

## ACYCLIC REDUCIBLE BOUNDS FOR OUTERPLANAR GRAPHS

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### Abstract

For a given graph  $G$  and a sequence  $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$  of additive hereditary classes of graphs we define an acyclic  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -colouring of  $G$  as a partition  $(V_1, V_2, \dots, V_n)$  of the set  $V(G)$  of vertices which satisfies the following two conditions:

1.  $G[V_i] \in \mathcal{P}_i$  for  $i = 1, \dots, n$ ,
2. for every pair  $i, j$  of distinct colours the subgraph induced in  $G$  by the set of edges  $uv$  such that  $u \in V_i$  and  $v \in V_j$  is acyclic.

A class  $\mathcal{R} = \mathcal{P}_1 \odot \mathcal{P}_2 \odot \dots \odot \mathcal{P}_n$  is defined as the set of the graphs having an acyclic  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -colouring. If  $\mathcal{P} \subseteq \mathcal{R}$ , then we say that  $\mathcal{R}$  is an *acyclic reducible bound* for  $\mathcal{P}$ .

In this paper we present acyclic reducible bounds for the class of outerplanar graphs.

**Keywords:** graph, acyclic colouring, additive hereditary class, outerplanar graph.

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## 1. INTRODUCTION

We consider finite undirected graphs without loops or multiple edges. Let  $\mathcal{I}$  denote the class of all such graphs. For a graph  $G \in \mathcal{I}$  we denote its vertex set by  $V(G)$  and its edge set by  $E(G)$ . For a vertex  $v \in V(G)$  its degree is denoted by  $d_G(v)$ , while the maximum degree of  $G$  is denoted by  $\Delta(G)$ . A *block* of a graph  $G$  is defined as a maximal connected subgraph of  $G$  without a cut-vertex.

A graph  $G$  is called *outerplanar* if it can be embedded in the plane so that no edges intersect and all the vertices belong to one face. An outerplanar graph  $G$  is called *maximal*, if for any edge  $e$  from the set  $E(\overline{G})$ , the graph  $G + e$  is not outerplanar,  $\overline{G}$  stands here for the complement of  $G$ .

Following Borowiecki *et al.* [3], we define a *class of graphs* to be any nonempty subset of  $\mathcal{I}$  which is closed under isomorphism. A class of graphs  $\mathcal{P}$  is called *hereditary* if  $G \in \mathcal{P}$  and  $H \subseteq G$  then also  $H \in \mathcal{P}$ , and *additive* if it is closed under disjoint union, i.e., if every component of  $G$  belongs to  $\mathcal{P}$ , then  $G \in \mathcal{P}$ . We list some additive hereditary classes:

$$\begin{aligned}\mathcal{O} &= \{G \in \mathcal{I} : E(G) = \emptyset\}, \\ \mathcal{S}_k &= \{G \in \mathcal{I} : \Delta(G) \leq k\}, \\ \mathcal{T}_2 &= \{G \in \mathcal{I} : G \text{ is outerplanar}\}.\end{aligned}$$

A hereditary class  $\mathcal{P}$  can be uniquely determined by the set of *minimal forbidden subgraphs* defined as follows:

$$\mathbf{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P}, \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\}.$$

Therefore we can define, for an arbitrary set  $\mathcal{F}$  of graphs, a class  $\mathcal{P} = \text{Forb}(\mathcal{F})$  as the set of all graphs having no subgraph isomorphic to any graph from  $\mathcal{F}$ . Clearly,  $\mathcal{P}$  is a hereditary class of graphs. If  $\mathcal{F} = \{H\}$  then we will write  $\text{Forb}(H)$  instead of  $\text{Forb}(\{H\})$ .

Let  $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$  be additive hereditary classes of graphs. A partition  $(V_1, V_2, \dots, V_n)$  of the vertex set  $V$  of  $G$  is called an *acyclic*  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -colouring of  $G$ , if  $G[V_i] \in \mathcal{P}_i$  for  $i = 1, \dots, n$ , and for every pair  $i, j$  ( $1 \leq i, j \leq n$ ) of distinct colours the subgraph induced in  $G$  by the set of edges  $uv$  such that  $u \in V_i$  and  $v \in V_j$  is acyclic. By  $\mathcal{P}_1 \odot \mathcal{P}_2 \odot \dots \odot \mathcal{P}_n$  we denote the set of all graphs having an acyclic  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -colouring. If  $\mathcal{R} = \mathcal{P}_1 \odot \dots \odot \mathcal{P}_n$  and  $\mathcal{P} \subseteq \mathcal{R}$ , then we say that  $\mathcal{R}$  is an *acyclic reducible bound* for  $\mathcal{P}$ .

The other specific terminology will be introduced in the text. The general concepts not defined in the paper can be found in [8, 10].

One can observe that the above presented definition of an acyclic  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -colouring corresponds to one presented in [1] and is a generalisation of a definition of an acyclic colouring of a graph, given by Grünbaum in [9] (it is enough to put each  $\mathcal{P}_i$  equal to  $\mathcal{O}$ ).

After being introduced by Grünbaum in 1973, the acyclic colouring has been widely studied over past thirty years by Burstein, see [7], Borodin [5], Borodin, Kostochka and Woodall [6], and many others. In 1999 Boiron, Sopena and Vignal considered the acyclic  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -colouring of planar and outerplanar graphs, see [1], and also of graphs with bounded degree, see [2]. In [1] it was proved that  $\mathcal{T}_2 \subseteq \mathcal{S}_5 \odot \mathcal{S}_5$ . In [4] another four acyclic reducible bounds for the class  $\mathcal{T}_2$  were presented. The aim of our paper is to provide new acyclic reducible bounds for the class of outerplanar graphs. To do this we propose a construction of special families of outerplanar graphs. This construction is presented in Section 2, while Section 3 contains main results. In Section 4 we discuss the relationship between acyclic reducible bounds given in [1] and [4], and ours.

## 2. CONSTRUCTION OF $\mathcal{H}_i$ AND ITS BASIC PROPERTIES

Let  $G = (V, E; L, S)$ ,  $L : V \rightarrow \{0, 1, \dots\}$ ,  $S : E \rightarrow \{0, 1, \dots\} \times \{+, 0, -\}$  be a graph with labels assigned to its vertices and edges. If  $S(e) = (k, \cdot)$ , then we say that the edge  $e$  has *level*  $k$  ( $\cdot$  stands for any of the signs from the set  $\{+, 0, -\}$ ). Similarly, the vertex  $v$  has *level*  $k$ , if  $L(v) = k$ . Moreover, we write  $S(e) = (\cdot, +)$ , if we mean that  $S(e) = (i, +)$ , but the value  $i$  is not important or unknown yet.

Let  $G = (V, E; L, S)$  and  $G' = (V', E'; L', S')$  be labelled graphs. If there is an isomorphism  $f : G \rightarrow G'$  such that  $L(v) = L'(f(v))$  for each  $v \in V$  and  $S(uv) = S'(f(u)f(v))$  for each  $uv \in E$ , then we say that the graphs are *LS-isomorphic*. Moreover, if  $F$  is a (labelled) graph and  $\mathcal{A}$  is a set of (labelled) graphs, then by  $F \in \mathcal{A}$  we mean  $F$  is (LS)-isomorphic to a member of  $\mathcal{A}$ . Let us remark that taking (induced) subgraphs preserves labels  $L$  and  $S$ .

We have introduced the labels  $L$  and  $S$  just for the technical purposes of the construction and the simplicity of the proofs, but because the main result of this paper concerns colouring of unlabelled graphs we must define an operator  $\sim$  which transforms a given labelled graph into unlabelled one. So, for a labelled graph  $G = (V, E; L, S)$  by  $\tilde{G}$  we mean the unlabelled graph  $(V, E)$ .

Similarly, if  $\mathcal{A}$  is a set of labelled graphs, then the set  $\tilde{\mathcal{A}}$  is defined as follows: a graph  $G \in \tilde{\mathcal{A}}$  if and only if  $G \simeq \tilde{H}$  for some  $H \in \mathcal{A}$ .

Let us define a family  $\mathcal{H}_0$  of labelled graphs as follows:  $\mathcal{H}_0 = \{H_0^1, H_0^2, H_0^3\}$ , where

$$H_0^1 = (\{u, v\}, \{uv\}; L^1, S^1) \text{ and } L^1(u) = L^1(v) = 0, S^1(uv) = (0, +),$$

$$H_0^2 = (\{u, v, w\}, \{uv, uw, vw\}; L^2, S^2) \text{ and } L^2(u) = L^2(v) = L^2(w) = 0,$$

$$S^2(uv) = (0, 0), S^2(uw) = S^2(vw) = (0, +),$$

$$H_0^3 = (\{u, v, w\}, \{uv, uw, vw\}; L^3, S^3) \text{ and } L^3(u) = L^3(v) = L^3(w) = 0,$$

$$S^3(uv) = S^3(vw) = (0, 0), S^3(uw) = (0, +).$$

The family  $\mathcal{H}_0$  is presented in Figure 1, labels of all the vertices are 0 and they are omitted.

