INTERSECTION GRAPH OF GAMMA SETS IN THE TOTAL GRAPH

T. TAMIZH CHELVAM

AND

T. ASIR

Department of Mathematics
Manonmaniam Sundaranar University
Tirunelveli- 627 012, Tamil Nadu, India

e-mail: {tamche59, asirjacob75}@gmail.com

Abstract

In this paper, we consider the intersection graph $I_\Gamma(\mathbb{Z}_n)$ of gamma sets in the total graph on $\mathbb{Z}_n$. We characterize the values of n for which $I_\Gamma(\mathbb{Z}_n)$ is complete, bipartite, cycle, chordal and planar. Further, we prove that $I_\Gamma(\mathbb{Z}_n)$ is an Eulerian, Hamiltonian and as well as a pancyclic graph. Also we obtain the value of the independent number, the clique number, the chromatic number, the connectivity and some domination parameters of $I_\Gamma(\mathbb{Z}_n)$.

Keywords: total graph, gamma sets, intersection graph, Hamiltonian, coloring, connectivity, domination number.

2010 Mathematics Subject Classification: 05C40,05C45, 05C69.

1. INTRODUCTION

In recent years, the interplay between ring structure and graph structure are studied by many researchers. For such kind of study, researchers define a graph whose vertices are set of elements of a ring or set of ideals in a ring and edges are defined with respect to a condition on the elements of the vertex set. A graph is defined out of non-zero zero divisors of a ring and is called zero-divisor graph of a ring [3]. Interesting variations are also defined like total graphs [2], unit graphs [4] and comaximal graphs [11] associated with rings. Also graphs are defined out of ideals of a ring, namely annihilating-ideal graph of a ring, intersection graph

of ideals of rings [6, 7] etc. The graphs constructed from rings help us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theoretic language and then the geometric properties of graphs help us to explore some interesting results related to algebraic structures of rings. Now, in this paper, we construct a graph called intersection graph of gamma sets in the total graph of a commutative ring R with vertex set as collection of all γ-sets of the total graph of R and two distinct vertices A and B are adjacent if and only if $A \cap B \neq \emptyset$. This graph is denoted by $I_\Gamma(R)$. We investigate the interplay between the graph-theoretic properties of $I_\Gamma(\mathbb{Z}_n)$ and the ring-theoretic properties of $\mathbb{Z}_n$.

Let A be a set and let S be a collection of nonempty subsets of A. The intersection graph of S is the graph whose vertices are the elements of S and where two vertices are adjacent if the subsets have a nonempty intersection [12]. Let R be a commutative ring, $Z(R)$ be its set of zero-divisors. Anderson, Badawi [2] introduced the concept of the total graph corresponding to a commutative ring. For futher research on total graphs, one can refer [1, 13]. The *total graph* of R, denoted by $T_\Gamma(R)$, is the undirected graph with vertices R, and for distinct $x, y \in R$, the vertices x and y are adjacent if $x + y \in Z(R)$. In this paper, we consider the intersection graph $I_\Gamma(R)$ of gamma sets in $T_\Gamma(R)$, where $R = \mathbb{Z}_n$.

Let $G = (V, E)$ be a graph. We say that G is connected if there is a path between any two distinct vertices of G. Let G_1 and G_2 be two graphs. The union of G_1 and G_2, which is denoted by $G_1 \cup G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge-set $E(G_1) \cup E(G_2)$. A graph G of order $m \geq 3$ is *pancyclic* if G contains cycles of all lengths from 3 to m. A set of vertices in G is *independent* if no two vertices in the set are adjacent. The *independent number $\beta_0(G)$*, is the maximum cardinality of an independent set in G. The *clique number $\omega(G)$*, is the number of vertices in a largest complete subgraph of G. For basic graph theory parameters, we refer to reader [5, 8, 9].

