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# FRACTIONAL DISTANCE DOMINATION IN GRAPHS

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**Abstract**

Let $G = (V, E)$ be a connected graph and let $k$ be a positive integer with $k \leq rad(G)$. A subset $D \subseteq V$ is called a distance $k$-dominating set of $G$ if for every $v \in V - D$, there exists a vertex $u \in D$ such that $d(u, v) \leq k$. In this paper we study the fractional version of distance $k$-domination and related parameters.

**Keywords:** domination, distance $k$-domination, distance $k$-dominating function, $k$-packing, fractional distance $k$-domination .

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## 1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For basic terminology in graphs we refer to Chartrand and Lesniak [3]. For basic terminology in domination related concepts we refer to Haynes *et al.* [9].

Let $G = (V, E)$ be a graph. A subset $D$ of $V$ is called a *dominating set* of $G$ if every vertex in $V - D$ is adjacent to at least one vertex in $D$. A dominating set $D$ is called a *minimal dominating set* if no proper subset of $D$ is a dominating set of $G$. The minimum (maximum) cardinality of a minimal dominating set of $G$ is called the *domination number* (*upper domination number*) of $G$ and is denoted by $\gamma(G)$ $(\Gamma(G))$. Let $A$ and $B$ be two subsets of $V$. We say that $B$ *dominates* $A$ if every vertex in $A - B$ is adjacent to at least one vertex in $B$. If $B$ dominates $A$, then we write $B \to A$. Meir and Moon [12] introduced the concept of a $k$-packing and distance $k$-domination in a graph as a natural generalisation of the concept of domination. Let $G = (V, E)$ be a graph and $v \in V$. For any positive integer $k$, let $N_k(v) = \{u \in V : d(u, v) \leq k\}$ and $N_k[v] = N_k(v) \cup \{v\}$. A set $S \subseteq V$ is a *distance $k$-dominating set* of $G$ if $N_k[v] \cap S \neq \emptyset$ for every vertex $v \in V - S$. The minimum (maximum) cardinality among all minimal distance $k$-dominating sets of $G$ is called the *distance $k$-domination number* (*upper distance $k$-domination number*) of $G$ and is denoted by $\gamma_k(G)$ ($\Gamma_k(G)$). A set $S \subseteq V$ is said to be an *efficient distance $k$-dominating set* of $G$ if $|N_k[v] \cap S| = 1$ for all $v \in V - S$. Clearly, $\gamma(G) = \gamma_1(G)$. A distance $k$-dominating set of cardinality $\gamma_k(G)$ ($\Gamma_k(G)$) is called a $\gamma_k$ ($\Gamma_k$)-set. Hereafter, we shall use the term $k$-domination for distance $k$-domination.

Note that, $\gamma_k(G) = \gamma(G^k)$, where $G^k$ is the $k^{th}$ power of $G$, which is obtained from $G$ by joining all pairs of distinct vertices $u, v$ with $d(u, v) \leq k$. A subset $S \subseteq V(G)$ of a graph $G = (V, E)$ is said to be a *$k$-packing* ([12]) of $G$, if $d(u, v) > k$ for all pairs of distinct vertices $u$ and $v$ in $S$. The *$k$-packing number* $\rho_k(G)$ is defined to be the maximum cardinality of a $k$-packing set in $G$. The *corona* of a graph $G$, denoted by $G \circ K_1$, is the graph formed from a copy of $G$ by attaching to each vertex $v$ a new vertex $v'$ and an edge $\{v, v'\}$. The *Cartesian product* of graphs $G$ and $H$, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G \Box H$ if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. For a survey of results on distance domination we refer to Chapter 12 of Haynes *et al.* [10].

Hedetniemi *et al.* [11] introduced the concept of fractional domination in graphs. Grinstead and Slater [6] and Domke *et al.* [5] have presented several results on fractional domination and related parameters in graphs. Arumugam *et al.* [1] have investigated the fractional version of global domination in graphs.

Let $G = (V, E)$ be a graph. Let $g : V \to \mathbb{R}$ be any function. For any subset $S$ of $V$, let $g(S) = \sum_{v \in S} g(v)$. The *weight* of $g$ is defined by $|g| = g(V) = \sum_{v \in V} g(v)$. For a subset $S$ of $V$, the function $\chi_S : V \to \{0, 1\}$ defined by

$$\chi_S(v) = \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{if } v \notin S, \end{cases}$$

is called the *characteristic function* of $S$.