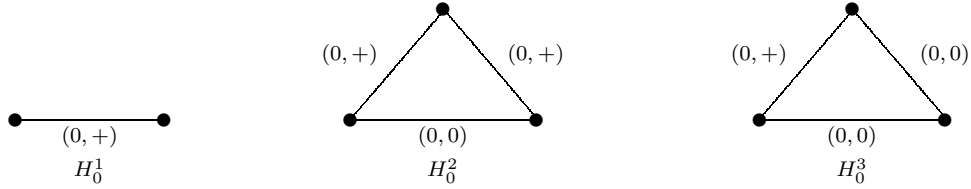


Figure 1. The family  $\mathcal{H}_0$  of graphs.

In order to define the family  $\mathcal{H}_i$ , for  $i \geq 1$ , we need to introduce the notion of a *child* of a labelled graph  $G$ . Namely, if  $G = (V, E; L, S)$  is a labelled graph, then we say that a graph  $G' = (V', E'; S', L')$  is a *child* of  $G$ , if  $G'$  is constructed in the following way:

- (a) set  $V' = V, E' = E, S' = S$  and  $L' = L$ ;
- (b) for each edge  $uv \in E'$  such that  $S'(uv) = (i, +)$ 
  - (b.0) set  $S'(uv) = (i, -)$ ;
  - (b.1) if  $\deg(u) = \deg(v) = 1$ , then add to  $G'$  vertices  $u', v', w'$ , edges  $u'u, v'u, v'v, w'v$  and set  $L'(u') = L'(v') = L'(w') = i+1, S'(u'u) = S'(w'v) = (i+1, +)$  and

$$\begin{aligned}
& ([S'(v'u) = (i+1, +), \quad S'(v'v) = (i+1, 0)] \text{ or} \\
& [S'(v'u) = (i+1, 0), \quad S'(v'v) = (i+1, +)] \text{ or} \\
& [S'(v'u) = S'(v'v) = (i+1, +)]);
\end{aligned}$$

- (b.2) if  $\deg(u) = 1$  and  $\deg(v) > 1$ , then add to  $G'$  vertices  $u', v'$ , edges  $u'u, v'u, v'v$  and set  $L'(u') = L'(v') = i+1$ ,  $S'(u'u) = (i+1, +)$  and

$$\begin{aligned}
& ([S'(v'u) = (i+1, +), \quad S'(v'v) = (i+1, 0)] \text{ or} \\
& [S'(v'u) = (i+1, 0), \quad S'(v'v) = (i+1, +)] \text{ or} \\
& [S'(v'u) = S'(v'v) = (i+1, +)]);
\end{aligned}$$

- (b.3) if  $\deg(u) > 1$  and  $\deg(v) > 1$ , then add to  $G'$  a vertex  $v'$ , edges  $v'u, v'v$  and set  $L'(v') = i+1$ ; let  $B$  be the block of  $G'$  which contains the edge  $uv$ ,

- (b.3.1) if  $B$  has an edge  $e, e \neq uv$ , such that  $S'(e) = (\cdot, +)$ , then set
- $$\begin{aligned}
& ([S'(v'u) = (i+1, +), \quad S'(v'v) = (i+1, 0)] \text{ or} \\
& [S'(v'u) = (i+1, 0), \quad S'(v'v) = (i+1, +)] \text{ or} \\
& [S'(v'u) = (i+1, 0), \quad S'(v'v) = (i+1, 0)]);
\end{aligned}$$

- (b.3.2) in the other case set

$$\begin{aligned}
& [S'(v'u) = (i+1, +), \quad S'(v'v) = (i+1, 0)] \text{ or} \\
& [S'(v'u) = (i+1, 0), \quad S'(v'v) = (i+1, +)];
\end{aligned}$$

- (c) for each edge  $uv \in E'$  such that  $S'(uv) = (i-1, 0)$  add to  $G'$  vertices  $u', v'$ , edges  $u'u, v'v$  and set

$$L'(u') = L'(v') = i+1, \quad S'(u'u) = S'(v'v) = (i+1, +).$$

Let us remark that if an edge has label  $(\cdot, +)$  in a labelled graph  $G$ , then it means that this edge plays a special role in  $G$  — it is used in the construction of a child of  $G$ .

If  $G$  is a labelled graph, then by  $\text{child}(G)$  we denote the set of all non- $LS$ -isomorphic graphs being the children of  $G$ . If a graph  $G' \in \text{child}(G)$ , then we say that the graph  $G$  is a *parent* of  $G'$ .

The family  $\mathcal{H}_i$ , for  $i \geq 1$ , is defined as follows:

$$\mathcal{H}_i = \bigcup_{H \in \mathcal{H}_{i-1}} \text{child}(H).$$

The family  $\mathcal{H}_1$  is presented in Figure 2, the vertices with label 1 are coloured white, while the vertices with label 0 are coloured black. Let us remark that the first two graphs on the picture are the children of  $H_0^1$ , the next five — the children of  $H_0^2$ , the last one is the child of  $H_0^3$ .

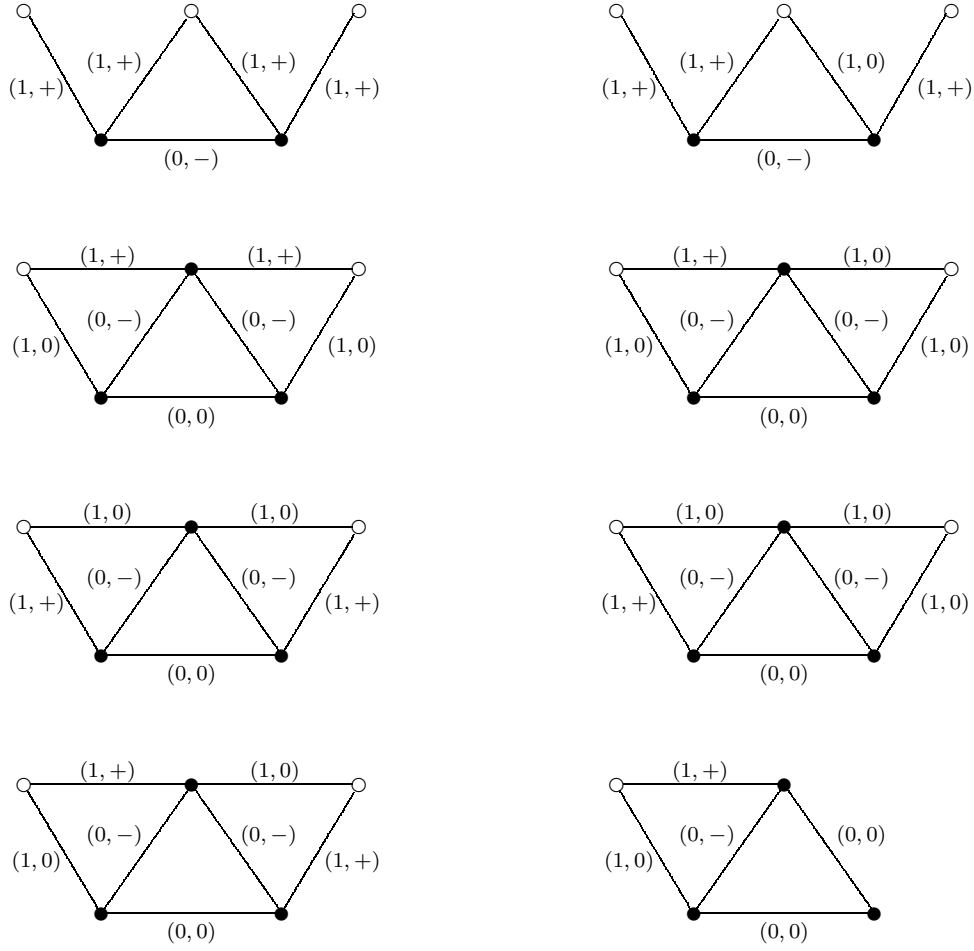


Figure 2. The family  $\mathcal{H}_1$  of graphs.

One can observe that if  $H \in \mathcal{H}_i$ , then  $H$  is outerplanar and each its block is maximal outerplanar.

In the next two lemmas we list another two properties of the graphs from  $\mathcal{H}_i$ ,  $i \geq 1$ . Both follows from the construction of  $\mathcal{H}_i$  and will be used in the proofs in Section 3.

