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OPTIMAL METHODS OF INTEGRATION IN THE CLASS OF DIFFERENTIABLE FUNCTIONS

1. Introduction. Denote by $W^{(r)}L_q(M; a, b)$ $(1 \leqslant q \leqslant \infty)$ the class of differentiable functions defined on the finite interval $[a, b]$ with piecewise continuous r-th derivative satisfying the condition

$$\Big\{\int_a^b |f^{(r)}(t)|^q dt\Big\}^{1/q} \leqslant M.$$

The purpose of this paper is to construct the optimal, in certain sense, method of approximation of the integral

$$I(f) = \int_a^b f(x)dx \quad \text{in } W^{(r)}L_q(M; a, b)$$

among all methods using as information the values of the function f and its derivatives $f', f'', \ldots, f^{(r-1)}$ at n distinct points in (a, b).

An analogous problem was studied in many works (see, e.g., [1] and [3]-[8]). But in all these papers the authors search the optimum linear method assuming additionally that

(i) it is exact for the polynomials of degree $r-1$, or

(ii) one considers only the functions $f \in W^{(r)}L_q(M; a, b)$ satisfying $f(a) = f'(a) = \ldots = f^{(r-1)}(a) = 0$.

In the present paper the optimization is extended over all admissible methods of integration without any restrictions.

2. Preliminary results and definitions. Let H be a linear metric space and F a convex centrally symmetrical set in H with the centre of symmetry 0. Let the real linear functionals $L(f), L_1(f), \ldots, L_N(f)$ be defined on F. An arbitrary method of approximation of the functional $L(f)$ using as information only the values $L_k(f)$ $(k = 1, 2, \ldots, N)$ can be given by a function S of N variables as follows:

$$L(f) \approx S\big(L_1(f), \ldots, L_N(f)\big).$$

Denote, for convenience, the approximate value $S(L_1(f), \ldots, L_N(f))$ of $L(f)$ by $S[f]$. The quantity

$$R(S; L_1, \ldots, L_N) = \sup_{f \in F} |L(f) - S[f]|$$

is called the *error of the method* S in the class F. The method S^* is said to be the *best* if

$$R(S^*; L_1, \ldots, L_N) = \inf_{S} R(S; L_1, \ldots, L_N)$$

holds and $\inf\limits_{S}$ is extended over all admissible methods of approximation of $L(f)$ that use only $L_k(f)$ $(k = 1, 2, \ldots, N)$.

Now we formulate a lemma due to Smolak [9] (see also [2]), needed in the sequel.

LEMMA 1. *Suppose that* $\sup\limits_{f \in F_0} L(f) < \infty$, *where*

$$F_0 \equiv \{f\colon f \in F,\ L_k(f) = 0\ (k = 1, 2, \ldots, N)\}.$$

Then there exist numbers $D_1, D_2, \ldots, D_N$ *such that*

$$\sup_{f \in F} \Big| L(f) - \sum_{k=1}^{N} D_k L_k(f) \Big| = \inf_{S} \sup_{f \in F} |L(f) - S[f]|,$$

i.e., there exists a linear method with a least estimate of the error.

The next two corollaries follow immediately from the proof of the lemma (see also [2]).

COROLLARY 1. *We have*

$$\sup_{f \in F_0} L(f) = R(S^*; L_1, \ldots, L_N).$$

Let ε be a fixed number and let $1 \leqslant k \leqslant N$. Write

$$F_\varepsilon^k \equiv \{f\colon f \in F, L_k(f) = \varepsilon, L_i(f) = 0, i \neq k\},$$

and assume that

$$\varphi_k(\varepsilon) = \sup_{f \in F_\varepsilon^k} L(f).$$

COROLLARY 2. *If* $\varphi_k'(0)$ *exists, then* $D_k = \varphi_k'(0)$.

Henceforth we concentrate on the special functional

$$I(f) = \int_a^b f(x)\,dx.$$

Finally, write

$$W_a \equiv \{f\colon f \in W^{(r)}L_q(M;\, a,\, b),\, f^{(k)}(a) = 0 \ (k = 0, 1, \ldots, r-1)\},$$

$$W_{ab} \equiv \{f\colon f \in W^{(r)}L_q(M;\, a,\, b),\, f^{(k)}(a) = f^{(k)}(b) = 0 \ (k = 0, 1, \ldots, r-1)\}.$$

We shall use the notation W instead of $W^{(r)}L_q(M;\, a,\, b)$ in some formulas.

