)  $c(XYZ) = 5$ ,  $c(XZW) = 7$ ,  $c(XZU) = 9$

**3. Polygonal areas in lattices with congruent tiles.** In this section we consider polygons  $P$  whose vertices are at the vertex points of one of the tilings shown in Fig. 2. For each tiling, we always assume each tile (square, triangle, or hexagon) to be of unit area. Thus the length of the edge of a tile is 1 or  $(4/3^{1/2})^{1/2}$  or  $[2/(3 \cdot 3^{1/2})]^{1/2}$ , respectively. The following theorem completely determines the value of the boundary characteristic and the area of any lattice polygon in the regular tilings by either squares or triangles.

**THEOREM 1.** *If  $P \in \mathcal{P}[(4)4]$ , then  $A(P) = n/2$  and  $c(P) = 8$ . If  $P \in \mathcal{P}[(6)3]$ , then  $A(P) = n$  and  $c(P) = 12$ .*

**Proof.** Pick's theorem asserts that  $A(P) = n/2$  in the square lattice. Let  $T$  be a linear transformation of the plane which maps the square lattice  $V[(4)4]$  onto the triangular lattice  $V[(6)3]$ . The unit square maps by  $T$  onto the union of 2 triangles, and the parameters  $b$  and  $i$  for any lattice polygon are preserved by  $T$  and its inverse. Since each triangle has unit area, the Jacobian of  $T$  is 2, and thus  $A(TP) = 2A(P)$  for any  $P \in \mathcal{P}[(4)4]$ . Hence  $A(P) = n$  for each  $P \in \mathcal{P}[(6)3]$ .

The following argument, determining  $c(P)$ , is due to Barry Stipe. The planar graphs of the tilings (4)4 and (6)3 have the property that, for each vertex-edge incidence, there is another edge incident with the vertex from the opposite direction. [Note that the graph of the hexagonal tiling does not have this property.] If  $P$  is any lattice polygon in a tiling with this property, then it is easily shown that  $g = (b-2)(d/2)$ . [The  $b$  lattice points on the boundary of  $P$  have interior angles which sum to  $180(b-2)$  degrees.] Hence, for any  $P$  in  $\mathcal{P}[(4)4]$  or  $\mathcal{P}[(6)3]$ ,

$$c = f - g = (f + g) - 2g = (bd) - 2(b-2)(d/2) = 2d.$$

For any  $P$  in  $\mathcal{P}[(4)4]$ , we have  $d = 4$ , so  $c(P) = 8$ . For any  $P$  in  $\mathcal{P}[(6)3]$ , we have  $d = 6$ , so  $c(P) = 12$ .

We next consider the lattice (3)6 with hexagons of unit area.

**LEMMA 1.** *Let  $Q$  be a polygon in  $\mathcal{C}[(3)6]$  whose interior is disjoint from the hexagonal tile  $P$ , and the common boundary  $P \cap Q$  is a path in the graph  $(V, E)$  of length at least one. (Thus  $P \cup Q$  is also in  $\mathcal{C}$ .) Then*

$$c(P \cup Q) = c(Q).$$

**Proof.** Consider how  $c(Q)$  is changed when the hexagon  $P$  is added to  $Q$  (see Fig. 4).

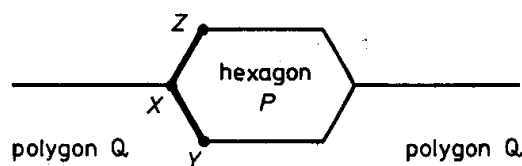


Fig. 4. Expanding polygon  $Q$  by one hexagon

At an end point  $X$  of the common border, the exterior edge  $XZ$  is lost and an interior edge  $XY$  is gained for a net decrease of 2 at each end point. At each of the other 4 vertex points of  $P$  (like  $Y$  or  $Z$  in Fig. 6) there is a net increase in  $c$  of one.

**THEOREM 2.** *If  $P \in \mathcal{C}[(3)6]$ , then  $A(P) = n/4$  and  $c(P) = 6$ .*

**Proof.** It is clear that  $c(P) = 6$  if  $P$  is a single hexagon. For each  $P$  in  $\mathcal{C}$  there is a *shelling* of  $P$  by regular lattice hexagons of unit area. That is, the lattice hexagons in  $P$  may be ordered in such a way that the union of the first  $k$  of them is a lattice polygon for each  $k$ . Use Lemma 1 to complete the induction argument. Thus  $c(P) = 6$  for all  $P$  in  $\mathcal{C}[(3)6]$ .