**Lemma 1.** *Let  $H \in \mathcal{H}_i$ ,  $i \geq 1$ , and let  $B$  be a block of  $H$ . Then*

1. *if  $\tilde{B} \simeq K_2$ , then  $e \in E(B)$  has label  $(i, +)$ ;*
2. *if  $\tilde{B} \simeq K_3$ , then the edges of  $B$  have labels  $(i, +), (i, +), (i - 1, -)$  or  $(i, +), (i, 0), (i - 1, -)$ ;*
3. *if  $B$  has more than three edges, then one of the following cases holds:*
  - (a)  *$B$  contains two adjacent edges  $e_1, e_2$  such that  $S(e_1) = (i, +)$  and  $S(e_2) = (i, 0)$  and all the other edges of  $B$  have labels neither  $(i, \cdot)$  nor  $(\cdot, +)$ ,*
  - (b)  *$B$  contains two adjacent edges  $e_1, e_2$  such that  $S(e_1) = (i, +)$  and  $S(e_2) = (i, 0)$  and another two adjacent edges  $e_3, e_4$  such that  $S(e_3) = (i, 0)$  and  $S(e_4) = (i, 0)$  and all the other edges of  $B$  have labels neither  $(i, \cdot)$  nor  $(\cdot, +)$ ,*
  - (c)  *$B$  contains two adjacent edges  $e_1, e_2$  such that  $S(e_1) = (i, +)$  and  $S(e_2) = (i, 0)$  and another two adjacent edges  $e_3, e_4$  such that  $S(e_3) = (i, +)$ ,  $S(e_4) = (i, 0)$  and all the other edges of  $B$  have labels neither  $(i, \cdot)$  nor  $(\cdot, +)$ .*

**Lemma 2.** *Let  $H = (V, E; L, S) \in \mathcal{H}_i$ ,  $i \geq 1$ , and let  $B$  be a block of  $H$ . Then*

1. *if  $\tilde{B} \simeq K_3$  with  $E(B) = \{e_1, e_2, e_3\}$  and  $S(e_1) = (i - 1, -)$ , then there are graphs  $H^1 = (V, E; L, S^1) \in \mathcal{H}_i$ ,  $H^2 = (V, E; L, S^2) \in \mathcal{H}_i$  and  $H^3 = (V, E; L, S^3) \in \mathcal{H}_i$  such that  $S^1(e) = S^2(e) = S^3(e) = S(e)$  for each edge  $e \in E$  different from  $e_2, e_3$  and  $S^1(e_2) = S^1(e_3) = (i, +)$ ,  $S^2(e_2) = (i, +)$  and  $S^2(e_3) = (i, 0)$ ,  $S^3(e_2) = (i, 0)$  and  $S^3(e_3) = (i, +)$ ; moreover, there is a graph  $H' \in \mathcal{H}_{i-1}$  such that  $H, H^1, H^2, H^3 \in \text{child}(H')$ ;*
2. *if  $B$  has more than three edges, then*
  - (a) *if  $B$  contains exactly two edges of level  $i$ , say  $e_1, e_2$ , then there are graphs  $H^1 = (V, E; L, S^1) \in \mathcal{H}_i$  and  $H^2 = (V, E; L, S^2) \in \mathcal{H}_i$  such that  $S^1(e) = S^2(e) = S(e)$  for each edge  $e \in E$  different from  $e_1, e_2$  and  $S^1(e_1) = (i, +)$  and  $S^1(e_2) = (i, 0)$ ,  $S^2(e_1) = (i, 0)$  and  $S^2(e_2) = (i, +)$ ; moreover, there is a graph  $H' \in \mathcal{H}_{i-1}$  such that  $H, H^1, H^2 \in \text{child}(H')$ ;*
  - (b) *if  $B$  contains exactly four edges of level  $i$ , say  $e_1, e_2, e_3, e_4$ , and if  $e_1, e_2$  are adjacent, and  $e_3, e_4$  are adjacent, then there are graphs  $H^j = (V, E; L, S^j) \in \mathcal{H}_i$  for  $j = 1, \dots, 8$  such that  $S^j(e) = S(e)$  for  $j = 1, \dots, 8$  and for each edge  $e \in E$  different from  $e_1, e_2, e_3, e_4$ , each of the edges*

$e_1, e_2, e_3, e_4$  has either label  $(i, +)$  or  $(i, 0)$  and at least one of them has label  $(i, +)$ ; moreover, if  $e_1$  has label  $(i, +)$ , then  $e_2$  has label  $(i, 0)$ , and vice versa; the same holds for  $e_3$  and  $e_4$ ; furthermore, there is a graph  $H' \in \mathcal{H}_{i-1}$  such that  $H, H^1, \dots, H^8 \in \text{child}(H')$ .

### 3. RESULTS

We start this section by reminding a special family of labelled graphs. In [4] it was proved that this family is a generator of the class of outerplanar graphs.

Let  $A_0 = (V_0, E_0; L_0, S_0)$  be a labelled graph such that  $V_0 = \{v_1, v_2, v_3\}$ ,  $E_0 = \{v_1v_2, v_2v_3, v_1v_3\}$ ,  $L_0(v) = 0$  for all  $v \in V_0$  and  $S_0(e) = (0, 0)$  for all  $e \in E_0$ . The graph  $A_{i+1} = (V_{i+1}, E_{i+1}; L_{i+1}, S_{i+1})$  is obtained from  $A_i = (V_i, E_i; L_i, S_i)$  by adding for each edge  $e = uv$  of level  $i$  one new vertex  $w_e$  and joining  $w_e$  with both  $u$  and  $v$ . The vertex  $w_e$  has level  $i + 1$  and both new added edges have labels  $(i + 1, 0)$ .

It is worth to mention that each  $A_i$  is maximal outerplanar and it has such an embedding in a plane in which all vertices belong to the outer face and only the edges of level  $i$  belong to the outer face. Therefore, when we talk about the graph  $A_i$  or its subgraphs, we always deal with this embedding.

The following lemma states that the family  $\{\widetilde{A}_0, \widetilde{A}_1, \dots\}$  of graphs is a generator of the class  $\mathcal{T}_2$  and render us possible to consider in our proofs only the graphs  $A_i$  instead of the whole class  $\mathcal{T}_2$ .

**Lemma 3.** [4] *For every  $G \in \mathcal{T}_2$  there exists an index  $i \geq 0$  such that  $G \subseteq \widetilde{A}_i$ .*

In [4] it was proved that each outerplanar graph  $G$  has an acyclic colouring  $(V_1, V_2)$  such that  $G[V_1] \in \mathcal{O}$  and  $G[V_2] \in \text{Forb}(\widetilde{A}_1)$ .

**Lemma 4.** [4]  $\mathcal{T}_2 \subseteq \mathcal{O} \odot \text{Forb}(\widetilde{A}_1)$ .

In the following lemmas, which easily follows from the construction of the graphs  $A_i$ , we present more properties of such graphs, which will be used later.

**Lemma 5.** *Let  $B$  be a 2-connected induced subgraph of  $A_i$ ,  $i \geq 0$ . Then  $B$  cannot contain any induced cycle of length greater than 3.*



**Lemma 6.** *An edge  $e \in E(A_i)$  belongs to the outer cycle of  $A_i$  if and only if  $S_i(e) = (i, 0)$ . Moreover, every vertex  $v \in V(A_i)$  such that  $L_i(v) = i$  is of degree 2.*

**Lemma 7.** *If  $H \in \mathcal{H}_i$ ,  $i \geq 0$ , then  $\widetilde{H}$  cannot contain a subgraph isomorphic to  $A_1$ .*

Next we prove that if  $\mathcal{P}$  is an additive hereditary class such that  $\widetilde{A_1} \notin \mathcal{P}$  and  $(V_1, V_2)$  is an acyclic  $(\mathcal{O}, \mathcal{P})$ -colouring of  $A_i$ ,  $i \geq 0$ , then there is a graph  $H_i \in \mathcal{H}_i$  such that  $\widetilde{A_i}[V_2] = \widetilde{H_i}$ .

Now we present some properties of such a colouring.

**Lemma 8.** *Let  $\mathcal{P}$  be an additive hereditary class and assume  $\widetilde{A_1} \notin \mathcal{P}$ . If  $(V_1, V_2)$  is an acyclic  $(\mathcal{O}, \mathcal{P})$ -colouring of  $A_i$ ,  $C$  is the outer cycle of  $A_i$  and  $B$  is a block of  $A_i[V_2]$ , then one of the following situations occurs:*

1.  *$E(B) \cap E(C)$  contains exactly four edges:  $uv, vw, u'v'$  and  $v'w'$ ; moreover,  $L_i(v) = L_i(v') = i$  and  $uw, u'w' \in E(A_i)$ ,*
2.  *$E(B) \cap E(C)$  contains exactly two edges:  $uv, vw$ ; moreover,  $L_i(v) = i$  and  $uw \in E(A_i)$*
3.  *$E(B) \cap E(C)$  contains exactly one edge.*

**Proof.** First observe that if  $B$  is a trivial block of  $A_i[V_2]$ , it means, if  $\widetilde{B} \simeq K_2$ , then from the construction of  $A_i$  and the fact that the colouring is acyclic it clearly follows that  $|E(B) \cap E(C)| = 1$ .