3. Best quadrature formulas with one and two knots. Throughout this section we shall denote by p the number for which $1/p+1/q = 1$.

THEOREM 1. *The quadrature formula*

$$I(f) \approx \sum_{k=0}^{r-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) \tag{1}$$

is a best method of integration in $W^{(r)}L_q(M;\, a,\, b)$ *among the methods that use as information only the values* $f^{(k)}(a)$ $(k = 0, 1, \ldots, r-1)$. *The error of the quadrature has the value*

$$\frac{M}{r!} \frac{(b-a)^{r+1/p}}{\sqrt[p]{rp+1}}.$$

Proof. By Corollary 1,

$$R = \sup_{f \in W_a} I(f).$$

Clearly, W_a is the class of all functions of the form

$$\frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} h(t)\, dt,$$

where $h(t)$ is a piecewise continuous function satisfying

$$\left\{\int_a^b |h(t)|^q dt\right\}^{1/q} \leqslant M. \tag{2}$$

Hence

$$R = \sup_h \int_a^b \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} h(t)\, dt\, dx$$

$$= \sup_h \int_a^b \frac{(b-t)^r}{r!} h(t)\, dt = \frac{M}{r!} \left(\int_a^b (b-t)^{rp} dt\right)^{1/p}$$

$$= \frac{M}{r!} \frac{1}{\sqrt[p]{rp+1}} (b-a)^{r+1/p}.$$

In order to determine the coefficients C_k $(k = 0, 1, \ldots, r-1)$ of quadrature (1) we use Corollary 2 and the Taylor formula. It is clear that the class

$$F_\varepsilon^k \equiv \{f\colon f\in W^{(r)}L_q(M; a, b), f^{(k)}(a) = \varepsilon,\ f^{(i)}(a) = 0,\ i \neq k\}$$

coincides with the class of all functions of the form

$$\frac{(x-a)^k}{k!}\varepsilon + \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1} h(t)\,dt,$$

where $h(t)$ is defined as in (2). Thus

$$\sup_{f\in F_\varepsilon^k} I(f) = \sup_h \left\{\int_a^b \left(\frac{(x-a)^k}{k!}\varepsilon + \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1} h(t)\,dt\right)dx\right\}$$

$$= \varepsilon\,\frac{(b-a)^{k+1}}{(k+1)!} + \frac{M}{r!}\left(\int_a^b (b-t)^{rp}\,dt\right)^{1/p}.$$

Differentiating the right-hand side of this equality and putting $\varepsilon = 0$, we get $C_k = (b-a)^{k+1}/(k+1)!$. This completes the proof.

In the same way we can prove the following

THEOREM 1′. *The quadrature formula*

$$I(f) \approx \sum_{k=0}^{r-1} (-1)^k \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(b)$$

is a best method of integration in $W^{(r)}L_q(M; a, b)$ among the methods that use as information only the values $f^{(k)}(b)$ $(k = 0, 1, \ldots, r-1)$. The error of the quadrature is the same as in Theorem 1.

Remark 1. Theorems 1 and 1′ show that the integration of the r-th partial sum of the Taylor expansion of the function is a best method of integration in $W^{(r)}L_q(M; a, b)$.

Throughout this paper we use the notation $U_{rp}(t; [a, b])$ to denote the r-th polynomial of the best L_p-approximation of zero in $[a, b]$. We have

$$\left\{\int_a^b |U_{rp}(t; [a, b])|^p\,dt\right\}^{1/p} = \inf_{\{a_k\}}\left\{\int_a^b |t^r + a_1 t^{r-1} + \ldots + a_r|^p\,dt\right\}^{1/p} = E_{rp}[a, b].$$

For simplicity, $U_{rp}(t) = U_{rp}(t; [-1, 1])$, and $E_{rp} = E_{rp}[-1, 1]$.