A similar inductive proof is the easiest way to show that  $A(P) = n/4$ . However, we will give a different argument which uses Theorem 1 and which will be useful later. We may embed the hexagonal lattice  $V[(3)6]$  in a triangular lattice  $V[(6)3]$  by adding the auxiliary set  $M$  of the centers of each hexagon. Since each original hexagon has unit area, the value of the Jacobian in the proof of Theorem 1 is  $k = 1/3$  for this new triangular lattice. The parameter  $i$  in Theorem 1 now becomes  $(i+h)$ , where  $i$  is the number of original (3)6-lattice points interior to  $P$ , and  $h$  is the number of auxiliary points of  $M$  interior to  $P$ . Note that no points of  $M$  lie on the boundary of  $P$ . Also  $A(P) = h$  since  $h$  gives the number of hexagons (each of unit area) in  $P$ . Thus, by our modified version of Theorem 1,

$$h = A(P) = (1/3)(b/2 + [i+h] - 1).$$

Solving this equation for  $h$  gives  $A(P) = n/4$  as desired.

We will next consider the more general class  $\mathcal{P}[(3)6]$  of polygons with vertices in the hexagonal lattice. Here the boundary characteristic  $c$  is no longer constant, as we have seen in Fig. 3, and it becomes a useful parameter in finding the area of certain lattice polygons. Extending the notation used in the proof of Theorem 2, we let  $M$  be the set of centers of the hexagons of (3)6, called the *auxiliary points*, when we embed (3)6 in the triangular tiling (6)3. Moreover, let  $h$  be the number of auxiliary points in the interior of  $P$ ,  $m$  be the number of auxiliary points on the boundary of  $P$ ,  $b' = b + m$  be the number of (6)3-lattice points on the boundary of  $P$ , and  $i' = i + h$  be the number of (6)3-lattice points in the interior of  $P$ .

**THEOREM 3.** *If  $P \in \mathcal{P}[(3)6]$ , then*

$$(1) \quad A(P) = (1/3)(b'/2 + i' - 1).$$

**Proof.** Let the Jacobian be  $k = 1/3$  in a modified Theorem 1.

We now come to a main theorem of this section, which, for a large class of hexagonal lattice polygons, gives the area as a function of only the boundary characteristic, and the number of lattice points in the interior and on the boundary of the polygon. To define this class, let  $\mathcal{D}[(3)6]$  denote the

class of all polygons in  $\mathcal{P}[(3)6]$  whose boundary consists of diagonals and/or edges of hexagons in (3)6.

THEOREM 4. *If  $P \in \mathcal{P}[(3)6]$ , then*

$$(2) \quad A(P) = (3n + c - 6)/12.$$

As an example, Theorem 4 asserts that the areas of triangles  $XYZ$ ,  $XZW$ , and  $XZU$  of Fig. 3c are, respectively,

$$(3+5-6)/12 = 1/6 \quad \text{and} \quad (3+7-6)/12 = 1/3,$$

and

$$(3+9-6)/12 = 1/2.$$

Also note that Theorem 2 is a special case of Theorem 4, since  $c = 6$  whenever  $P$  is in  $\mathcal{C}[(3)6]$ . By combining (1) and (2) it is easy to show that Theorem 4 is equivalent to the following result, which is a purely combinatorial result, and not dependent on the scale (area of a hexagonal tile) of the lattice.

THEOREM 4'. *For any  $P$  in  $\mathcal{P}[(3)6]$ ,*

$$c = 2m + 4h - b - 2i + 8.$$

**Proof of Theorem 4.** Referring to the triangles in Fig. 3c, each polygon  $P$  in  $\mathcal{P}$  is either congruent to the triangle  $XZU$  or else there is a shelling of  $P$  by lattice triangles congruent to  $XYZ$  and/or  $XZW$ . (Fig. 5 shows an example of such a shelling.) We have checked above that Theorem 4 holds for these three special triangles. Thus we may proceed with an induction argument using the shelling. Suppose a triangle congruent to  $XYZ$  is adjoined to a lattice polygon  $Q$ . If the triangle and  $Q$  have one edge in common, it is easy to show that in every case  $b(Q)$  is one less than  $b$  of the union, and  $i$  does not change. If the triangle and  $Q$  have two edges in common, then  $b(Q)$  is one more than  $b$  of the union, and  $i(Q)$  is one less than  $i$  of the union. In either case  $c(Q)$  is one more than  $c$  of the union. So adding the triangle increases  $n$  by one, decreases  $c$  by one, and thus increases the area (2) by  $1/6$ . Similarly, if a triangle congruent to  $XZW$  is adjoined to a lattice polygon  $Q$ , it is easy to show that in every case  $c$  and  $n$  both increase by 1, so the area of  $Q$  is increased by  $1/3$ . Thus induction on the shelling preserves the area (2) and the theorem is proved.

