

NOTE

## SOLUTION TO THE PROBLEM OF KUBESA

MARIUSZ MESZKA

*Faculty of Applied Mathematics*  
*AGH University of Science and Technology*  
*Mickiewicza 30, 30-059, Kraków, Poland*

**e-mail:** meszka@agh.edu.pl

### Abstract

An infinite family of  $T$ -factorizations of complete graphs  $K_{2n}$ , where  $2n = 56k$  and  $k$  is a positive integer, in which the set of vertices of  $T$  can be split into two subsets of the same cardinality such that degree sums of vertices in both subsets are not equal, is presented. The existence of such  $T$ -factorizations provides a negative answer to the problem posed by Kubesa.

**Keywords:** tree,  $T$ -factorization, degree sequence.

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### 1. INTRODUCTION

Let  $K_{2n}$  be the complete graph on  $2n$  vertices and  $T$  be its spanning tree. A  $T$ -factorization of  $K_{2n}$  is a collection of edge disjoint factors  $T_1, T_2, \dots, T_n$  of  $K_{2n}$ , each of which being isomorphic to  $T$ .

At the workshop in Krynica in 2004 D. Fronček presented the following problem originally posed by M. Kubesa [2].

**Problem.** Suppose that there exists a  $T$ -factorization of  $K_{2n}$ . Is it true that the vertex set of  $T$  can be split into two subsets,  $V_1$  and  $V_2$ , such that  $|V_1| = |V_2| = n$  and  $\sum_{v \in V_1} \deg(v) \neq \sum_{v \in V_2} \deg(v)$ ?

Notice that there is no requirement on connectness or disconnectness of graphs induced by  $V_1$  or  $V_2$ .

Recently, N.D. Tan [3] solved the problem in the affirmative for two narrow classes of trees.

## 2. CONSTRUCTIONS

A tree which becomes a star after removal of its pendant edges is called a *snowflake*. Its central vertex (ie. the central vertex of a star obtained in such a way) is called a *root*, whilst remaining vertices of degrees greater than one are called *inner vertices*.

We define a family of snowflakes  $\tilde{T}_{2n}$  of order  $2n = 56k$ , for every positive integer  $k$ . There are 7 vertices of degrees:  $28k - 18$ ,  $28k - 20$ , 11, 10, 8, 7, 7, the remaining  $56k - 7$  are leaves. The vertex of degree 11 is the root of  $\tilde{T}_{2n}$ .

**Lemma 1.** *For every positive integer  $k$ , the complete graph  $K_{56k}$  has  $\tilde{T}_{56k}$ -factorization.*

**Proof.** The snowflake  $\tilde{T}_{56k}$  is defined by listing its edges; we use the notation  $u \prec u_1, u_2, \dots, u_m$  if all the vertices  $u_1, u_2, \dots, u_m$  are adjacent to  $u$ . Consider two cases.

*Case I.  $k = 1$ .* Let  $V(K_{56}) = U \cup X \cup Y \cup Z$ , where  $U = \{u_0, u_1, \dots, u_{13}\}$ ,  $X = \{x_0, x_1, \dots, x_{13}\}$ ,  $Y = \{y_0, y_1, \dots, y_{13}\}$  and  $Z = \{z_0, z_1, \dots, z_{13}\}$ . Edges of  $K_{56}$  with both endvertices either in  $U$  or  $X$  or  $Y$  or  $Z$  are called *pure* edges; the remaining ones are *mixed* edges. To indicate a required  $\tilde{T}_{56}$ -factorization we prescribe 28 snowflakes split into two classes:  $\{T_i : i = 0, 1, \dots, 13\}$  and  $\{T'_i : i = 0, 1, \dots, 13\}$ , each  $T_i$  and  $T'_i$  being isomorphic to  $\tilde{T}_{56}$ .