THEOREM 2. *The quadrature formula*

$$I(f) \approx \sum_{k=0}^{r-1} \left(A_k f^{(k)}(a) + B_k f^{(k)}(b)\right),$$

where

$$A_k = \frac{(-1)^{k+1}}{r!}\left(\frac{b-a}{2}\right)^{k+1} U_{rp}(-1) \quad and \quad B_k = \frac{(-1)^k}{r!}\left(\frac{b-a}{2}\right)^{k+1} U_{rp}(1),$$

is a best method of integration in $W^{(r)}L_q(M; a, b)$ among the methods that use as information only the values $f^{(k)}(a)$ and $f^{(k)}(b)$ $(k = 0, 1, \ldots, r-1)$. The error R of the quadrature has the value

$$R = \frac{M}{r!}\left(\frac{b-a}{2}\right)^{r+1/p} E_{rp}.$$

First, we prove

LEMMA 2. *We have*

$$\sup_{f \in W_{ab}} \int_a^b f(x)\,dx = \frac{M}{r!}\left\{\int_a^b |U_{rp}(t; [a, b])|^p dt\right\}^{1/p}.$$

Proof. According to Lemma 1, we have

$$R = \sup_{f \in W_{ab}} \int_a^b f(x)\,dx = \inf_{\{B_k\}} \sup_{f \in W_a}\left(\int_a^b f(x)\,dx - \sum_{k=0}^{r-1} B_k f^{(k)}(b)\right).$$

But it can be shown that

$$f^{(k)}(b) = \frac{1}{(r-k-1)!}\int_a^b (b-t)^{r-k-1} f^{(r)}(t)\,dt \qquad (k = 0, 1, \ldots, r-1).$$

That gives

$$R = \inf_{\{B_k\}} \sup_{f \in W_a} \int_a^b \left(\frac{(b-t)^r}{r!} - \sum_{k=0}^{r-1} \frac{B_k}{(r-k-1)!}(b-t)^{r-k-1}\right) f^{(r)}(t)\,dt$$

$$= \inf_{\{B_k\}} \frac{M}{r!}\left\{\int_a^b \left|(b-t)^r - \sum_{k=0}^{r-1} \frac{B_k r!}{(r-k-1)!}(b-t)^{r-k-1}\right|^p dt\right\}^{1/p}$$

$$= \frac{M}{r!}\left\{\int_a^b |U_{rp}(t; [a, b])|^p dt\right\}^{1/p}.$$

The proof of the lemma is complete.

Proof of Theorem 2. By Lemma 2 and an easy computation we get

$$R = \frac{M}{r!}\left(\frac{b-a}{2}\right)^{r+1/p} E_{rp}.$$

Denote by σ the set of all methods of integration of f that use only the values $f^{(k)}(a)$ and $f^{(k)}(b)$ $(k = 0, 1, \ldots, r-1)$, and write

$$S_0[f] = \sum_{k=0}^{r-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a).$$

We have

$$R = \inf_{S\in\sigma} \sup_{f\in W} |I(f) - S[f]|$$
$$= \inf_{S\in\sigma} \sup_{f\in W} \left| \int_a^b \frac{(b-t)^r}{r!} f^{(r)}(t)\,dt - (S[f] - S_0[f]) \right|.$$

Notice that the set of methods S_1 of the form $S_1 = S - S_0$, where $S \in \sigma$, coincides with σ. Thus

$$R = \inf_{S_1\in\sigma} \sup_{f\in W} \left| \int_a^b \frac{(b-t)^r}{r!} f^{(r)}(t)\,dt - S_1[f] \right|.$$

Let $P(t)$ be a polynomial with real coefficients of degree less than or equal to $r-1$, i.e. $P(t) \in \pi_{r-1}$. Integration by parts produces the formula

$$\int_a^b P(t) f^{(r)}(t)\,dt = \sum_{k=0}^{r-1} (-1)^{r-k-1} \big(P^{(r-k-1)}(b) f^{(k)}(b) - P^{(r-k-1)}(a) f^{(k)}(a)\big).$$

Therefore, the method

$$I(f) \approx \int_a^b P(t) f^{(r)}(t)\,dt$$

belongs to σ. That gives

$$R \leqslant \inf_{P\in\pi_{r-1}} \sup_{f\in W} \left| \int_a^b \frac{(b-t)^r}{r!} f^{(r)}(t)\,dt - \frac{1}{r!} \int_a^b P(t) f^{(r)}(t)\,dt \right|$$
$$= \inf_{P\in\pi_{r-1}} \frac{M}{r!} \left\{ \int_a^b |(b-t)^r - P(t)|^p\,dt \right\}^{1/p} = \frac{M}{r!} \left\{ \int_a^b |U_{rp}(t; [a, b])|^p\,dt \right\}^{1/p}.$$