A subset S of V is called a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. A dominating set S is called a *perfect domination set* if every vertex in $V - S$ is adjacent to exactly one vertex in S. The *domination number $\gamma(G)$* is defined to be the minimum cardinality of a dominating set in G and the corresponding dominating set is called as a γ-set of G. In a similar way, we define the *perfect domination number $\gamma_p(G)$*, *independent dominating number $i(G)$*, *total domination number $\gamma_t(G)$*, *connected domination number $\gamma_c(G)$* and *clique domination number $\gamma_{cl}(G)$*. A graph G is called *excellent* if for every vertex $v \in V(G)$ there is a γ-set S containing v. A *domatic partition* of G is a partition of $V(G)$, all of whose class are dominating sets in G. The maximum number of classes of a domatic partition of G is called the *domatic number* of G and is denoted by $d(G)$. A graph G is called *domatically full* if $d(G) = \delta(G) + 1$ and G is called *well-covered* if $\beta_0(G) = i(G)$. For basic domination parameters,

we refer to reader [10].

The purpose of this article is to study the basic graph theoretical properties of the new graph $I_\Gamma(\mathbb{Z}_n)$. In Section 2, we obtain the degree of each vertex, diameter and girth of $I_\Gamma(\mathbb{Z}_n)$. Also we characterize the values of n for which $I_\Gamma(\mathbb{Z}_n)$ is complete, bipartite, cycle, chordal and planar. In Section 3, we prove that $I_\Gamma(\mathbb{Z}_n)$ is an Eulerian, Hamiltonian and pancyclic graph. In Section 4, we obtain the values of independent number and clique number of $I_\Gamma(\mathbb{Z}_n)$. In Section 5, we obtain the values of chromatic numbers of $I_\Gamma(\mathbb{Z}_n)$ and characterize the values of n for which $I_\Gamma(\mathbb{Z}_n)$ is a perfect graph. Also the intersection graph of gamma sets in $T_\Gamma(\mathbb{Z}_n)$ represent very reliable networks, which means that the vertex connectivity $k(I_\Gamma(\mathbb{Z}_n))$ equals the degree of regularity of $I_\Gamma(\mathbb{Z}_n)$. In Section 6, we find several domination parameters of $I_\Gamma(\mathbb{Z}_n)$.

Throughout this paper, we denote the intersection graph of gamma sets in $T_\Gamma(\mathbb{Z}_n)$ as $I_\Gamma(\mathbb{Z}_n)$ for a positive integer n. Also p_1 denotes the smallest prime divisor of n, $|S|$ denotes number of elements in S, $\langle S \rangle$ denotes the subgraph induced by S, $\mathbb{Z}^+$ denotes the set of all positive integers and K_m denotes the complete graph with m vertices. For convenience, we use the notation x or y to denote an element of $\mathbb{Z}_n$ and u, v or w for the vertices of $I_\Gamma(\mathbb{Z}_n)$. (i.e, u, v and w represent the γ-sets of $T_\Gamma(\mathbb{Z}_n)$).

The following results are part of a paper submitted for publication [14]. In order to understand them, they are presented along with the proof.

Lemma 1. *Let $n > 1$ be an integer. Then*
(i) *$T_\Gamma(\mathbb{Z}_n)$ is regular if and only if n is even. Moreover $deg(v) = n - \phi(n) - 1$, for every $v \in T_\Gamma(\mathbb{Z}_n)$.*
(ii) *$\Delta[T_\Gamma(\mathbb{Z}_n)] = \delta[T_\Gamma(\mathbb{Z}_n)] + 1$ if and only if n is odd and in this case $\Delta[T_\Gamma(\mathbb{Z}_n)] = n - \phi(n)$. In particular $deg(v) = \delta$ if $v \in Z(\mathbb{Z}_n)$ and $deg(v) = \Delta$ if $v \notin Z(\mathbb{Z}_n)$.*

Theorem 2. *Let n be a composite integer and p_1 be the smallest prime divisor of n. Then $\gamma(T_\Gamma(\mathbb{Z}_n)) = p_1$.*

Proof. Let $G = T_\Gamma(\mathbb{Z}_n)$. Assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where p_i's are primes with $2 \le p_1 < \cdots < p_m$ and $\alpha_i \ge 1$ for $1 \le i \le m$.

Case 1. Let n be even and so $p_1 = 2$. Let x be an even number and y be an odd number in $\mathbb{Z}_n$. Since $Z(\mathbb{Z}_n)$ contains all the even numbers in $\mathbb{Z}_n$, $\{x, y\}$ is a dominating set in $T_\Gamma(\mathbb{Z}_n)$. Since no vertex in $T_\Gamma(\mathbb{Z}_n)$ has degree $n - 1$, $\gamma(T_\Gamma(\mathbb{Z}_n)) = p_1 = 2$.