On the other hand, if  $B$  is a non-trivial block of  $A_i[V_2]$ ,  $u, v$  are two consecutive vertices of the outer cycle of this block and the edge  $uv$  does not belong to the cycle  $C$ , then for each such a pair of vertices there is one vertex  $w \in V(A_i) - V(B)$ , adjacent to both  $u$  and  $v$ , which follows from the construction of  $A_i$ . Therefore, if  $E(B) \cap E(C) = \emptyset$ , then the colouring  $(V_1, V_2)$  cannot be acyclic.

Hence consider an edge  $e = uv \in E(B) \cap E(C)$ . Without loss of generality we can assume that  $L_i(v) = i$ , therefore, by Lemma 6,  $d(v) = 2$ . Let  $w$  be the second neighbour of  $v$  in  $A_i$ . If the edge  $e_1 = vw$  does not belong to  $B$ , then  $\widetilde{B}$  is isomorphic to  $K_2$ . Hence  $e_1 \in E(B)$ . If  $E(B) \cap E(C) = \{e, e_1\}$ , then the second situation occurs.

Therefore, we can assume that there is an edge  $e' = u'v' \in E(B)$ ,  $e' \neq e$ ,  $e' \neq e_1$ , which belongs to the cycle  $C$ . Similarly as above, we can assume that  $v'$  has level  $i$  in  $A_i$ . Hence, by Lemma 6,  $d(v') = 2$ . Moreover, if  $w'$  is the remaining neighbour of  $v'$ , then the edge  $e'_1 = w'v' \in E(C) \cap E(B)$ , which is clear because  $B$  is a block. Moreover, from the construction of  $A_i$

it follows that  $e'_1 \neq e$  and  $e'_1 \neq e_1$ . If  $E(B) \cap E(C) = \{e, e_1, e', e'_1\}$ , then we have the first situation.

Observe that if there is another edge  $e'' = v''u'' \in E(B) \cap E(C)$ ,  $e'' \notin \{e, e_1, e', e'_1\}$ , then assuming that  $L(v'') = i$ , we will have that there is exactly one vertex  $w'' \neq u''$  adjacent to  $v''$  in  $A_i$ . Moreover,  $w'' \in V(B)$  and  $u''w'' \in E(B)$ . But it is quite easy to see that in this case  $\widetilde{B}$  must contain a subgraph isomorphic to  $\widetilde{A}_1$ , which follows from the construction of  $A_i$ , a contradiction. ■

From the proof of Lemma 8 and the construction of  $A_i$  we can conclude the following.

**Remark 1.** Let  $\mathcal{P}$  be an additive hereditary class such that  $\widetilde{A}_1 \notin \mathcal{P}$ . Consider an acyclic  $(\mathcal{O}, \mathcal{P})$ -colouring  $(V_1, V_2)$  of  $A_i$ . Let  $C$  be the outer cycle of  $A_i$ . Furthermore, assume that  $v$  is a vertex such that  $v \in V_2$  and  $L_i(v) = i$ . Clearly,  $d(v) = 2$ . Hence, let  $u$  and  $w$  be the neighbours of  $v$ . Obviously,  $vu, vw \in E(C)$ . Moreover, if  $B$  is the block of  $A_i[V_2]$  containing  $v$ , then either  $\widetilde{B} \simeq K_2$  or  $vu, vw \in E(B)$ .

**Theorem 1.** Let  $\mathcal{P}$  be an additive hereditary class and assume that  $\widetilde{A}_1 \notin \mathcal{P}$ . If  $(V_1, V_2)$  is an acyclic  $(\mathcal{O}, \mathcal{P})$ -colouring of  $A_i$ , then there is a graph  $H_i \in \mathcal{H}_i$  such that  $\widetilde{A_i[V_2]} = \widetilde{H_i}$ .

**Proof.** We prove a stronger statement, namely we prove that if  $(V_1, V_2)$  is any acyclic  $(\mathcal{O}, \mathcal{P})$ -colouring of  $A_i = (V_i, E_i; L_i, S_i)$ , then there is a graph  $H_i = (V, E; L, S) \in \mathcal{H}_i$  such that  $\widetilde{A_i[V_2]} = \widetilde{H_i}$  and for each vertex  $t \in V_2$  we have  $L_i(t) = L(t)$ .

We use induction on  $i$ . It is easy to check that the theorem is true for  $A_0$  and  $A_1$ .

Let  $i \geq 1$ . Assume the theorem holds for every  $A_j$  ( $0 \leq j \leq i$ ), we will prove it for  $A_{i+1}$ .

Consider the graph  $A_{i+1} = (V_{i+1}, E_{i+1}; L_{i+1}, S_{i+1})$  and the subgraph  $A_i = (V_i, E_i; L_i, S_i)$  induced in  $A_{i+1}$  by the vertices of levels  $0, \dots, i$ . Let  $(V_1, V_2)$  be any acyclic  $(\mathcal{O}, \mathcal{P})$ -colouring of  $A_i$ . By the induction hypothesis it follows that there is the graph  $H_i = (V, E; L, S) \in \mathcal{H}_i$  such that  $\widetilde{A_i[V_2]} = \widetilde{H_i}$  and moreover for each vertex  $t \in V_2$  we have  $L_i(t) = L(t)$ . It is sufficient to prove that for any acyclic  $(\mathcal{O}, \mathcal{P})$ -colouring  $(V'_1, V'_2)$  of  $A_{i+1}$  such that  $V_1 \subseteq V'_1$ ,  $V_2 \subseteq V'_2$ , there is also a graph  $H_{i+1} = (V', E'; L', S') \in \mathcal{H}_{i+1}$  such that  $\widetilde{A_{i+1}[V'_2]} = \widetilde{H_{i+1}}$  and for each vertex  $t \in V'_2$  we have  $L_{i+1}(t) = L'(t)$ .

At the beginning observe that for any  $s \geq 0$ , level of an edge  $uv$  of a graph  $F \in \mathcal{H}_s$  is the maximum of levels of the vertices  $u$  and  $v$ . Therefore, from the induction hypothesis and the fact that for any  $l \geq 0$  all edges of the outer cycle of  $A_l$  have level  $l$ , it clearly follows that:

**Observation 1.** *If  $e \in E$ , then  $S(uv) = (i, \cdot)$  if and only if  $uv$  belongs to the outer cycle of  $A_i$ .*

Furthermore, from the construction of  $A_i$  and the facts that the colouring  $(V_1, V_2)$  is acyclic and the set  $V_1$  is independent it follows that  $A_i[V_2]$  is connected.

The rest of the proof is divided into two steps. In the first step we use Lemma 8 to prove that there is a graph  $H_{i+1} = (V', E'; L', S') \in \mathcal{H}_{i+1}$  which satisfies  $\widetilde{A_{i+1}[V_2]} \subseteq \widetilde{H_{i+1}}$  and for each vertex  $t \in V_2'$  we have  $L_{i+1}(t) = L'(t)$ .

First, we choose an arbitrary graph  $H_{i+1} = (V', E'; L', S') \in \text{child}(H_i)$ . Clearly, from the induction hypothesis it follows that it is sufficient to prove that for each vertex  $x \in V_2'$  which has level  $i + 1$  in  $A_{i+1}$  there is a suitable vertex of level  $i + 1$  in  $H_{i+1}$ .

From the construction of the graph  $A_{i+1}$  it follows that each such  $x$  has a unique neighbour  $v$  of level  $i$  in  $A_{i+1}$  and there is also a unique vertex  $y$ ,  $y \neq x$ , of level  $i + 1$  in  $A_{i+1}$  which is adjacent to  $v$ . We will consider each such pair of vertices  $x$  and  $y$ . Obviously,  $d_{A_{i+1}}(x) = d_{A_{i+1}}(y) = 2$ .

Assume at the beginning that  $v \in V_2$ . Let  $B$  be the block of  $A_i[V_2]$  containing  $v$ . Furthermore, let  $C$  be the outer cycle of  $A_i$ . According to Lemma 8 we have to distinguish three cases.

*Case 1.* If  $E(B) \cap E(C) = \{uv, vw, u'v', v'w'\}$ , then notice that  $S_i(uv) = S_i(vw) = S_i(u'v') = S_i(v'w') = (i, 0)$  and moreover  $L_i(v) = L_i(v') = i$ . Hence  $L(v) = L(v') = i$ . Therefore  $v, v'$  are of degree two in  $A_i$  and in  $H_i$ . Moreover,  $uw \in E(A_i)$  and  $u'w' \in E(A_i)$ . Let  $x', y'$  be the neighbours of  $v'$  in  $A_{i+1}$  which have level  $i + 1$ . Without loss of generality we can assume that  $xu, yw, x'w', y'w' \in E(A_{i+1})$ . It is evident that the block  $B$  has at least 5 vertices. Furthermore, from the definition of  $A_i$  it follows that there is a vertex  $z \in V(A_i)$ , different from  $u, v, w$ , such that  $uz, wz \in E(A_i)$ . The vertex  $z$  is unique, for otherwise we will have a subgraph isomorphic to  $K_{2,3}$  in  $\widetilde{A_i}$ , which is impossible. Similarly, there is exactly one vertex  $z' \in V(A_i)$ , different from  $u', v', w'$ , such that  $u'z', w'z' \in E(A_i)$ . Since  $B$  is a block, we have  $z, z' \in V(B)$ .

