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A GENERALIZATION OF BRANCH WEIGHT CENTROIDS*

Abstract. The *weight* of a point p in a finite aligned space $(X, \mathscr{C})$ is defined as

$$\mathrm{wt}(p) = \max\{|K|:\ p \notin K \text{ and } K \in \mathscr{C}\}$$

and the *centroid of* $(X, \mathscr{C})$ is the set of points with the minimum weight. We prove that every centroid is convex and it is a free set in a convex geometry. We define the *centroid of a graph* G as the centroid of the monophonic alignment of G and we show that the centroid of a connected chordal graph induces a complete subgraph.

1. Introduction. In a simple connected graph $G = (V, E)$, the *distance* between vertices u and v, denoted by $d(u, v)$, is the smallest number of edges in a path connecting u and v. Then the *distance of vertex* u is defined by

$$d(u) = \sum_{v \in V} d(u, v),$$

and the *median* of G is the set of those vertices v for which $d(v)$ is minimal.

If T is a tree and $u \in V(T)$, then the *branch weight* of u, denoted by $\mathrm{bw}(u)$, is the largest number of vertices in a component of $T \setminus \{u\}$. Evidently,

$$\mathrm{bw}(u) = \max\{|K|:\ u \notin K \text{ and } K \in \mathscr{C}\}, \tag{1}$$

where $\mathscr{C}$ is the family of subsets $K \subseteq V(T)$ such that the induced subgraph $T[K]$ is connected. The *branch weight centroid* of T is the set of vertices for which the function bw has the minimal possible value.

The concept of a branch weight centroid has been extended by Slater in two different ways so that it can be defined for arbitrary graphs. The definition of security center [7] is based upon the idea that "vertex u is more central than vertex v if u is closer to more vertices than v is". The accretion center [8] is defined using the notion of a sequential labelling. For a tree, these two centers coincide with the branch weight centroid.

The above notions are closely related to the problem of optimal location of facilities in a graph, where an optimal location is assumed to be in a sense "central".

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In this paper we introduce yet another measure of centrality in a very general case. Namely, we define a centroid for an arbitrary finite set X with a distinguished family $\mathscr{C}$ of "convex" subsets of X. Then we consider the centroid for the chordless path convexity in any graph and show that, for a connected chordal graph G, the centroid induces a complete subgraph of G. This generalizes the well-known Jordan's theorem [6].

2. Centroids of aligned spaces. An *alignment* on a finite set X is a family $\mathscr{C}$ of subsets of X, which is closed under intersections and which contains both X and the empty set. The pair $(X, \mathscr{C})$ is called an *aligned space* and members of $\mathscr{C}$ are regarded as *convex* subsets of X. The smallest convex set containing a set $S \subseteq X$ is denoted by $c(S)$ and called a (convex) *hull* of S.

The *weight* of a point p in an aligned space $(X, \mathscr{C})$ is defined as

$$\mathrm{wt}(p) = \max\{|K|\colon\ p \notin K \text{ and } K \in \mathscr{C}\}, \tag{2}$$

and then the *centroid* of $(X, \mathscr{C})$ consists of all points with the smallest weight.

Let us put $A_m = \{x \in X\colon\ \mathrm{wt}(x) \leqslant m\}$ for all $m \in N$.

LEMMA 1. *The set A_m is convex for all $m \in N$.*

Proof. Let $x \in c(A_m)$ and suppose that K is a convex set such that $x \notin K$ and $|K| = \mathrm{wt}(x)$. The set A_m is not a subset of K, since otherwise $x \in c(A_m) \subseteq c(K) = K$. So, there is an element $a \in A_m \setminus K$ and we have

$$m \geqslant \mathrm{wt}(a) \geqslant |K| = \mathrm{wt}(x).$$

Thus $x \in A_m$, which completes the proof.

An element p of a set $Y \in \mathscr{C}$ is an *extreme point* of Y if $Y \setminus \{p\}$ is convex. The set of all extreme points of Y is denoted by $\mathrm{ext}\, Y$. A set $S \in \mathscr{C}$ is *free* if $S = \mathrm{ext}\, S$ (or, equivalently, every subset of S is convex). A *convex geometry* (antimatroid [1], extremally detachable space [4]) is an aligned space satisfying the following additional property:

KREIN-MILMAN PROPERTY. *Every convex set is the hull of its extreme points.*

LEMMA 2. *If $(X, \mathscr{C})$ is a convex geometry, then $A_m \setminus A_{m-1} \subseteq \mathrm{ext}\, A_m$.*

Proof. Let $x \in A_m \setminus A_{m-1}$ and suppose on the contrary that $x \in c(A_m \setminus \{x\})$. If K is a convex set such that $x \notin K$ and $|K| = \mathrm{wt}(x)$, then there exists an element $a \neq x$ such that $a \in A_m \setminus K$, since otherwise

$$x \in c(A_m \setminus \{x\}) \subseteq c(K) = K.$$

It is a simple observation (cf. [3]) that, in a convex geometry, if S is a maximal convex set not containing an element u, then the set $S \cup \{u\}$ is convex. Using this observation we obtain

$$m \geqslant \mathrm{wt}(a) \geqslant |K \cup \{x\}| = |K| + 1 = \mathrm{wt}(x) + 1,$$

which contradicts the assumption $\mathrm{wt}(x) = m$. Therefore $x \notin c(A_m \setminus \{x\})$ and, consequently, x is an extreme point of A_m.