This inequality and Lemma 2 show that the method $S^* = S_1 + S_0$, where

$$S_1[f] = \frac{(-1)^r}{r!} \int_a^b \big((t-b)^r - U_{rp}(t; [a, b])\big) f^{(r)}(t)\,dt,$$

is the best one. Since

$$\frac{1}{r!}\int_a^b (b-t)^r f^{(r)}(t)\,dt = I(f)-S_0[f],$$

we obtain

$$S^*[f] = S_1[f]+S_0[f] = -\frac{(-1)^r}{r!}\int_a^b U_{rp}(t;[a,b])f^{(r)}(t)\,dt+I(f)$$

$$= \sum_{k=0}^{r-1}(-1)^k\big(U_{rp}^{(r-k-1)}(b;[a,b])f^{(k)}(b)-U_{rp}^{(r-k-1)}(a;[a,b])f^{(k)}(a)\big)$$

which completes the proof of the theorem.

COROLLARY 3. *The quadrature formula given in Theorem 2 is precise for polynomials of degree not greater than* $r-1$.

Indeed, we have

$$I(f)-S^*[f] = \frac{(-1)^r}{r!}\int_a^b U_{rp}(t;[a,b])f^{(r)}(t)\,dt$$

and the corollary follows.

4. Optimal quadrature formula. The methods using as information only the values $f^{(k)}(x_i)$ $(i = 1, 2, \ldots, n;\ k = 0, 1, \ldots, r-1)$ will be considered in this section and the knots $\{x_k\}_1^n$ will be required to satisfy only

$$a < x_1 < x_2 < \ldots < x_n < b.$$

THEOREM 3. *The quadrature formula*

$$I(f) \approx \sum_{k=0}^{r-1}\sum_{i=1}^{n} A_i^{(k)} f^{(k)}(x_i), \tag{3}$$

where

(4)

$$\begin{cases} A_1^{(k)} = -\dfrac{(a-x_1)^{k+1}}{(k+1)!}+\dfrac{(-1)^{k+1}}{r!}\left(\dfrac{x_2-x_1}{2}\right)^{k+1}U_{rp}^{(r-k-1)}(-1),\\[2ex] A_n^{(k)} = \dfrac{(b-x_n)^{k+1}}{(k+1)!}+\dfrac{(-1)^k}{r!}\left(\dfrac{x_n-x_{n-1}}{2}\right)^{k+1}U_{rp}^{(r-k-1)}(1),\\[2ex] A_i^{(k)} = \dfrac{(-1)^k}{r!}\left[\left(\dfrac{x_i-x_{i-1}}{2}\right)^{k+1}U_{rp}^{(r-k-1)}(1)-\left(\dfrac{x_{i+1}-x_i}{2}\right)^{k+1}U_{rp}^{(r-k-1)}(-1)\right],\end{cases}$$

is best among all admissible methods of this type. The error of the quadrature has the value

$$(5)\quad R(x_1, \ldots, x_n) = \frac{M}{r!}\left\{\frac{(x_1-a)^{rp+1}}{rp+1}+\frac{(b-x_n)^{rp+1}}{rp+1}+\sum_{k=1}^{n-1}\left(\frac{x_{k+1}-x_k}{2}\right)^{rp+1}E_{rp}^{p}\right\}^{1/p}.$$

Proof. In order to construct the best method (3) we divide the integral $I(f)$ into $n+1$ parts

$$I(f)=\int_a^{x_1} f(x)\,dx+\int_{x_n}^{b} f(x)\,dx+\sum_{k=1}^{n-1}\int_{x_k}^{x_{k+1}} f(x)\,dx,$$

and then apply the best methods given by Theorems 1 and 1′ for the first two integrals and the best method of Theorem 2 to the others. It is easily seen that the method is of type (3). The coefficients $A_i^{(k)}$ are sums of the corresponding coefficients calculated in Theorems 1, 1′ and 2.