If  $x, y, x', y' \in V_1'$ , then  $(V_1', V_2')$  cannot be acyclic, which follows from the proof of Lemma 8.

If we assume that  $x, y \in V'_2$  or  $x', y' \in V'_2$ , then  $\widetilde{A_{i+1}}[V'_2]$  contains a subgraph isomorphic to  $\widetilde{A_1}$ , namely a subgraph induced by the vertices  $\{x, y, u, v, w, z\}$  or  $\{x', y', u', v', w', z'\}$ .

If exactly one of the vertices  $x, y, x', y'$ , say  $x'$ , belongs to  $V'_2$ , and the rest of them belong to  $V'_1$ , then from the fact that  $L_i(v) = i$  we have  $L(v) = i$ . Hence  $S(uw) = (i - 1, -)$ . Likewise,  $S(u'w') = (i - 1, -)$ . Therefore by Lemma 2 there is a graph  $H'_i = (V, E; L, S'') \in \mathcal{H}_i$  such that  $S''(e) = S(e)$  for every edge  $e \in E$ ,  $e \notin \{u'v', vw, uv, v'w'\}$  and  $S''(u'v') = (i, +)$ ,  $S(vw) = S(uv) = S(v'w') = (i, 0)$ . We set  $H_i = (V, E; L, S'')$ . In the step (b.3) of the construction of a graph  $H_{i+1}$ , we add a vertex  $a$  adjacent to both  $u'$  and  $v'$  with  $L'(a) = i + 1$ . We set  $x' = a$ . Furthermore, from Observation 1 we conclude that in  $B' = H_i[V(B)]$  only the edge  $u'v'$  has label  $(\cdot, +)$ .

In the remaining case, if exactly two of the vertices  $x, y, x', y'$  are in  $V'_2$  and the two others are in  $V'_1$ , then without loss of generality we can assume that  $x, x' \in V'_2$ . Similarly as above, from the fact that  $L_i(v) = i$  we have  $L(v) = i$  and  $S(uw) = (i - 1, -)$ . Likewise,  $S(u'w') = (i - 1, -)$ . Therefore, by Lemma 2 there is a graph  $H'_i = (V, E; L, S'') \in \mathcal{H}_i$  such that  $S''(e) = S(e)$  for every edge  $e \in E$ ,  $e \notin \{u'v', vw, uv, v'w'\}$  and  $S''(uv) = S''(u'v') = (i, +)$  and  $S''(vw) = S''(v'w') = (i, 0)$ . We set  $H_i = (V, E; L, S'')$ . In the step (b.3) of the construction of a graph  $H_{i+1}$ , we add two new vertices  $a$  and  $b$  (one for the edge  $uv$  and another for  $u'v'$ ), such that  $a$  is adjacent to both  $u$  and  $v$ , and  $b$  is adjacent to both  $u'$  and  $v'$  with  $L'(a) = L'(b) = i + 1$ . We set  $x = a$ ,  $x' = b$ . Observation 1 implies that in  $B' = H_i[V(B)]$  there are no edges with labels  $(\cdot, +)$ , except of  $uv$  and  $u'v'$ .

*Case 2.* Assume that  $E(B) \cap E(C) = \{uv, vw\}$ . It is easy to observe that the edges  $uv$  and  $vw$  both have level  $i$  in  $A_i$ , since  $L_i(v) = i$ . Hence  $L(v) = i$ . Therefore,  $d_{A_i}(v) = 2$  and also  $d_{H_i}(v) = 2$ . Besides,  $uw \in E_i$ . Without loss of generality we can assume that  $xu \in E_{i+1}$  and  $yw \in E_{i+1}$ .

If  $x, y \in V'_1$ , then the colouring  $(V'_1, V'_2)$  cannot be acyclic, which follows from the proof of Lemma 8.

On the other hand, if both  $x, y$  are in  $V'_2$ , but  $B$  has a vertex  $z$  adjacent to both  $w$  and  $u$ ,  $z \neq v$ , then the graph  $\widetilde{A_{i+1}}[\{x, y, u, v, w, z\}]$  is isomorphic to  $\widetilde{A_1}$ , which is impossible.

If we assume that  $x, y \in V'_2$  and  $B$  does not have such a vertex  $z$ , then  $\widetilde{B} \simeq K_3$  and, by Lemma 1, the edge  $uw$  has label  $(i - 1, -)$  in  $H_i$ . Therefore by Lemma 2 there is a graph  $H'_i = (V, E; L, S'') \in \mathcal{H}_i$  such that  $S''(e) = S(e)$  for every edge  $e \in E$ ,  $e \neq uv$ ,  $e \neq vw$  and  $S''(uv) = S''(vw) = (i, +)$ .

We set  $H_i = (V, E; L, S'')$ . Notice that in the step (b.3) of the construction of a graph  $H_{i+1}$  we add two new vertices  $a$  and  $b$  (one for the edge  $uv$  and another for  $vw$ ), such that  $a$  is adjacent to both  $u$  and  $v$ , and  $b$  is adjacent to both  $v$  and  $w$  with  $L'(a) = L'(b) = i + 1$ . We set  $x = a$ ,  $y = b$ . From Observation 1 we conclude that in  $B' = H_i[V(B)]$  there are no edges with labels  $(\cdot, +)$ , except of  $uv$  and  $vw$ .

It remains to consider the case  $x \in V'_2$  and  $y \in V'_1$ . Clearly, from the fact that  $L_i(v) = i$  and  $v$  is adjacent to both  $u$  and  $w$ , we conclude that  $S(uw) = (i - 1, -)$ . Therefore, by Lemma 2 there is a graph  $H'_i = (V, E; L, S'') \in \mathcal{H}_i$  such that  $S''(e) = S(e)$  for every edge  $e \in E$ ,  $e \neq uv$ ,  $e \neq vw$  and  $S''(uv) = (i, +)$ ,  $S''(vw) = (i, 0)$ . We set  $H_i = (V, E; L, S'')$ . Notice that in the step (b.3) of the construction of a graph  $H_{i+1}$ , we add a new vertex  $a$  adjacent to both  $u$  and  $v$  with  $L'(a) = i + 1$ . We set  $x = a$ . From Observation 1 we conclude that in  $B' = H_i[V(B)]$  only the edge  $uv$  has label  $(\cdot, +)$ . Similar considerations can be applied to the case when  $x \in V'_1$  and  $y \in V'_2$ .

*Case 3.* If  $E(B) \cap E(C) = \{uv\}$ , then  $\tilde{B} \simeq K_2$ . It is easy to observe that in  $A_i$  there is a vertex  $w$  such that  $uw, vw \in E(A_i)$ . As  $\tilde{B} \simeq K_2$  we have  $w \in V_1$ . Moreover, there is no loss of generality in assuming that  $xu, yw \in E(A_{i+1})$ , which follows from the definition of  $A_{i+1}$ .

If either  $x$  or  $y$  belongs to  $V'_1$ , then either the colouring  $(V'_1, V'_2)$  cannot be acyclic or the set  $V'_1$  cannot be independent.

On the other hand, if  $x, y \in V'_2$ , then from the fact that  $L_i(v) = i$  it follows that  $L(v) = i$ . Moreover,  $v$  is of degree 1 in  $B$  and cannot be adjacent in  $A_i$  to any vertex from  $V_2$ , except  $u$ . Thus  $v$  is of degree 1 in  $H_i$ . Therefore,  $S(uv) = (i, +)$ . In the step (b.2) of the construction of a graph  $H_{i+1}$ , we add two new vertices  $a$  and  $b$  such that  $a$  is adjacent to both  $u$  and  $v$ ,  $b$  is adjacent to  $v$ . We set  $x = a$ ,  $y = b$ . Moreover, from Observation 1 we can conclude that in  $B' = H_i[V(B)]$  only the edge  $uv$  has label  $(\cdot, +)$ .