Case 2. Let n be odd. Take $r = \frac{n - p_1}{p_1}$. Note that 0 is adjacent to $p_1, \ldots, rp_1$ in G and 1 is adjacent to $p_1 - 1, 2p_1 - 1, \ldots, rp_1 - 1, n - 1$. Inductively $p_1 - 2$ is adjacent to $2, p_1 + 2, 2p_1 + 2, \ldots, (r-1)p_1 + 2, n - p_1 + 2$ and $p_1 - 1$ is adjacent to $1, p_1 + 1, 2p_1 + 1, \ldots, (r-1)p_1 + 1, n - p_1 + 1$. Therefore $\{0, 1, \ldots, p_1 - 1\}$ is

a dominating set in G. Suppose $S = \{x_1, x_2, \ldots, x_{p_1-1}\}$ is a dominating set of G with $p_1 - 1$ elements. Let $r_i \equiv x_i \pmod{p_1}$ and $R = \{r_1, r_2, \ldots, r_{p_1-1}\}$. We claim that $|N[S]| \leq |N[R]|$.

(a) If $r_j \neq r_k$ for all $j \neq k$, then $|S| = |R|$. As mentioned in Lemma 1, $deg(0) = \delta$ and $deg(r_t) = \Delta$ for all $0 \neq r_t \in R$. If $0 \in R$, then there exists an $x_\ell \in S$ such that $deg(x_\ell) = \delta$. In view these remarks, one can conclude that R contains exactly one vertex of minimum degree and all other vertices are of maximum degree, where as S may contain more than one vertex with minimum degree. Thus

(1) $$\sum_{x_j \in S} deg(x_j) \leq \sum_{r_j \in R} deg(r_j).$$

We show that $|N[r_j] \cap N[r_k]| \leq |N[x_j] \cap N[x_k]|$ for all j, k with $j \neq k$. Let $A_i = \{0, p_i, 2p_i, \ldots, n - p_i\}$, for $i = 1, \ldots, m$. For $a \in \mathbb{Z}_n$, define $N_{A_i}[a] = -a + A_i$, for $1 \leq i \leq m$. Clearly $N[a] = N_{A_1}[a] \cup N_{A_2}[a] \cup \cdots \cup N_{A_m}[a]$. Further $N[r_j] \cap N[r_k] = \{x : x \in N_{A_r}[r_j] \cap N_{A_s}[r_k],\ r \neq s\}$ and $N[x_j] \cap N[x_k] = \{x : x \in N_{A_r}[x_j] \cap N_{A_s}[x_k],\ r \neq s\} \cup \{x : x \in N_{A_r}[x_j] \cap N_{A_r}[x_k],\ r \neq 1\}$. Now we claim that $|N_{A_r}[r_j] \cap N_{A_s}[r_k]| = |N_{A_r}[x_j] \cap N_{A_s}[x_k]|$. Without loss of generality, assume that $r < s$. Clearly $|N_{A_s}[x]| = |-x + A_s| = \frac{n}{p_s}$. One may note that for every consecutive p_r elements in the set $\{-x + A_s\}$ one element is common with $N_{A_r}[x]$ and so $|N_{A_r}[r_j] \cap N_{A_s}[r_k]| = \frac{n}{p_r p_s}$. Similarly $|N_{A_r}[x_j] \cap N_{A_s}[x_k]| = \frac{n}{p_r p_s}$. Therefore $|N_{A_r}[r_j] \cap N_{A_s}[r_k]| = |N_{A_r}[x_j] \cap N_{A_s}[x_k]|$ and hence

(2) $$|N[r_j] \cap N[r_k]| \leq |N[x_j] \cap N[x_j]|.$$

Thus from $|S| = |R|$ and from equations (1) and (2), we get $|N[S]| \leq |N[R]|$.