Denote by $R^*(x_1, \ldots, x_n)$ the error of this method in $W^{(r)}L_q(M; a, b)$. For convenience,

$$x_0 = b,\qquad x_{n+1} = b,$$

$$e_0=\frac{1}{r!}\,\frac{(x_1-a)^{r+1/p}}{\sqrt[p]{rp+1}},\qquad e_n=\frac{1}{r!}\,\frac{(b-x_n)^{r+1/p}}{\sqrt[p]{rp+1}},$$

$$e_k=\frac{1}{r!}\left(\frac{x_{k+1}-x_k}{2}\right)^{r+1/p}E_{rp}\qquad (k=1,2,\ldots,n-1).$$

It follows from the previous sections that

$$R^*(x_1, \ldots, x_n)=\sup_{f\in W}\sum_{k=0}^{n} e_k\left\{\int_{x_k}^{x_{k+1}} |f^{(r)}(t)|^q dt\right\}^{1/q}.$$

By the Bunjakovski-Schwartz inequality,

$$(6)\quad R^*(x_1, \ldots, x_n)\leqslant\left\{\sup_{f\in W}\left[\sum_{k=0}^{n} e_k^p\right]^{1/p}\right\}\left\{\int_a^b |f^{(r)}(t)|^q dt\right\}^{1/q}=M\left\{\sum_{k=0}^{n} e_k^p\right\}^{1/p}.$$

On the other hand, by Corollary 1, the best method in $W^{(r)}L_q(M; a, b)$ has the error $R(x_1, \ldots, x_n)$ greater than $\int_a^b f(x)\,dx$ for all f such that

$$f^{(k)}(x_i)=0\qquad (k=0,1,\ldots,r-1;\ i=1,2,\ldots,n).$$

Let $\varepsilon < 1$ be an arbitrary positive number. By Corollary 1, there exists a function $g_i(x) \in W^{(r)}L_q(M_i; x_i, x_{i+1})$ for which

$$\int_{x_i}^{x_{i+1}} g_i(t)\,dt \geqslant \varepsilon M_i e_i,$$

$$g_i^{(k)}(x_i) = g^{(k)}(x_{i+1}) = 0 \quad (k = 0, 1, \ldots, r-1) \quad \text{for } i = 1, 2, \ldots, n-1,$$

and

$$g_0^{(k)}(x_1) = g_n^{(k)}(x_n) = 0.$$

$M_i e_i$ is the error of the best method in the class $W^{(r)}L_q(M_i; x_i, x_{i+1})$ $(i = 0, 1, \ldots, n)$ (described in Theorems 1, 1′ and 2, respectively). Determine M_i by

$$M_i = \frac{M e_i^{p-1}}{\left(\sum\limits_{i=0}^{n} e_i^p\right)^{1/q}}.$$

The function $g(x) = g_i(x)$ for $x_i \leqslant x \leqslant x_{i+1}$ belongs to the class $W^{(r)}L_q(M; a, b)$. Indeed, by the definition of g_i, we have

$$\left\{\int_{x_i}^{x_{i+1}} |g^{(r)}(t)|^q\,dt\right\}^{1/q} \leqslant M_i.$$

Then

$$\left\{\int_a^b |g^{(r)}(t)|^q\,dt\right\}^{1/q} = \left\{\sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} |g_i^{(r)}(t)|^q\,dt\right\}^{1/q} \leqslant \left\{\sum_{i=0}^{n} M_i^q\right\}^{1/q} = M.$$

Since $g(t)$ satisfies the conditions

$$g^{(k)}(x_i) = 0 \quad (k = 0, 1, \ldots, r-1;\ i = 1, 2, \ldots, n),$$

by Corollary 1 we get

$$R(x_1, \ldots, x_n) \geqslant \int_a^b g(t)\,dt \geqslant \sum_{i=0}^{n} \varepsilon M_i e_i = \varepsilon M \left\{\sum_{i=0}^{n} e_i^p\right\}^{1/p}.$$

This inequality and (6) complete the proof of the theorem.

We notice that, according to Remark 1 and Corollary 3, the best method of Theorem 3 is precise for piecewise polynomial functions with discontinuity at the knots.

Now, we minimize the error $R(x_1, \ldots, x_n)$ varying the knots $\{x_k\}_1^n$ in (a, b). Let

$$R(\xi_1, \ldots, \xi_n) = \inf_{\{x_k\}} R(x_1, \ldots, x_n).$$

The best method of integration with these extremal knots is said to be *optimal*.