Let us assume now that  $v \notin V_2$ . Hence  $v \notin V(H_i)$ . By the definition of  $A_i$  we have that there are vertices  $u, w \in V(A_i)$  such that  $uv, vw, wu \in E(A_i)$  and  $ux, wy \in E(A_{i+1})$ . Obviously,  $u, w \in V_2$ . Let  $z \in V(A_i)$  be the unique vertex different from  $v$  which is adjacent to both  $u$  and  $w$  in  $A_i$ . The uniqueness of the vertex  $z$  follows from the definition of  $A_i$ . The colouring  $(V_1, V_2)$  is acyclic so  $z \in V_2$ . If either  $x \in V'_1$  or  $y \in V'_1$ , then  $V'_1$  cannot be independent. Therefore, we can assume  $x, y \in V'_2$ . Moreover, one

of the vertices  $u, w$ , say  $u$ , has level  $i - 1$  in  $A_i$ , and another one, it means  $w$ , has level lower than or equal to  $i - 1$  in  $A_i$ . Hence  $L(u) = i - 1$  and  $L(w) \leq i - 1$ . Thus  $S(uw) = (i - 1, \cdot)$ . Since  $z \in V_2$ , we have  $z \in V(H_i)$ . Moreover,  $z$  has level less than or equal to  $i - 1$  in  $A_i$ . Furthermore, the edge  $uw$  cannot have label  $(i - 1, -)$  in  $H_i$ , because in this case we would have a vertex  $a$  in  $H_i$ , adjacent to both  $u$  and  $w$ , and therefore, since  $\widetilde{H_i} = \widetilde{A_i}[V_2]$ , we would have  $a \in V_2$  and  $L_i(a) = i$ . This would contradict the fact that the vertex  $v \in V_1$  is the only neighbour of both  $u$  and  $w$  which has  $L_i$  equal to  $i$ . So we must have  $S(uw) = (i - 1, 0)$ . Therefore, in the step (c) of the construction of  $H_{i+1}$  we add vertices  $b$  and  $c$  and edges  $ub$  and  $uc$ . We set  $x = b$ ,  $y = c$ .

Notice that we have just obtained certain graphs  $H_i = (V, E; L, S)$  and  $H_{i+1} = (V', E'; L', S')$ . We use this particular graphs to finish the proof by showing that  $\widetilde{A_{i+1}[V'_2]} = \widetilde{H_{i+1}}$ .

Clearly, it is sufficient to prove that

- if a vertex  $t \in V'$  and  $L'(t) = i + 1$ , then  $t \in V'_2$  and
- if an edge  $e \in E'$  and  $S'(e) = (i + 1, \cdot)$ , then  $e \in E(A_{i+1}[V'_2])$ .

Let  $a \in V(H_{i+1})$  be a vertex satisfying  $L'(a) = i + 1$ . Obviously,  $a$  can be either of degree 2 or 1 in  $H_{i+1}$ .

Assume that  $a$  is of degree 2 in  $H_{i+1}$  and let  $b, c$  be its neighbours in  $H_{i+1}$ . Clearly,  $S'(bc) = (i, -)$ . Let  $B'$  be the block of  $H_{i+1}$  containing the edge  $bc$ . Moreover, let  $B = H_{i+1}[\{t \in V(B') : L'(t) < i + 1\}]$ . Obviously,  $B$  is a block of the graph  $H_i$  and  $S(bc) = (i, +)$ . Hence by Observation 1,  $bc$  belongs to the outer cycle of  $A_i$ . Therefore, according to the procedure described in *Case 1* and *Case 2*, we have  $a \in V'_2$  and  $ab, ac \in E(A_{i+1}[V'_2])$ . Now we assume that  $a$  is of degree 1 in  $H_{i+1}$  and let  $b \in V(H_{i+1})$  be its neighbour. It is clear that  $L'(b) < i + 1$ . Hence  $b \in V(H_i)$ .

If  $L(b) = i$ , then  $b$  is of degree 1 in  $H_i$ . Moreover, if  $c$  is the neighbour of  $b$  in  $H_i$ , then  $S(bc) = (i, +)$ . Therefore, we have the situation described in *Case 3* and clearly  $a \in V'_2$  and  $ab \in E(A_{i+1}[V'_2])$ .

In the opposite case, if  $L(b) = j < i$ , then according to the construction of  $H_{i+1}$  we see that there is an edge  $bc$  in  $H_i$  such that  $S(bc) = (i - 1, 0)$ . Hence  $bc$  has level  $i - 1$  in  $A_i$ . Therefore, in  $A_i$  there is a vertex  $v$  of level  $i$ , adjacent to both  $b$  and  $c$ . The vertex  $v$  must be in  $V_1$  since  $\widetilde{A_i}[V_2] = \widetilde{H_i}$  and in  $H_i$  we do not have any vertex of level  $i$  adjacent to both  $b$  and  $c$ , because

$S(bc) = (i-1, 0)$ . Hence in  $A_{i+1}$  there is a vertex  $x$ , adjacent to both  $v$  and  $b$ . We have  $x \in V_2'$  since  $V_1'$  is independent and we can set  $a = x$ . Clearly,  $ab \in E(A_{i+1}[V_2'])$ . ■

If  $G$  is a graph and  $(V_1, V_2)$  is a colouring of  $G$ , then a cycle  $C = (c_1, \dots, c_n)$  of a graph  $G$  is called *alternating* (with respect to the colouring  $(V_1, V_2)$ ), if  $n$  is even and  $c_1, c_3, \dots, c_{n-1} \in V_2$  and  $c_2, c_4, \dots, c_n \in V_1$ . Such a cycle  $C$  is called *minimal alternating*, if there is no other alternating cycle  $C'$  (with respect to the colouring  $(V_1, V_2)$ ) satisfying  $V(C') \subset V(C)$ .

In the proof of the next theorem we will use the following lemma, which presents a certain property of minimal alternating cycles with respect to an  $(\mathcal{O}, \mathcal{P})$ -colouring of  $A_i$ , where  $\mathcal{P}$  is a given additive hereditary class.

**Lemma 9.** *Let  $(V_1, V_2)$  be a colouring of  $A_i$  such that  $\widetilde{A}_i[V_1] \in \mathcal{O}$  and let  $C = (c_1, c_2, \dots, c_k)$  be any minimal alternating cycle of length  $k \geq 6$  with respect to the colouring  $(V_1, V_2)$  such that  $c_1 \in V_2$ . Then  $V_2 \cap V(C)$  is a cycle of  $A_i$ . Moreover, the graph  $A_i[V_2 \cap V(C)]$  is a block in  $A_i[V_2]$ .*

**Proof.** Let  $(V_1, V_2)$  be a colouring of  $A_i$  satisfying  $\widetilde{A}_i[V_1] \in \mathcal{O}$  and let  $C = (c_1, c_2, \dots, c_k)$  ( $k \geq 6$ ) be any minimal alternating cycle with respect to this colouring such that  $c_1 \in V_2$ . By Lemma 5 we have the set  $V(C)$  induces the graph  $G_C$  in  $A_i$ , which cannot contain any induced cycle of length greater than 3. From the fact that the set  $V_1$  is independent we can conclude that if  $u, v \in \{c_2, c_4, \dots, c_k\}$ , then  $u$  and  $v$  cannot be adjacent in  $A_i$ . If the vertex  $c_{2l}$ ,  $1 \leq l \leq k/2$ , is adjacent to a certain vertex  $c_{2l'-1}$ ,  $1 \leq l' \leq k/2$ , then the cycle  $C$  cannot be minimal alternating. Therefore the vertex  $c_{2l-1}$  is adjacent to  $c_{2l+1}$ , for all  $l = 1, \dots, k/2 - 1$ , and  $c_{k-1}$  is adjacent to  $c_1$ , since otherwise in  $G_C$  it would be an induced cycle of length greater than 3. The fact that  $A_i[V_2 \cap V(C)]$  is a block in  $A_i[V_2]$  follows from Lemma 5 and the definition of  $A_i$ . ■

**Theorem 2.** *Let  $H_i \in \mathcal{H}_i$ . Then there is an acyclic colouring  $(V_1, V_2)$  of  $A_i$  such that  $\widetilde{A}_i[V_1] \in \mathcal{O}$  and  $\widetilde{A}_i[V_2] = H_i$ .*

**Proof.** We prove a little stronger statement. Namely, we prove that for any graph  $H_i = (V, E; L, S) \in \mathcal{H}_i$  there is an acyclic  $(\mathcal{O}, \mathcal{P})$ -colouring  $(V_1, V_2)$  of  $A_i = (V_i, E_i; L_i, S_i)$  such that  $\widetilde{A}_i[V_2] = \widetilde{H}_i$  and for each vertex  $v \in V$  we have  $L(v) = L_i(v)$ .

We use induction on  $i$ . It is easy to check that the theorem is true for all  $H \in \mathcal{H}_0$  and  $H \in \mathcal{H}_1$ .

Let  $i \geq 1$ . Assume the theorem is true for all  $H_j \in \mathcal{H}_j$  and every  $j \leq i$ . We will prove it for  $i + 1$ .