A function $g : V \to [0, 1]$ is called a *dominating function* ($DF$) of the graph $G = (V, E)$ if $g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ for all $v \in V$. For functions $f, g$ from $V \to [0, 1]$ we write $f \leq g$ if $f(v) \leq g(v)$ for all $v \in V$. Further, we write $f < g$ if $f \leq g$ and $f(v) < g(v)$ for some $v \in V$. A $DF$ $g$ of $G$ is *minimal* ($MDF$) if $f$ is not a $DF$ for all functions $f : V \to [0, 1]$ with $f < g$.

The *fractional domination number* $\gamma_f(G)$ and the *upper fractional domination number* $\Gamma_f(G)$ are defined as follows:

$$\gamma_f(G) = \min\{|g| : g \text{ is a minimal dominating function of } G\},$$
$$\Gamma_f(G) = \max\{|g| : g \text{ is a minimal dominating function of } G\}.$$

For a dominating function $f$ of $G$, the *boundary set* $\mathcal{B}_f$ and the *positive set* $\mathcal{P}_f$ are defined by $\mathcal{B}_f = \{u \in V(G) : f(N[u]) = 1\}$ and $\mathcal{P}_f = \{u \in V(G) : f(u) > 0\}$. A function $g : V \to [0,1]$ is called a *packing function* ($PF$) of the graph $G = (V,E)$ if $g(N[v]) = \sum_{u \in N[v]} g(u) \leq 1$ for all $v \in V$. The *lower fractional packing number* $p_f(G)$ and the *fractional packing number* $P_f(G)$ are defined as follows:

$$p_f(G) = \min\{|g| : g \text{ is a maximal packing function of } G\},$$
$$P_f(G) = \max\{|g| : g \text{ is a maximal packing function of } G\}.$$

It was observed in Chapter 3 of [10] that for every graph $G$, $1 \leq \gamma_f(G) = P_f(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_f(G)$. We need the following theorems:

**Theorem 1.1** [5]**.** *For a graph $G$, $p_f(G) \leq \rho_2(G) \leq P_f(G)$.*

**Theorem 1.2** [2]**.** *A DF $f$ of $G$ is an MDF if and only if $\mathcal{B}_f \to \mathcal{P}_f$.*

**Theorem 1.3** [2]**.** *If $f$ and $g$ are MDFs of $G$ and $0 < \lambda < 1$ then $h_\lambda = \lambda f + (1-\lambda)g$ is an MDF of $G$ if and only if $\mathcal{B}_f \cap \mathcal{B}_g \to \mathcal{P}_f \cup \mathcal{P}_g$.*

**Theorem 1.4** [5]**.** *If $G$ is an $r$-regular graph of order $n$, then $\gamma_f(G) = \frac{n}{r+1}$.*

**Theorem 1.5** [4]**.** *Let $G$ be a block graph. Then for any integer $k \geq 1$, we have $\rho_{2k}(G) = \gamma_k(G)$.*

For other families of graphs satisfying $\rho_2(G) = \gamma(G)$, we refer to Rubalcaba *et al.* [13].

**Definition 1.6** [15]**.** A *linear Benzenoid chain* $B(h)$ of length $h$ is the graph obtained from $P_2 \Box P_{h+1}$ by subdividing exactly once each edge of the two copies of $P_{h+1}$. Hence $B(h)$ is a subgraph of $P_2 \Box P_{2h+1}$. The graph $B(4)$ is given in Figure 1.

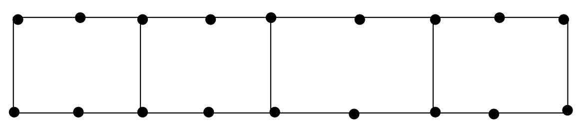

Figure 1. $B(4)$.

**Theorem 1.7** [15]**.** *For the linear benzenoid chain $B(h)$, we have*

$$\gamma_k(B(h)) = \begin{cases} \lceil \frac{h+1}{k} \rceil & \text{if } k \neq 2, \\ \lceil \frac{h+2}{k} \rceil & \text{if } k = 2. \end{cases}$$

We refer to Scheinerman and Ullman [14] for fractionalization techniques of various graph parameters. Hattingh *et al.* [8] introduced the distance $k$-dominating function and proved that the problem of computing the upper distance fractional domination number is NP-complete. In this paper we present further results on fractional distance $k$-domination.