We construct the first class. The vertex  $u_{12}$  of degree 11 is the root of  $T_0$  and its inner vertices:  $u_0, x_1, x_2, y_0, y_1, z_7$  have degrees 8, 8, 7, 10, 7, 10, respectively. The remaining pendant edges are:  $u_{12} \prec u_1, u_2, u_4, u_7, u_{11}$ ;  $u_0 \prec x_8, x_9, x_{11}, x_{12}, x_{13}, y_4, y_5$ ;  $x_1 \prec u_5, u_9, u_{10}, u_{13}, y_3, y_8, y_{10}$ ;  $x_2 \prec x_3, x_4, x_5, x_6, x_7, x_{10}$ ;  $y_0 \prec u_6, u_8, z_0, z_1, z_2, z_3, z_4, z_6, z_9$ ;  $y_1 \prec y_2, y_6, y_7, y_{11}, y_{12}, y_{13}$ ;  $z_7 \prec u_3, x_0, y_9, z_5, z_8, z_{10}, z_{11}, z_{12}, z_{13}$ . Snowflakes  $T_1, T_2, \dots, T_{13}$  can be obtained from  $T_0$  by applying the cyclic permutation  $\varphi = (0, 1, \dots, 13)$  in parallel on the indices of vertices in the sets  $U$ ,  $X$ ,  $Y$  and  $Z$ . One can easily check that the lengths 1, 2, 3, 4, 5, 6 of all pure edges in  $K_{56}$  have been already covered, as well as the following lengths of mixed edges for types:  $UX$ : 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13;  $UY$ : 2, 3, 4, 5, 6, 8;  $UZ$ : 4, 9;  $XY$ : 2, 7, 9;  $XZ$ : 7;  $YZ$ : 0, 1, 2, 3, 4, 6, 9, 12.

To construct the second class we need the snowflake  $T'_0$ . Let the vertex  $u_7$  of degree 11 be the root and  $x_0, x_8, y_2, y_3, z_0, z_1$  be the inner vertices of degrees 8, 8, 7, 7, 10, 10, respectively. The remaining pendant edges are:  $u_7 \prec u_0, z_3, z_4, z_5, z_6$ ;  $x_0 \prec x_7, y_0, y_8, y_{10}, y_{11}, y_{12}, y_{13}$ ;  $x_8 \prec z_2, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}$ ;  $y_2 \prec x_1, x_{10}, x_{11}, x_{12}, x_{13}, y_9$ ;  $y_3 \prec u_2, u_3, u_4, u_5, u_6, u_{10}$ ;  $z_0 \prec u_8, u_9, u_{11}, u_{12}, u_{13}, y_1, y_6, y_7, z_7$ ;  $z_1 \prec u_1, x_2, x_3, x_4, x_5, x_6, x_9, y_4, y_5$ . Six snowflakes  $T'_i$ , for  $i = 2, 4, \dots, 12$ , can be obtained from  $T'_0$  by applying  $i$ th power of  $\varphi$  in parallel on the sets  $U, X, Y$  and  $Z$ . Thus the length 7 of all pure edges is covered completely and still remaining lengths of mixed edges, except the lengths 0 of type  $UX$  and 5 of type  $YZ$ , are covered in a half. Seven remaining snowflakes  $T'_j$  for  $j = 1, 3, \dots, 13$  are obtained from  $T'_0$  by replacing the edges  $u_0 u_7, x_0 x_7, y_2 y_9$  and  $z_0 z_7$  with the edges  $u_0 x_0, u_7 x_7, y_2 z_7$  and  $y_9 z_0$ , respectively, and then by applying the permutation  $(\varphi)^j$  in parallel on the sets  $U, X, Y$  and  $Z$ . Notice that such a replacement does not result in changing the structure of snowflake, i.e., all  $T'_j$  are isomorphic to  $T'_0$ . In this way we cover all remaining lengths of mixed edges.

*Case II.*  $k \geq 2$ . Let  $V(K_{56k}) = \bigcup_{l=1}^k (U^l \cup X^l \cup Y^l \cup Z^l)$ , where  $U^l = \{u_0^l, u_1^l, \dots, u_{13}^l\}$ ,  $X = \{x_0^l, x_1^l, \dots, x_{13}^l\}$ ,  $Y = \{y_0^l, y_1^l, \dots, y_{13}^l\}$  and  $Z = \{z_0^l, z_1^l, \dots, z_{13}^l\}$ ,  $l = 1, 2, \dots, k$ . In what follows subscripts should be read modulo 14.