Let us consider the graph  $H_{i+1} = (V', E'; L', S') \in \mathcal{H}_{i+1}$  and let  $H_i = (V, E; L, S)$  be a parent of  $H_{i+1}$ . Clearly,  $H_i \in \mathcal{H}_i$ . Therefore by the induction hypothesis we have that  $A_i$  has an acyclic  $(\mathcal{O}, \mathcal{P})$ -colouring  $(V_1, V_2)$ , which satisfies the condition  $\widehat{A_i[V_2]} = \widehat{H_i}$  and such that for each vertex  $v \in V$  we have  $L(v) = L_i(v)$ . We prove that there is an acyclic colouring  $(V'_1, V'_2)$  of  $A_{i+1}$ , such that  $\widehat{V_1} \subseteq \widehat{V'_1}$  and  $\widehat{V_2} \subseteq \widehat{V'_2}$ , and which satisfies the condition  $\widehat{A_{i+1}[V'_1]} \in \mathcal{O}$ ,  $\widehat{A_{i+1}[V'_2]} = \widehat{H_{i+1}}$  and for each vertex  $v \in V'$  we have  $L'(v) = L_{i+1}(v)$ .

Now we show how we construct the colouring  $(V'_1, V'_2)$ . First, we colour every vertex of level at most  $i$  in  $A_{i+1}$  according to the colouring  $(V_1, V_2)$ . Next, we will extend this colouring to the whole graph  $A_{i+1}$ . Clearly, it is sufficient to consider only the vertices of level  $i + 1$  in  $A_{i+1}$ .

Let  $x, y$  be a pair of vertices of level  $i + 1$  in  $A_{i+1}$ , such that there is a vertex  $v$  of level  $i$  in  $A_i$  adjacent to both  $x$  and  $y$ . Besides let  $u, w$  be the vertices such that  $uw, vw, vu, xu, yw \in E(A_{i+1})$ . Furthermore, let  $z$  be a vertex different from  $v$  such that  $uz, wz \in E(A_{i+1})$ . According to the definition of  $A_{i+1}$  we see that these vertices exist and moreover, all  $u, w, z$  have neither level  $i + 1$  nor  $i$ . Since the colouring  $(V_1, V_2)$  is acyclic and the set  $V_1$  is independent, at most one of the vertices  $u, v, w$  can belong to  $V_1$ . Hence there are four cases to consider.

*Case 1.* If  $u, v, w \in V_2$  and  $z \in V_1$ , then by the definition of  $A_i$ , if  $B$  is a block in the graph  $H_i$  which contains  $u, v, w$ , then  $\widehat{B} \simeq K_3$ . Therefore, by Lemma 1, at least one of the edges  $uv, vw$  has label  $(i, +)$  in  $H_i$ . Assume, without loss of generality, that  $L(uv) = (i, +)$ . We put  $x$  into  $V'_2$ . If  $L(vw) = (i, +)$ , then we put  $y$  into  $V'_2$ , otherwise into  $V'_1$ .

*Case 2.* If  $u, v, w \in V_2$  and  $z \in V_2$  then let  $B$  be a block of  $H_i$  which contains  $u, v, w, z$ . From Lemma 8 it follows that  $B$  has either two or four common edges with the outer cycle of  $A_i$ .

*Subcase 2.1.* If there are only two common edges  $uv$  and  $vw$ , then there are no other edges of level  $i$  in  $B$ . Therefore, by Lemma 1, exactly one of this two edges, say  $uv$ , has label  $(i, +)$ . Hence in the construction of  $H_{i+1}$  we add a vertex  $a$ , adjacent to both  $u$  and  $v$ . Therefore we put  $x$  into  $V'_2$ . Since the edge  $vw$  has label  $(i, -)$ , we put  $y$  into  $V'_1$ .

*Subcase 2.2.* If there are four common edges:  $uv, vw, u'v'$  and  $w'v'$ , then in the block  $B$  there are no other edges of level  $i$ . Hence, again by



Lemma 1, at least one of these four edges has label  $(i, +)$  in  $H_i$ . If the edge  $uv$  has label  $(i, +)$ , then we put  $x$  into  $V'_2$ , otherwise into  $V'_1$ . If the edge  $vw$  has label  $(i, +)$ , then we put  $y$  into  $V'_2$ , otherwise into  $V'_1$ .

*Case 3.* If  $v \in V_1$ , then clearly  $u, w, z \in V_2$ , since the colouring  $(V_1, V_2)$  is acyclic and the set  $V_1$  is independent. Moreover, the edge  $uw$  has label  $(i - 1, 0)$  in  $H_i$ . Therefore, in the construction of the graph  $H_{i+1}$  we add two new vertices adjacent to both  $u$  and  $w$ , respectively. We put  $x, y$  into  $V'_2$ .

*Case 4.* If  $w \in V_1$  (or, similarly,  $u \in V_1$ ), then  $z \in V_2$ , because the set  $V_1$  is independent. Furthermore, the edge  $uv$  is a trivial block in  $H_i$ . Therefore,  $L(uv) = (i, +)$  in  $H_i$  and the vertex  $v$  is of degree 1 in  $H_i$ . Hence in the construction of the graph  $H_{i+1}$  we add one new vertex adjacent to both  $u$  and  $v$  and another one adjacent to  $v$ . We put  $x, y$  into  $V'_2$ .

From the above it clearly follows that  $\widetilde{A_{i+1}[V'_2]} \subseteq \widetilde{H_{i+1}}$ ,  $\widetilde{A_{i+1}[V'_1]} \in \mathcal{O}$  and each vertex from the set  $V'_2$ , which has level  $l$  in  $H_{i+1}$ , has level  $l$  in  $A_{i+1}$ .

In order to finish the proof it remains to show that the colouring  $(V'_1, V'_2)$  is acyclic and that  $\widetilde{A_{i+1}[V'_2]} = \widetilde{H_{i+1}}$ .

Assume, on the contrary, that the colouring  $(V'_1, V'_2)$  is not acyclic. From the fact that  $V_1 \subseteq V'_1$  and  $V_2 \subseteq V'_2$  it follows that any alternating cycle must contain a vertex of level  $i + 1$  in  $A_{i+1}$ . Notice, that each vertex of level  $i + 1$  is of degree 2 in  $A_{i+1}$ . Next, observe that putting a vertex  $x$  into  $V'_2$ , we cannot create an alternating cycle. Moreover, in all the cases, except of *Subcase 2.2*, a given vertex  $x$  is put into  $V'_1$  only if the following two conditions hold:  $|N_{A_{i+1}}(x) \cap V_2| = 2$  and at least one neighbour of  $x$  does not have any neighbour in  $V'_1$ . Clearly, we cannot obtain an alternating cycle in this way. Therefore, there is only one situation when an alternating cycle can occur. Namely, in *Subcase 2.2* when both  $x$  and  $y$  have level  $i + 1$  and belong to  $V'_1$ . Observe that each such alternating cycle must contain at least 6 vertices. Assume that  $C$  is the shortest one. By Lemma 9 it follows that if  $W_2 = V'_2 \cap V(C)$ , then the vertices of  $W_2$  create a cycle in  $A_{i+1}$  and moreover, they induce a block in  $A_{i+1}[V'_2]$ . But, as it was described in *Subcase 2.2*, at least one of the vertices from the set  $W_2$  has level  $i + 1$  in  $A_{i+1}$ , so is of degree 2. Hence both its neighbours also belong to  $W_2$ . This fact clearly implies that  $C$  is not alternating.

Now we prove that  $\widetilde{A_{i+1}[V'_2]} = \widetilde{H_{i+1}}$ . On the contrary, assume that there is a vertex  $a$  of level  $i + 1$  in  $H_{i+1}$ , such that  $a \notin V'_2$ . Observe that

from the above it follows that  $a$  cannot be of degree 2. Therefore, we can assume, that  $a$  is of degree 1. Let  $b$  be the neighbour of  $a$  in  $H_{i+1}$ . Clearly,  $b$  does not have level  $i + 1$  in  $H_{i+1}$ .

If  $b$  has level  $i$  in  $H_i$ , then  $b$  is of degree 1 in  $H_i$ . Moreover, if  $c$  is the neighbour of  $b$  in  $H_i$ , then  $L(bc) = (i, +)$ . Thus we have the situation described in *Case 4* and the vertex  $a \in V'_2$ , a contradiction.

On the other hand, if  $b$  has level  $j < i$  in  $H_i$ , then according to the construction of  $H_{i+1}$  there is an edge  $bc$  in  $H_i$  such that  $L(bc) = (i - 1, 0)$ . If  $bc$  is of level  $i - 1$  in  $H_i$ , then it is also of level  $i - 1$  in  $A_i$ . Hence, as in *Case 3*, the vertex  $a \in V'_2$ , a contradiction. ■

Let  $\mathbf{H} = \{H_0, H_1, H_2, \dots\}$  be a set of graphs such that each  $H_i \in \mathcal{H}_i$  and each  $H_i$ , for  $i > 0$ , is a child of  $H_{i-1}$ . We define a class  $\mathcal{P}_{\mathbf{H}}$  of graphs as follows: a graph  $G$  belongs to  $\mathcal{P}_{\mathbf{H}}$  if and only if  $G$  is a subgraph of the disjoint union of some graphs from the set  $\tilde{\mathbf{H}}$ . Clearly, the class  $\mathcal{P}_{\mathbf{H}}$  is additive and hereditary.