## 2. Distance $k$-dominating Function

Hattingh *et al.* [8] introduced the following concept of fractional distance $k$-domination.

**Definition 2.1.** A function $g : V \to [0,1]$ is called a *distance $k$-dominating function* or simply a *$k$-dominating function* ($kDF$) of a graph $G = (V,E)$, if for every $v \in V$, $g(N_k[v]) = \sum_{u \in N_k[v]} g(u) \geq 1$. A $k$-dominating function ($kDF$) $g$ of a graph $G$ is called a *minimal $k$-dominating function* ($MkDF$) if $f$ is not a $k$-dominating function of $G$ for all functions $f : V \to [0,1]$ with $f < g$. The *fractional $k$-domination number* $\gamma_{kf}(G)$ and the *upper fractional $k$-domination number* $\Gamma_{kf}(G)$ are defined as follows:

$$\gamma_{kf}(G) = \min\{|g| : g \text{ is an } MkDF \text{ of } G\},$$
$$\Gamma_{kf}(G) = \max\{|g| : g \text{ is an } MkDF \text{ of } G\}.$$

We observe that if $k \geq rad(G)$, then $\Delta(G^k) = n - 1$ and $\gamma_{kf}(G) = 1$. Hence throughout this paper, we assume that $k < rad(G)$.

**Lemma 2.2** [8]**.** *Let $f$ be a $k$-dominating function of a graph $G = (V,E)$. Then $f$ is minimal $k$-dominating if and only if whenever $f(v) > 0$ there exists some $u \in N_k[v]$ such that $f(N_k[u]) = 1$.*

**Remark 2.3.** The characteristic function of a $\gamma_k$-set and that of a $\Gamma_k$-set of a graph $G$ are $MkDF$s of $G$. Hence it follows that $1 \leq \gamma_{kf}(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq \Gamma_{kf}(G)$.

**Definition 2.4.** A function $g : V \to [0,1]$ is called a *distance $k$-packing function* or simply a *$k$-packing function* of a graph $G = (V,E)$, if for every $v \in V$, $g(N_k[v]) \leq 1$. A $k$-packing function $g$ of a graph $G$ is *maximal* if $f$ is not a $k$-packing function of $G$ for all functions $f : V \to [0,1]$ with $f > g$. The *fractional $k$-packing number* $p_{kf}(G)$ and the *upper fractional $k$-packing number* $P_{kf}(G)$ are defined as follows:

$$p_{kf}(G) = \min\{|g| : g \text{ is a maximal k-packing function of } G\},$$
$$P_{kf}(G) = \max\{|g| : g \text{ is a maximal k-packing function of } G\}.$$

**Observation 2.5.** The fractional $k$-domination number $\gamma_{kf}(G)$ is the optimal solution of the following linear programming problem (LPP).

Minimize $z = \sum_{i=1}^{n} f(v_i)$ , subject to

$$\sum_{u \in N_k[v]} f(u) \geq 1 \text{ and } 0 \leq f(v) \leq 1 \text{ for all } v \in V.$$

The dual of the above LPP is

Maximize $z = \sum_{i=1}^{n} f(v_i)$, subject to

$$\sum_{u \in N_k[v]} f(u) \leq 1 \text{ and } 0 \leq f(v) \leq 1 \text{ for all } v \in V.$$

The optimal solution of the dual LPP is the upper fractional $k$-packing number $P_{kf}(G)$. It follows from the strong duality theorem that $P_{kf}(G) = \gamma_{kf}(G)$. Hence if there exists a minimal $k$-dominating function $g$ and a maximal $k$-packing function $h$ with $|g| = |h|$, then $P_{kf}(G) = |h| = |g| = \gamma_{kf}(G)$.

**Lemma 2.6.** *For any graph $G$ of order $n$ we have $\gamma_{kf}(G) \leq \frac{n}{k+1}$ and the bound is sharp.*

***Proof.*** Since $|N_k[u]| \geq k + 1$ for all $u \in V$, it follows that the constant function $f$ defined on $V$ by $f(v) = \frac{1}{k+1}$ for all $v \in V$, is a $k$-dominating function with $|f| = \frac{n}{k+1}$. Hence $\gamma_{kf}(G) \leq \frac{n}{k+1}$. To prove the sharpness of this bound, consider the graph $G$ consisting of a cycle of length $2k$ with a path of length $k$ attached to each vertex of the cycle. Clearly $n = 2k(k + 1)$. Further the set $S$ of all pendant vertices of $G$ forms an efficient $k$-dominating set of $G$ and hence $\sum_{u \in N_k[v]} f(u) = 1$ for all $v \in V$ where $f$ is the characteristic function of $S$. Hence $\gamma_k(G) = \gamma_{kf}(G) = 2k = \frac{n}{k+1}$. ■

**Observation 2.7.** We observe that $\gamma_{kf}(G) = \gamma_f(G^k)$. Hence the following is an immediate consequence of Theorem 1.2.