In order to construct  $28k$  factors, each isomorphic to  $\tilde{T}_{56k}$ , we proceed in the following way. First, for every snowflake  $T_i$ ,  $i = 0, 1, \dots, 13$ , in the  $\tilde{T}_{56}$ -factorization of  $K_{56}$  constructed in Case I we make  $k$  copies  $T_i^l$ ,  $l = 1, 2, \dots, k$ , by copying every edge  $st$  of  $T_i$  into  $k$  edges  $s^l t^l$ , each being an edge of appropriate  $T_i^l$ , where  $s, t \in U \cup X \cup Y \cup Z$ . Moreover, for every  $T_i^l$  among  $14k$  trees obtained in this way, where  $i = 0, 1, \dots, 13$  and  $l = 1, 2, \dots, k$ , we add  $56(k-1)$  edges:  $u_i^l \prec u_j^p, x_j^p, y_j^r, z_j^r, y_i^l \prec u_j^r, x_j^r, y_j^p, z_j^p$ , where  $l < p \leq k$ ,  $1 \leq r < l$ ,  $j = 0, 1, \dots, 13$ . Thus every  $T_i^l$  is a snowflake with the root  $u_{12+i}^l$  of degree 11, and six inner vertices  $u_i^l, x_{1+i}^l, x_{2+i}^l, y_i^l, y_{1+i}^l, z_{7+i}^l$  of degrees  $28k - 20, 8, 7, 28k - 18, 7, 10$ , respectively.

Similarly, for every snowflake  $T'_i$  constructed in Case I,  $i = 0, 1, \dots, 13$ , we built  $k$  copies  $T_i'^l$ ,  $l = 1, 2, \dots, k$ , by copying every edge  $st$  of  $T'_i$  into  $k$  edges  $s^l t^l$ ,  $s, t \in U \cup X \cup Y \cup Z$ . Analogously to the above, for every  $T_i'^l$  of  $14k$  trees just obtained,  $i = 0, 1, \dots, 13$  and  $l = 1, 2, \dots, k$ , new  $56(k-1)$  edges are added:  $x_i^l \prec u_j^p, x_j^p, y_j^r, z_j^r, z_i^l \prec u_j^r, x_j^r, y_j^p, z_j^p$ , where  $l < p \leq k$ ,  $1 \leq r < l$ ,  $j = 0, 1, \dots, 13$ . Every  $T_i'^l$  obtained in this way is a snowflake

with the root  $u_{7+i}^l$  of degree 11, and six inner vertices  $x_i^l, x_{8+i}^l, y_{2+i}^l, y_{3+i}^l, z_i^l, z_{1+i}^l$  of degrees  $28k - 20, 8, 7, 7, 28k - 18, 10$ , respectively. ■

**Lemma 2.** *For every set  $\bar{V} \subset V(\tilde{T}_{56k}) = V(K_{56k})$  such that  $|\bar{V}| = 28k$ ,  $\sum_{v \in \bar{V}} \deg(v) \neq 56k - 1$ .*

**Proof.** One can check that there are only four sequences of length  $28k$  whose terms are degrees of  $\tilde{T}_{56k}$  and whose sum of terms is  $56k - 1$ :

- (1)  $28k - 18, 10, 10, 1, 1, \dots, 1$ ,
- (2)  $28k - 18, 7, 7, 7, 1, 1, \dots, 1$ ,
- (3)  $28k - 20, 11, 11, 1, 1, \dots, 1$ ,
- (4)  $28k - 20, 8, 8, 7, 1, 1, \dots, 1$ .

None of these sequences is a subsequence of degree sequence of  $\tilde{T}_{56k}$ . Thus the assertion holds. ■

Notice that every of the sequences (1)–(4) indeed appears as a set of degrees for some vertex in factors of  $\tilde{T}_{56k}$ -factorization of  $K_{56k}$ . It is easily seen that all terms of (1) are degrees of the vertex  $z_i^l$  in  $\tilde{T}_{56k}$ -factorization, similarly (2) is a set of degrees for  $y_i^l$ , (3) for  $u_i^l$  and (4) for  $x_i^l$ ,  $i = 0, 1, \dots, 13$ ,  $l = 1, 2, \dots, k$ .

It is still possible that a similar example for the order  $2n < 56$  exists. Nevertheless, a computer was used to check that in that case  $2n$  cannot be smaller than 38.

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