Clearly,  $\mathcal{C} \subset \mathcal{D} \subset \mathcal{P}$ . It would be interesting to characterize the class  $\mathcal{E}$  of lattice polygons for which (2) holds. The example in Fig. 6 shows that  $\mathcal{E} \neq \mathcal{P}$ . Theorem 4 applies to the parallelogram  $XYZW$ , where  $c = 6$ ,  $i = 3$ ,  $b = 8$ , and thus  $n = 12$  and the area is 3. However, the triangle  $XYZ$  of Fig. 6 is in  $\mathcal{P}$  but not  $\mathcal{D}$ , so Theorem 4 does not apply, its area is clearly  $1.5$ .

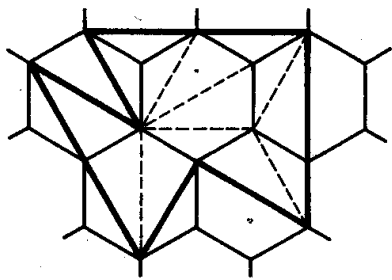
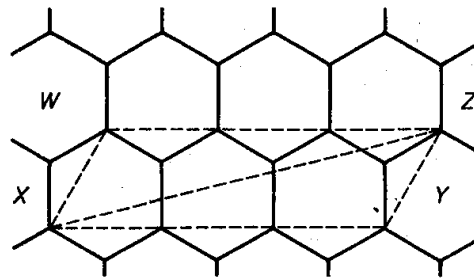
Fig. 5. Shelling in  $\mathcal{D}[(3)6]$ 

Fig. 6. A counterexample

(half of the area of the parallelogram), and  $c = 7$ ,  $i = 1$ ,  $b = 5$ . Thus, for the triangle  $XYZ$ ,

$$(3n + c - 6)/12 = 4/3$$

and equation (2) is not valid.

We next extend the above results from [3] to get a broader class of polygons satisfying (2). We will define a *lattice segment* to be a line segment  $S = [a, b]$  in a (3)6-tiling of the plane by hexagons of unit area, where  $[a, b] \cap V = \{a, b\}$ . There is a discrete increasing sequence  $\{a_1, a_2, \dots\}$  of all possible real values that may be realized as the length of a lattice segment. The numbers  $a_1, a_2, a_3$  are the lengths of an edge, a minor diagonal, and a major diagonal of a hexagon in the tiling, respectively. Lattice segments of lengths  $a_4$  to  $a_{11}$  are shown with solid lines in Fig. 7. The diagonal  $XZ$  in Fig. 6 also has length  $a_{11}$ . For each lattice segment of length  $a_i$  ( $4 \leq i \leq 11$ ) define its *alternative path* to be the path with the same end vertices as the segment, as shown with dotted lines in Fig. 7. The symmetry of the (3)6-tiling assures that the alternative path is well defined for any edge of length  $a_i$  ( $4 \leq i \leq 11$ ). Note that each alternative path consists of only edges and/or diagonals of the lattice hexagons. The vertices of each alternative path, except for the end vertices, are called the vertices *associated* with the lattice segment.