Let $G$ be a graph and let $A, B \subseteq V$. We say that $A$, *$k$-dominates* $B$ if $N_k[v] \cap A \neq \emptyset$ for all $v \in B$ and we write $A \rightarrow_k B$. Now for any $kDF$ $f$ of $G$ let $\mathcal{P}_f = \{u \in V(G) : f(u) > 0\}$ and $\mathcal{B}_f = \{u \in V(G) : f(N_k[u]) = 1\}$. Then $f$ is an $MkDF$ of $G$ if and only if $\mathcal{B}_f \rightarrow_k \mathcal{P}_f$.

**Observation 2.8.** If $f$ and $g$ are $kDFs$ of a graph $G = (V, E)$ and $\lambda \in (0, 1)$, then the convex combination of $f$ and $g$ defined by $h_\lambda(v) = \lambda f(v) + (1 - \lambda)g(v)$ for all $v \in V$ is a $kDF$ of $G$. However, the convex combination of two $MkDFs$ of a graph $G$ need not be minimal, as shown in the following example.

Consider the cycle $G = C_7 = (u_1 u_2 \ldots u_7 u_1)$ with $k = 2$. The function $f : V(G) \rightarrow [0, 1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_1, u_5\}, \\ 0 & \text{otherwise}, \end{cases}$$

is a minimal 2-dominating function of $G$ with $\mathcal{P}_f = \{u_1, u_5\}$, $\mathcal{B}_f = \{u_1, u_2, u_4, u_5\}$. Also, the function $g : V(G) \rightarrow [0, 1]$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x \in \{u_3, u_6\}, \\ 0 & \text{otherwise}, \end{cases}$$

is a minimal 2-dominating function of $G$ with $\mathcal{P}_g = \{u_3, u_6\}$, $\mathcal{B}_g = \{u_2, u_3, u_6, u_7\}$. Let $h = \frac{1}{2}f + \frac{1}{2}g$. Then $h(u_1) = h(u_3) = h(u_5) = h(u_6) = \frac{1}{2}$, $h(u_2) = h(u_4) = h(u_7) = 0$, $h(N_2[u_i]) = \frac{3}{2}$ for $i \neq 2$ and $h(N_2[u_2]) = 1$. Hence $\mathcal{P}_h = \{u_1, u_3, u_5, u_6\}$ and $\mathcal{B}_h = \{u_2\}$. Since $u_5, u_6 \notin N_2[u_2]$ we have $\mathcal{B}_h$ does not 2-dominate $\mathcal{P}_h$ and hence the $kDF$ $h$ is not minimal.

**Observation 2.9.** If $f$ and $g$ are $MkDFs$ of $G$ and $0 < \lambda < 1$, then $h_\lambda = \lambda f + (1-\lambda)g$ is an $MkDF$ of $G$ if and only if $\mathcal{B}_f \cap \mathcal{B}_g \to_k \mathcal{P}_f \cup \mathcal{P}_g$.

**Observation 2.10.** For the cycle $C_n$, the graph $G = C_n^k$ is $2k$-regular and hence it follows from Theorem 1.4 that $\gamma_{kf}(C_n) = \frac{n}{2k+1}$.

We now proceed to determine the fractional $k$-domination number of several families of graphs.

**Proposition 2.11.** *For the hypercube $Q_n$, $\gamma_{kf}(Q_n) = \frac{2^n}{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{k}}$.*

***Proof.*** For any two vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $Q_n$, $d(x, y) \leq k$ if and only if $x$ and $y$ differ in at most $k$ coordinates and hence $Q_n^k$ is $r$-regular where $r = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$. Hence by Theorem 1.4, we have $\gamma_{kf}(Q_n) = \frac{2^n}{r+1} = \frac{2^n}{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{k}}$. ■

**Proposition 2.12.** *For the graph $G = P_2 \square C_n$, we have*

$$\gamma_{kf}(G) = \begin{cases} \frac{8}{7} & \text{if } n = 4 \text{ and } k = 2, \\ \frac{n}{2k} & \text{if } n \geq 5. \end{cases}$$