THEOREM 4. *The quadrature formula*

$$I(f) \approx \sum_{k=0}^{r-1} \sum_{i=1}^{n} C_i^{(k)} f^{(k)}(\xi_i),$$

where

(7)

$$\begin{cases} \xi_i = a + A(r,p)h + 2(i-1)h \quad (i = 1, 2, \ldots, n), \\ h = \frac{b-a}{2}\big(n-1+A(r,p)\big)^{-1} \quad A(r,p) = \left(\frac{E_{rp}^p}{2}(rp+1)\right)^{1/rp}, \\ C_i^{(k)} = \frac{(-1)^k}{r!} h^{k+1}\big(U_{rp}^{(r-k-1)}(1) - U_{rp}^{(r-k-1)}(-1)\big) \quad (i = 2, 3, \ldots, n-1), \end{cases}$$

$$C_1^{(k)} = \frac{(-1)^k}{(k+1)!}\big(A(r,p)h\big)^{k+1} + \frac{(-1)^{k+1}}{r!} h^{k+1} U_{rp}^{(r-k-1)}(-1),$$

$$C_n^{(k)} = \frac{1}{(k+1)!}\big(A(r,p)h\big)^{k+1} + \frac{(-1)^k}{k!} h^{k+1} U_{rp}^{(r-k-1)}(1),$$

is the optimal method of integration in $W^{(r)}L_q(M; a, b)$. The error of the method is

$$R(\xi_1, \ldots, \xi_n) = \frac{M}{r!} h^r \left(\frac{b-a}{2}\right)^{1/p} E_{rp}.$$

Proof. It follows from (5) that the extremal knots satisfy the equations

$$(x_1 - a)^{rp} - (rp+1)\left(\frac{x_2 - x_1}{2}\right)^{rp} \frac{E_{rp}^p}{2} = 0,$$

$$-(b - x_n)^{rp} + (rp+1)\left(\frac{x_n - x_{n-1}}{2}\right)^{rp} \frac{E_{rp}^p}{2} = 0,$$

$$x_{i+1} - x_i = x_i - x_{i-1} \quad (i = 2, 3, \ldots, n-1).$$

The system has a unique solution $\{\xi_i\}_1^n$ determined by

$$\xi_{i+1} - \xi_i = 2h \quad (i = 2, 3, \ldots, n-1),$$

$$\xi_1 = a + A(r,p)h, \quad \xi_n = b - A(r,p)h,$$

$$\xi_1 - a + \sum_{k=1}^{n-1} (\xi_{k+1} - \xi_k) + b - \xi_n = b - a,$$

which produced (7).

The proof of the theorem is completed by inserting the values of $\{\xi_k\}_1^n$ in (4) and (5).

Since $E_{rp} = \big(2/(rp+1)\big)^{1/p} U_{rp}(1)$, it is seen that $A(r,p) = \big(U_{rp}(1)\big)^{1/r}$ which implies $C_1^{(r-1)} = C_n^{(r-1)} = 0$ if r is an even integer. Note else that $C_i^{(k)} = 0$ $(i = 2, 3, \ldots, n-1)$ if k is an odd integer.

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OPTYMALNE METODY CAŁKOWANIA W KLASIE FUNKCJI RÓŻNICZKOWALNYCH

STRESZCZENIE

Rozpatrywana jest klasa funkcji $f(x)$, określonych na skończonym przedziale $[a, b]$ i takich, że $r-1$ pochodna jest przedziałami ciągła w tym przedziale, a r-ta pochodna spełnia warunek $\|f^{(r)}\|_{L_q} \leqslant M$. W pracy konstruuje się sposób całkowania funkcji $f \in W^{(r)}L_q(M; a, b)$, optymalny wśród wszystkich metod, które jako informację wykorzystują wartości $f^{(k)}(x_i)$ $(k = 0, 1, \ldots, r-1;\ i = 1, 2, \ldots, n)$, gdzie $a \leqslant x_1 < x_2 < \ldots < x_n \leqslant b$. Praca zawiera także dokładne oszacowanie błędów metody optymalnej. Okazuje się, że otrzymana w pracy optymalna metoda całkowania jest metodą liniową i dokładną dla wielomianów stopnia co najwyżej $r-1$. Praca stanowi zatem kontynuację prac [4] i [6].