THEOREM 1. *The centroid of an aligned space $(X, \mathscr{C})$ is always convex. Moreover, if $(X, \mathscr{C})$ is a convex geometry, then its centroid is a free set.*

Proof. If $m_0 = \min\{\mathrm{wt}(p)\colon p \in X\}$, then the centroid of $(X, \mathscr{C})$ coincides with the set A_{m_0}. By Lemma 1, A_{m_0} is convex.

If $(X, \mathscr{C})$ is a convex geometry, then, according to Lemma 2,

$$A_{m_0} \setminus A_{m_0 - 1} \subseteq \mathrm{ext}\, A_{m_0}.$$

Since $A_{m_0 - 1} = \emptyset$, we have $A_{m_0} \subseteq \mathrm{ext}\, A_{m_0}$. The converse inclusion is always true.

3. Centroids of graphs. Let G be a simple graph (undirected without loops and multiple edges). A path P in G is *chordless* if the only pairs of vertices in P that are adjacent in G are consecutive along P. A set K of vertices is *monophonically convex* if K contains every vertex on every chordless path between vertices in K. A graph G is *chordal* if it contains no cycle of length greater than 3 as an induced subgraph.

THEOREM 2 (Jamison [2]). *The monophonic alignment of a graph G is a convex geometry if and only if G is chordal.*

A vertex v is called *simplicial* if its neighborhood induces a complete subgraph. It is not difficult to see that v is an extreme point of a monophonically convex set K if and only if v is simplicial in $G[K]$ (the graph induced by K). We now define a *centroid of a graph* G to be the centroid of the monophonic alignment of G. Note that if G is a tree, then the centroid defined above coincides with the branch weight centroid. This follows from the fact that in trees monophonically convex sets are exactly connected subgraphs.

THEOREM 3. *The centroid of a graph G is a monophonically convex set. Moreover, if G is a connected chordal graph, then the centroid of G is a complete subgraph.*

Proof. By Theorems 1 and 2, the centroid of G is monophonically convex, and if G is chordal, it is a free set in the monophonic alignment of G. Moreover, it is easy to see that a connected graph, whose every vertex is simplicial, must be complete.

COROLLARY (Jordan [6]). *The branch weight centroid of a tree consists of one vertex or two adjacent vertices.*

Zelinka proved in [9] that branch weight centroids and medians coincide in trees. The graph in Fig. 1 shows that, in general, centroids and medians may be different.

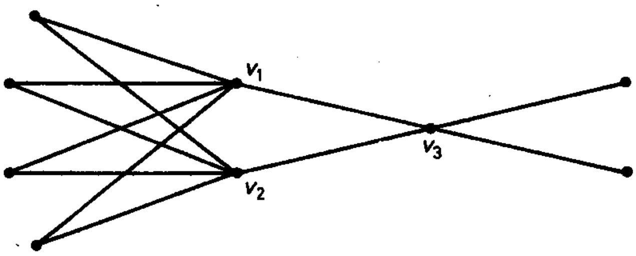

Fig. 1. A graph with the median $\{v_1, v_2\}$ and the centroid $\{v_3\}$

4. Conclusions. If in the above definition of convexity on graphs we replace "chordless" by "shortest", we get the geodesic alignment, which may appear to be a more natural notion of convexity. However, the one we prefer is often more useful. For example, if the geodesic alignment of G is a convex geometry, then it coincides with the monophonic alignment of G (cf. [2]). Moreover, it was shown [5] that, for the monophonic alignment in any graph G, the Helly number equals the size of a maximum clique of G, while even bipartite graphs can have arbitrarily large Helly numbers in the geodesic alignment.

Although we concentrated our attention on graphs, discussion in Section 2 leads to an easy description of centroids in other convex geometries such as order and semilattice alignments or finite subsets of Euclidean space.

We conclude this paper with three standard problems in the area of investigation of "central" subgraphs of a graph:

1. Which graphs may occur as centroids?
2. Which graphs are selfcentroidal?
3. Describe centroids in particular classes of graphs.

Some answers to these problems will appear elsewhere.

Remark. The author is indebted to the referee for his improvements. The referee has suggested the following interesting problem:

In a convex geometry X, define recursively

$$J^{n+1}(X) = J^n(X) \setminus \operatorname{ext}(J^n(X)), \quad \text{where } J^0(X) = X.$$

The set $J^n(X)$, non-empty for the largest n, is a free set and also consists of very "central" points. How does this compare with the branch weight centroid?

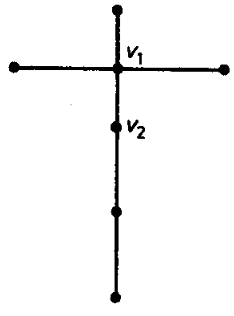

Fig. 2. A tree with the centroid $\{v_1\}$ and the set $J^2(V(T)) = \{v_2\}$

It is not hard to see that in general the above notion of centrality is different from the one we have introduced. Moreover, the example in Fig. 2 shows that even in trees these two concepts do not coincide.

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