***Proof.*** If $n = 4$ and $k = 2$, then $G^2$ is a 6-regular graph and hence $\gamma_{2f}(G) = \frac{8}{7}$. If $n \geq 5$, $G^k$ is a $(4k-1)$-regular graph and hence $\gamma_{kf}(G) = \frac{2n}{4k-1+1} = \frac{n}{2k}$. ■

**Theorem 2.13.** *Let $G = C_n \circ K_1$. Then $\gamma_{kf}(G) = \frac{n}{2k-1}$.*

***Proof.*** Let $C_n = (v_1 v_2 \ldots v_n v_1)$. Let $u_i$ be the pendant vertex adjacent to $v_i$. Clearly, $|N_k[u_i] \cap V(C_n)| = 2k - 1$ and $N_k[u_i] \subset N_k[v_i]$, $1 \leq i \leq n$. Hence the function $g : V(G) \to [0, 1]$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x = u_i, \\ \frac{1}{2k-1} & \text{if } x = v_i \end{cases}$$

is a minimal $k$-dominating function of $G$ with $|g| = \frac{n}{2k-1}$. Also we have $|N_k[v_i] \cap \{u_j : 1 \leq j \leq n\}| = 2k - 1$, $1 \leq i \leq n$. Hence the function $h : V(G) \to [0, 1]$ defined by

$$h(x) = \begin{cases} \frac{1}{2k-1} & \text{if } x = u_i, \\ 0 & \text{if } x = v_i \end{cases}$$

is a maximal $k$-packing function of $G$ with $|h| = \frac{n}{2k-1}$. Hence by Observation 2.5, we have $\gamma_{kf}(G) = \frac{n}{2k-1}$. ■

**Theorem 2.14.** *For the grid $G = P_2 \square P_n$, we have*

$$\gamma_{kf}(G) = \begin{cases} \frac{n(n+2k)}{2k(n+k)} & \textit{if } n \equiv 0 \pmod{2k}, \\ \lceil \frac{n}{2k} \rceil & \textit{otherwise.} \end{cases}$$

***Proof.*** Let $P_2 = (u_0, u_1)$ and $P_n = (v_0, v_1, \ldots, v_{n-1})$, so that $V(G) = \{(u_i, v_j) : i = 0, 1,\ 0 \le j \le n-1\}$.

*Case* 1. $n \equiv 0 \pmod{2k}$. Let $n = 2kp$, $p > 1$. Define $f : V(G) \to [0,1]$ by

$$f((u_i, v_j)) = \begin{cases} (\frac{1}{2p+1})(p - \lfloor \frac{j}{2k} \rfloor) & \text{if } j \equiv (k-1) \pmod{2k}, \\ (\frac{1}{2p+1})(\lfloor \frac{j}{2k} \rfloor + 1) & \text{if } j \equiv k \pmod{2k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f$ is a $k$-dominating function of $G$. Also, since $f((u_0, v_j)) = f((u_1, v_j))$ for all $j$, we have $|f| = 2(\sum_{j=0}^{n-1} f((u_0, v_j))) = \frac{2}{2p+1}[(p + (p-1) + \cdots + 3 + 2 + 1) + (1 + 2 + 3 + \cdots + p)] = \frac{2p(p+1)}{2p+1} = \frac{n(n+2k)}{2k(n+2)}$. Now consider the function $h : V(G) \to [0,1]$ defined by

$$h((u_i, v_j)) = \begin{cases} (\frac{1}{2p+1})(p - \lfloor \frac{j}{2k} \rfloor) & \text{if } j \equiv 0 \pmod{2k}, \\ (\frac{1}{2p+1})(\lfloor \frac{j}{2k} \rfloor + 1) & \text{if } j \equiv (2k-1) \pmod{2k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $h$ is a $k$-packing function of $G$ with $|h| = \frac{2p(p+1)}{2p+1} = \frac{n(n+2k)}{2k(n+2k)}$. Hence $\gamma_{kf}(G) = \frac{n(n+2k)}{2k(n+k)}$.