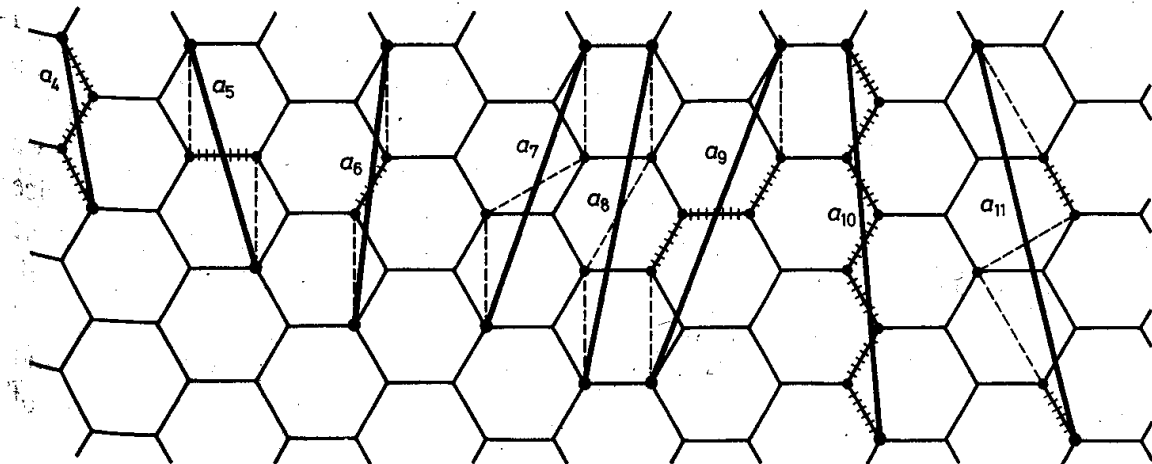


Fig. 7. The first few lattice segments and alternative paths

**THEOREM 5.** *If  $P \in \mathcal{P}[(3)6]$ , if each lattice segment in the boundary of  $P$  has length at most*

$$a_{10} = 3.77376\dots,$$

*and if  $P$  is sufficiently rotund so that no lattice vertex is associated with more than one lattice segment in  $\partial P$ , then*

$$A(P) = (3n + c - 6)/12.$$

**Proof.** Sequentially replace each lattice segment in the boundary of  $P$  by its alternative path if the length of the segment is greater than  $a_3$ . Since no lattice vertex is associated with more than one lattice segment in  $\partial P$ , this sequential replacement will always give a new polygon whose boundary is a simple closed curve. After all replacements are made, the resulting polygon  $P'$  will be in  $\mathcal{P}[(3)6]$ , and by Theorem 4 its area will be

$$A(P') = (3n + c - 6)/12.$$

Each alternative path for a lattice segment of length  $a_i$  for  $4 \leq i \leq 11$  is centrally symmetric about its midpoint. Thus the replacement of a lattice segment by its alternative path preserves the area of the polygon. It follows that  $A(P) = A(P')$ . The proof is completed by observing that the parameters  $n$  and  $c$  remain invariant under each replacement of a lattice segment by its alternative path, provided the length of the path is at most  $a_{10}$ . (It is easy to check that the last statement is not true if the lattice segment has length  $a_{11}$ ;  $n$  is invariant but  $c$  is not.)

In view of the above theorem, we see that the counterexample given in Fig. 6 is, in a certain sense, the best possible.

**4. Related results and applications.** Since its appearance in 1899, the theorem of Pick [9] has been proved in many ways; see [1], [2], [4], [7], [8], [14]. The simplicity and elegant beauty of the theorem have led to many articles of a popular nature in journals concerning the teaching of mathematics. Gaskell et al. [5], among others, point out that Pick's theorem is a topological result, and not dependent on areas. A number of authors ([2], [4], [7], [10]) have shown that it is equivalent to Euler's theorem or have used the Euler–Poincaré characteristic in generalizations to polygons whose boundaries are not simple closed polygonal curves with vertices at the lattice points. Scott [13] has established a number of inequalities between the parameters  $b$  and  $i$  for various classes of polygons. Reeve [10] has generalized Pick's theorem to 3-dimensional lattice polyhedra, but was forced to use an adjunct lattice as well as the lattice which contains the vertices. Ding and Reay [3] extend Pick's theorem to lattice polyhedra that are the union of tiles for each of the 11 Archimedean tilings of the plane.



The paper by Grünbaum and Shephard [6] is an excellent introduction to results and open problems concerning the Archimedean and other regular tilings.

We next suggest a possible application to computer graphics. Computer-generated images, in a raster-scan display, frequently have characteristic stair-step or jagged polygonal edges. To improve the display of an image, the resolution of the raster can be increased. But there is a practical limit of about 2000 pixels per scan line in present CRT raster-scan devices. The display of smooth boundaries can also be improved by varying the color or the intensity at each pixel that is on, or very near, the boundary of the area displayed. If we think of each display pixel as a region (usually rectangular or hexagonal), its intensity or color may be made a function of the area of the intersection of the pixel region and the image being displayed. The area of this intersection may be computed in several ways. If a precise mathematical expression for the boundary of the image is known, then one standard algorithm uses convolution integrals whose kernels depend on the display pixel. A second technique is to tile the display pixel with subpixels to form a higher resolution raster. Each subpixel is then determined to be either within or outside of the image. The set of subpixels within the image therefore is a polygon in  $\mathcal{C}$ , and the theorems of this paper could be used to find its area if the subpixels came from the hexagonal tiling. See Chapter 2 of [11] for more detailed descriptions and further references.

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HABEI TEACHER'S UNIVERSITY  
SHIJIAZHUANG  
PEOPLE'S REPUBLIC OF CHINA

WESTERN WASHINGTON UNIVERSITY  
BELLINGHAM, WASHINGTON  
U.S.A.

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