**Theorem 3.** *Let  $\mathcal{P}_{\mathbf{H}}$  be the class of graphs defined as above. Then*

$$\mathcal{T}_2 \subseteq \mathcal{O} \odot \mathcal{P}_{\mathbf{H}}.$$

**Proof.** It clearly follows from Lemma 3 and Theorem 2. ■

#### 4. CONCLUDING REMARKS

Boiron, Sopena and Vignal proved in [1], that  $\mathcal{T}_2 \subseteq \mathcal{S}_5 \odot \mathcal{S}_5$ , where  $\mathcal{S}_5$  is the class of all graphs of maximum degree at most 5. We prove that this bound and our bounds are incomparable.

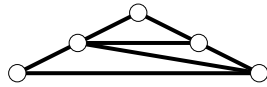


Figure 3. Graph  $B_0$

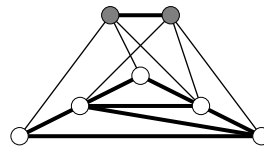


Figure 4. Graph  $B_1$

Let us consider at the beginning the graph  $B_0$  presented in Figure 3. We show that in every acyclic colouring  $(V_1, V_2)$  of the graph  $A_3$  either  $B_0 \subseteq \widetilde{A_3}[V_1]$  or  $B_0 \subseteq \widetilde{A_3}[V_2]$ .

Assume on the contrary, that there is an acyclic colouring  $(V_1, V_2)$  of  $A_3$ , such that  $B_0 \not\subseteq \widetilde{A_3}[V_1]$  and  $B_0 \not\subseteq \widetilde{A_3}[V_2]$ . Assume at the beginning, that all the vertices of level 0 belong to  $V_1$ . Clearly, at least two of the vertices of level 1 have to be in the set  $V_2$ , since otherwise  $B_0 \subseteq \widetilde{A_3}[V_1]$ . Moreover, at least one vertex of level 1 has to be in  $V_1$ , because the colouring is acyclic. Let  $x$  and  $y$  be the vertices of level 2 such that each of them is adjacent to two vertices from  $V_1$ .

The colouring is acyclic, hence both  $x$  and  $y$  cannot be in  $V_2$ . Therefore, either  $x \in V_1$  or  $y \in V_1$ , but in this case  $B_0 \subseteq \widetilde{A_3}[V_1]$ , a contradiction.

On the other hand, if we assume that two vertices of level 0, say  $x$  and  $y$ , are in  $V_1$ , and the remaining one in  $V_2$ , then let  $z$  be the vertex of level 1 adjacent to both  $x$  and  $y$ . Clearly,  $z \in V_1$ , because the colouring is acyclic. Now we can proceed as in the previous case, starting from the triangle  $x, y, z$ .

Let  $\mathcal{P}$  and  $\mathcal{Q}$  be additive hereditary classes of graphs such that  $\mathcal{T}_2 \subseteq \mathcal{P} \odot \mathcal{Q}$ . Clearly, from the above it follows that either  $B_0 \in \mathcal{P}$  or  $B_0 \in \mathcal{Q}$ . Moreover, if we assume additionally that the classes  $\mathcal{P}$  and  $\mathcal{Q}$  are both different from  $\mathcal{O}$ , then the graph  $B_1$ , presented in Figure 4, admits an acyclic  $(\mathcal{P}, \mathcal{Q})$ -colouring. Now we show that  $B_1$  does not have any acyclic  $(\mathcal{O}, \mathcal{T}_2)$ -colouring. Assume on the contrary, that  $(V_1, V_2)$  is an acyclic  $(\mathcal{O}, \mathcal{T}_2)$ -colouring of  $B_1$ . Observe at the beginning, that if we remove any vertex from  $B_1$ , then the remaining graph is not outerplanar. Hence we have to put at least two vertices to  $V_1$ , but in this case the colouring is not acyclic.

**Remark 2.** If  $\mathcal{P}$  and  $\mathcal{Q}$  are additive hereditary classes of graphs, both different from  $\mathcal{O}$ , and such that  $\mathcal{T}_2 \subseteq \mathcal{P} \odot \mathcal{Q}$ , then  $\mathcal{P} \odot \mathcal{Q} \not\subseteq \mathcal{O} \odot \mathcal{T}_2$ .

In particular, we can put both  $\mathcal{P}$  and  $\mathcal{Q}$  equal to  $\mathcal{S}_5$  in the above remark. Moreover, instead of the class  $\mathcal{T}_2$  we can take the class  $\mathcal{P}_{\mathbf{H}}$ , with  $\mathcal{P}_{\mathbf{H}}$  defined as at the end of Section 3. Hence,  $\mathcal{S}_5 \odot \mathcal{S}_5 \not\subseteq \mathcal{O} \odot \mathcal{P}_{\mathbf{H}}$ .

Next we show that there is a graph which belongs to  $\mathcal{O} \odot \mathcal{P}_{\mathbf{H}} \setminus \mathcal{S}_5 \odot \mathcal{S}_5$ , where the class  $\mathcal{P}_{\mathbf{H}}$  is defined as at the end of Section 3. Consider the graph  $B_3 = K_1 + 6K_{1,6}$ . It is obvious that  $B_3$  has an acyclic  $(\mathcal{O}, \mathcal{P}_{\mathbf{H}})$ -colouring  $(V_1, V_2)$ , because we can put the vertex of maximal degree to  $V_1$  and the other vertices to  $V_2$ . On the other hand,  $B_3$  does not admit any acyclic  $(\mathcal{S}_5, \mathcal{S}_5)$ -colouring, which follows from that fact that any copy of the graph  $K_{1,6}$  cannot be monochromatic, hence the vertex  $v$  of maximum degree is adjacent to at least six vertices of the same colour, as colour of  $v$ .

**Remark 3.** If  $\mathcal{P}_{\mathbf{H}}$  is the class of graphs defined as above, then  $\mathcal{O} \odot \mathcal{P}_{\mathbf{H}} \not\subseteq \mathcal{S}_5 \odot \mathcal{S}_5$ .

Let us recall that a maximal outerplanar graph  $G$  with at least 3 vertices is called a *2-path* of even order  $n = 2p$ , if  $G$  consists of two paths  $P_1 = (x_1, x_2, \dots, x_p)$  and  $P_2 = (y_1, y_2, \dots, y_p)$  and additional edges:  $x_i y_i$ ,  $i = 1, \dots, p$  and  $x_j y_{j+1}$  for  $j = 1, \dots, p-1$ . A 2-path of odd order is defined as  $H = G - x_p$ , where  $G$  is a 2-path of even order. A maximal outerplanar graph  $G$  with at least 3 vertices is called a *fan* of order  $n$ , if  $G$  is the join of  $K_1$  and a path of order  $n-1$ .

In [4] it was proved that if  $G$  is a 2-path (or a fan) of order 6, then  $\mathcal{T}_2 \subseteq \mathcal{O} \odot \text{Forb}(A_1, G)$ . They also proved that  $\mathcal{T}_2 \subseteq \mathcal{O} \odot \mathcal{FT}$  and  $\mathcal{T}_2 \subseteq \mathcal{O} \odot \mathcal{PT}$ , where classes  $\mathcal{FP}$  and  $\mathcal{PT}$  are defined as follows:

$\mathcal{PT} = \{G \in \mathcal{I} \text{ such that each its block is a subgraph of a 2-path}\},$

$\mathcal{FT} = \{G \in \mathcal{I} \text{ such that each its block is a subgraph of a fan}\}.$

One can observe that there is a set  $\mathbf{H}' = \{H'_0, H'_1, \dots\}$  of graphs such that each  $H'_i$  satisfies the following three conditions:

- $H'_i \in \mathcal{H}_i$ ,
- $H'_i$  is a child of the graph  $H'_{i-1}$ , for  $i \geq 1$ ,
- each its block is a 2-path.

Clearly, we have  $\mathcal{P}_{\mathbf{H}'} = \mathcal{PT}$ , where  $\mathcal{P}_{\mathbf{H}'}$  is defined as a class of graphs such that a graph  $\widetilde{F}$  belongs to  $\mathcal{P}_{\mathbf{H}'}$  if and only if  $F$  is a subgraph of a graph from the set  $\mathbf{H}'$ .

If we consider the class  $\mathcal{FT}$ , then it is easy to see that there is a set  $\mathbf{H}'' = \{H''_0, H''_1, \dots\}$  of graphs such that each  $H''_i$  satisfies the following three conditions:

- $H''_i \in \mathcal{H}_i$ ,
- $H''_i$  is a child of the graph  $H''_{i-1}$ , for  $i \geq 1$ ,
- each its block is a fan.

Clearly, we have  $\mathcal{P}_{\mathbf{H}''} \subset \mathcal{FT}$ , where  $\mathcal{P}_{\mathbf{H}''}$  is a class of graphs such that a graph  $F \in \mathcal{P}_{\mathbf{H}''}$  if and only if  $F$  is a subgraph of a graph from the set  $\widetilde{\mathbf{H}''}$ .

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