*Case* 2. $n \not\equiv 0 \pmod{2k}$. Let $n = 2kq + r$, $1 \le r \le 2k-1$. Let $S = S_1 \cup S_2$ and

$$S_1 = \begin{cases} \{(u_0, v_j) : j \equiv 0 \pmod{4k}\} & \text{if } 1 \le r \le k, \\ \{(u_0, v_j) : j \equiv (k-1) \pmod{4k}\} & \text{if } k+1 \le r \le 2k-1. \end{cases}$$

$$S_2 = \begin{cases} \{(u_1, v_j) : j \equiv 2k \pmod{4k}\} & \text{if } 1 \le r \le k, \\ \{(u_1, v_j) : j \equiv (3k-1) \pmod{4k}\} & \text{if } k+1 \le r \le 2k-1. \end{cases}$$

Let $f$ be the characteristic function of $S$. Since $d(x,y) \ge 2k+1$ for all $x, y \in S$, it follows that $f(N_k[u]) = 1$ for all $u \in V(G)$. Thus $f$ is both a minimal $k$-dominating function and a maximal $k$-packing function of $G$ and hence $\gamma_{kf}(G) = |f| = |S| = \lceil \frac{n}{2k} \rceil$. ■

A special case of the above theorem gives the following result of Hare [7].

**Corollary 2.15.** *For the grid graph $G = P_2 \square P_n$, we have*

$$\gamma_f(G) = \begin{cases} \frac{n(n+2)}{2(n+1)} & \textit{if } n \textit{ is even,} \\ \lceil \frac{n}{2} \rceil & \textit{if } n \textit{ is odd.} \end{cases}$$

## 3. Graphs with $\gamma_{kf}(G) = \gamma_k(G)$

In this section we obtain several families of graphs for which the fractional $k$-domination number and the $k$-domination number are equal.

**Lemma 3.1.** *If a graph $G$ has an efficient $k$-dominating set, then $\gamma_{kf}(G) = \gamma_k(G)$.*

***Proof.*** Let $D$ be an efficient $k$-dominating set of $G$. Then $|N_k[u] \cap D| = 1$ for all $u \in V(G)$. Hence the characteristic function of $D$ is both a minimal $k$-dominating function and a maximal $k$-packing function of $G$ and so $\gamma_{kf}(G) = \gamma_k(G)$. ■

**Lemma 3.2.** *For any graph $G$, $\gamma_{kf}(G) = 1$ if and only if $\gamma_k(G) = 1$.*

***Proof.*** Suppose $\gamma_k(G) = 1$. Since $\gamma_{kf}(G) \leq \gamma_k(G)$, it follows that $\gamma_{kf}(G) = 1$. Conversely, let $\gamma_{kf}(G) = 1$. Then $\gamma_f(G^k) = 1$ and hence $\gamma(G^k) = 1$. Since $\gamma(G^k) = \gamma_k(G)$ the result follows. ■

**Lemma 3.3.** *For any graph $G$, $p_{kf}(G) \leq \rho_{2k}(G) \leq P_{kf}(G)$.*

***Proof.*** Let $u \in V(G)$. Since $N_k[u] = N_{G^k}[u]$, we have $p_{kf}(G) = p_f(G^k)$, $P_{kf}(G) = P_f(G^k)$ and $\rho_{2k}(G) = \rho_2(G^k)$.

Hence the result follows from Theorem 1.1. ■

**Corollary 3.4.** *For any graph $G$, $1 \leq p_{kf}(G) \leq \rho_{2k}(G) \leq P_{kf}(G) = \gamma_{kf}(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq \Gamma_{kf}(G)$.*

**Corollary 3.5.** *If $G$ is any graph with $\rho_{2k}(G) = \gamma_k(G)$, then $\gamma_{kf}(G) = \gamma_k(G)$.*

**Corollary 3.6.** *If $G$ is a block graph, then $\gamma_{kf}(G) = \gamma_k(G)$.*

***Proof.*** It follows from Theorem 1.5 that $\rho_{2k}(G) = \gamma_k(G)$ and hence the result follows. ■

**Corollary 3.7.** *For any tree $T$, we have $\gamma_{kf}(T) = \gamma_k(T)$.*

**Theorem 3.8.** *For the graph $G = P_{k+1} \square P_n$ where $n \equiv 1 \pmod{(k+1)}$, $k \geq 1$, we have $\gamma_{kf}(G) = \gamma_k(G) = \lceil \frac{n}{k+1} \rceil$.*

***Proof.*** Let $n = (k+1)q + 1$, $q \geq 1$. Clearly $|V(G)| = n(k+1) = (k+1)^2 q + (k+1)$. Let $P_{k+1} = (u_0, u_1, u_2, \ldots, u_k)$ and $P_n = (v_0, v_1, \ldots, v_{n-1})$ so that $V(G) = \{(u_i, v_j) : 0 \leq i \leq k,\ 0 \leq j \leq n-1\}$.

Now let $S_1 = \{(u_0, v_i) : i \equiv 0 \pmod{2(k+1)}\}$, $S_2 = \{(u_k, v_i) : i \equiv (k+1) \pmod{2(k+1)}\}$ and $S = S_1 \cup S_2$. Clearly, $d(x, y) = (2k+1)r,\ r \geq 1$, for all $x, y \in S$ and $|S| = \lceil \frac{n}{k+1} \rceil = q + 1$. Also, $(u_0, v_0)$ and exactly one of the vertices $(u_0, v_{n-1})$ or $(u_k, v_{n-1})$ are in $S$ and each of these two vertices $k$-dominates $\frac{(k+1)(k+2)}{2}$ vertices of $G$. Also, if $u \in N_k[x] \cap N_k[y]$, where $x, y \in S$, then $d(u,x) \leq k$, $d(u,y) \leq k$ and so $d(x,y) \leq d(x,u) + d(u,y) \leq 2k$, which is a contradiction. Thus $N_k[x] \cap N_k[y] = \emptyset$ for all $x, y \in S$. Each of the remaining vertices of $S$ $k$-dominates $(k+1)^2$ vertices of $G$. Further, $|V(G)| - (k+1)(k+2)$ is a multiple of $(k+1)^2$ and hence it follows that $S$ is an efficient $k$-dominating set of $G$. Hence, by Lemma 3.1, we have $\gamma_{kf}(G) = \gamma_k(G) = |S| = \lceil \frac{n}{k+1} \rceil$. ∎

**Theorem 3.9.** *For the graph $G = P_3 \square P_n$, we have $\gamma_{2f}(G) = \gamma_2(G) = \lceil \frac{n}{3} \rceil$.*

***Proof.*** If $n \equiv 1 \pmod 3$, then the result follows from Theorem 3.8. Suppose $n \equiv 0 \pmod 3$ or $2 \pmod 3$. Let $n = 3q$, $q \geq 1$ or $n = 3q+2$, $q \geq 0$. Let $P_3 = (u_0, u_1, u_2)$ and $P_n = (v_0, v_1, \ldots, v_{n-1})$ so that $V(G) = \{(u_i, v_j) : 0 \leq i \leq 2,\ 0 \leq j \leq n-1\}$. Now $D = \{(u_1, v_j) : j \equiv 1 \pmod 3\}$ is a $\gamma_2$-set of $G$ with $|D| = \lceil \frac{n}{3} \rceil$ and hence $\gamma_2(G) = \lceil \frac{n}{3} \rceil$. Further $f = \chi_D$ is a 2-dominating function of $G$ with $|f| = \lceil \frac{n}{3} \rceil$. Also let $S_1 = \{(u_0, v_j) : j \equiv 0 \pmod 6\}$, $S_2 = \{(u_2, v_j) : j \equiv 3 \pmod 6\}$ and $S = S_1 \cup S_2$. Then $g = \chi_S$ is a 2-packing function of $G$ with $|g| = \lceil \frac{n}{3} \rceil$. Hence $\gamma_{2f}(G) = \lceil \frac{n}{3} \rceil$. ∎

**Observation 3.10.** The graph $G = P_3 \square P_5$ does not have an efficient 2-dominating set. In fact the set $S = \{(u_0, v_0), (u_2, v_3)\}$ efficiently 2-dominates 14 vertices of $G$ and the vertex $(u_0, v_4)$ is not 2-dominated by $S$. Further if $S$ is any 2-dominating set of $G$ with $|S| = \gamma_2(G) = 2$, then at least one vertex of $G$ is 2-dominated by both vertices of $S$. This shows that the converse of Lemma 3.1 is not true.

**Theorem 3.11.** *For the linear benzenoid chain $G = B(h)$, we have*

$$\gamma_{kf}(G) = \gamma_k(G) = \begin{cases} \frac{h}{2} + 1 & \text{if } k = 2 \text{ and } h \equiv 0 \pmod 2, \\ \lceil \frac{h}{k} \rceil & \text{if } k \geq 3 \text{ and } h \equiv \lfloor \frac{k}{2} \rfloor \pmod k. \end{cases}$$

***Proof.*** Since $G = B(h)$ is a subgraph of $P_2 \square P_{2h+1}$, we take $V(G) = \{(u_i, v_j) : i = 0, 1,\ 0 \leq j \leq 2h\}$, where $P_2 = (u_0, u_1)$ and $P_{2h+1} = (v_0, v_1, \ldots, v_{2h})$. Clearly, $|V(G)| = 4h + 2$. Any vertex $u \in V(G)$ $k$-dominates at most $4k$ vertices of $G$ and hence $\gamma_k(G) \geq \lceil \frac{4h+2}{4k} \rceil$.

*Case* 1. $k = 2$ and $h \equiv 0 \pmod 2$. In this case we have $\gamma_2(G) \geq \lceil \frac{4h+2}{8} \rceil = \frac{h}{2} + 1$. Now let $S_1 = \{(u_0, v_j) : j \equiv 0 \pmod 8\}$, $S_2 = \{(u_1, v_j) : j \equiv 4 \pmod 8\}$ and $S = S_1 \cup S_2$. Clearly, for any $x, y \in S$, $d(x,y) \geq 5$ and hence $N_2[x] \cap N_2[y] = \emptyset$. Also $|S| = \lceil \frac{2h+1}{4} \rceil = \frac{h}{2} + 1$. Now $(u_0, v_0)$ and exactly one of the vertices $(u_0, v_{2h})$ or $(u_1, v_{2h})$ is in $S$ and each of these two vertices 2-dominates exactly 5 vertices of $G$. Each of the remaining vertices of $S$ 2-dominates 8 vertices of $G$. Further $|V(G)| - 10 = 4h - 8 = 8(\frac{h}{2} - 1)$, which is a multiple of 8 and hence it follows that $S$ is an efficient 2-dominating set of $G$. Hence $\gamma_{2f}(G) = \gamma_2(G) = |S| = \frac{h}{2} + 1$.

*Case* 2. $k \geq 3$ and $h \equiv \lfloor \frac{k}{2} \rfloor \pmod{k}$. Let $h = kq + \lfloor \frac{k}{2} \rfloor$, $q \geq 1$. In this case we have $\gamma_k(G) \geq \lceil \frac{4h+2}{4k} \rceil = \lceil \frac{h}{k} \rceil$. Now let $S_1 = \{(u_0, v_j) : j \equiv (k-1) \pmod{4k}\}$, $S_2 = \{(u_1, v_j) : j \equiv (3k-1) \pmod{4k}\}$ and $S = S_1 \cup S_2$. Clearly, $d(x, y) = (2k+1)r$, $r \geq 1$ for all $x, y \in S$, hence $N_k[x] \cap N_k[y] = \emptyset$. Also $|S| = \lceil \frac{2h-(k-1)}{2k} \rceil = \lceil \frac{h}{k} \rceil$.

Now, when $k$ is odd, exactly one of the vertices $(u_0, v_{2h})$ or $(u_1, v_{2h})$ is in $S$ and it $k$-dominates $2k+1$ vertices. When $k$ is even, exactly one of the vertices $(u_0, v_{2h-1})$ or $(u_1, v_{2h-1})$ are in $S$ and it $k$-dominates $2k+3$ vertices. The vertex $(u_0, v_{k-1})$ $k$-dominates $4k-1$ vertices. In both cases the number of vertices of $G$ which are not $k$-dominated by these two vertices is a multiple of $4k$ and each of the remaining vertices of $S$ $k$-dominates $4k$ vertices of $G$. Hence it follows that $S$ is an efficient $k$-dominating set of $G$ so that $\gamma_{kf}(G) = \gamma_k(G) = |S| = \lceil \frac{h}{k} \rceil$. ■

**Conclusion.** In this paper we have determined the fractional $k$-domination number of several families of graphs. We have also obtained several families of graphs for which $\gamma_{kf}(G) = \gamma_k(G)$. The study of the fractional version of distance $k$-irredundance and distance $k$-independence remains open. Slater has mentioned several efficiency parameters such as redundance and influence in Chapter 1 of [10]. One can investigate these parameters for fractional distance domination. The following are some interesting problems for further investigation.

1. Characterize the class of graphs $G$ for which $\gamma_{kf}(G) = \frac{n}{k+1}$.

2. Characterize the class of graphs $G$ with $\gamma_{kf}(G) = \gamma_k(G)$.

3. Determine $\gamma_{kf}(P_r \Box P_s)$ for $r, s \geq 4$.

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