(b) Now assume that $r_j = r_k$ for some j, k and $j \neq k$. Then $N[x_j] \cap N[x_k] = \{x : x \in N_{A_r}[x_j] \cap N_{A_s}[x_k],\ r \neq s\} \cup \{x : x \in N_{A_r}[x_j] \cap N_{A_r}[x_k],\ \text{for all } r\}$. Therefore $|N[S]| \leq |N[R]|$. From this to conclude that S is not a dominating set it is enough to prove that R is not a dominating set. Note that all elements of R are less than p_1 and there exists one i such that $0 \leq i \leq p_1 - 1$ and $i \notin R$. Note that $i \notin R$ and $N[n - i] = \{\ldots, i - p_1, i, i + p_1, \ldots\}$ and so $n - i \notin N[R]$, i.e., $n - i$ is not dominated by any of the vertices in R. Thus R is not a dominating set. Hence $\gamma(T_\Gamma(\mathbb{Z}_n)) = p_1$. ∎

The results given below identify all γ-sets in $T_\Gamma(\mathbb{Z}_n)$ for all values of n.

Theorem 3. *Let n be a composite integer and p_1 be the smallest prime divisor of n. A set $S = \{x_1, x_2, \ldots, x_{p_1}\} \subset V(T_\Gamma(\mathbb{Z}_n))$ is a γ-set of $T_\Gamma(\mathbb{Z}_n)$ if and only if $x_i + lp_1 \notin S$ for all $i = 1, \ldots, p_1$ and $l \in \mathbb{Z}^+$.*

Proof. Let $S = \{x_1, x_2, \ldots, x_{p_1}\}$ with $x_i + lp_1 \notin S$ for all $1 \leq i \leq p_1$. Let $A = \{0, p_1, 2p_1, \ldots, n - p_1\}$. Since $x_j \neq x_i + lp_1$ for all $i \neq j$, $1 \leq i, j \leq p_1$, the

cosets $x_i + A$ and $x_j + A$ are distinct. Note that $|-x_i + A| = \frac{n}{p_1}$. Since each x_i is adjacent to all the elements of the coset $-x_i + A$ and $|S| = p_1$, $|N[S]| = n$ and so S is a dominating set of $T_\Gamma(\mathbb{Z}_n)$. By Theorem 2, S is a γ-set of $T_\Gamma(\mathbb{Z}_n)$.

Conversely, assume that $S = \{x_1, x_2, \ldots, x_{p_1}\}$ is a γ-set. Suppose there exist $i \neq j$ such that $x_j = x_i + lp_1$ where $l \in \mathbb{Z}^+$. From this there exist some k ($0 \leq k \leq p_1 - 1$) such that $x_m \not\equiv k \pmod{p_1}$ for all $x_m \in S$. Then as in the proof of Theorem 2, S is not a dominating set, which is a contradiction. ∎

Lemma 4. *Let p be a prime number. Then the following are true:*
(i) $\gamma(T_\Gamma(\mathbb{Z}_p)) = \frac{p+1}{2}$.
(ii) *Every γ-set of $T_\Gamma(\mathbb{Z}_p)$ contains 0. Further $S = \{0, x_1, \ldots, x_{\frac{p-1}{2}}\} \subset V(T_\Gamma(\mathbb{Z}_p))$ is a γ-set of H if and only if $x_i \neq 0$ and $x_i \neq -x_j$ for all $i \neq j$.*

2.   Basic Properties of $I_\Gamma(\mathbb{Z}_n)$

In this section, we study some basic properties of the intersection graph of gamma sets in $T_\Gamma(\mathbb{Z}_n)$ and the same is denoted by $I_\Gamma(\mathbb{Z}_n)$. Actually we characterize, when $I_\Gamma(\mathbb{Z}_n)$ is complete, bipartite, cycle, chordal and planar. Also we find the diameter and girth of $I_\Gamma(\mathbb{Z}_n)$. Note that if $n = 1$ or 2, then $I_\Gamma(\mathbb{Z}_n) = K_1$ and so hereafter we assume that $n > 2$. Also note that by Theorem 13, $I_\Gamma(\mathbb{Z}_n)$ is a connected graph. First we start the section with the cardinality of the vertex set and the degree of each vertex of the new graph $I_\Gamma(\mathbb{Z}_n)$.

Lemma 5. *Let $n > 2$ be any positive integer and p_1 be the smallest prime divisor of n. Then*
(i) *the number of gamma sets in $T_\Gamma(\mathbb{Z}_n)$ is given by*
$$|V(I_\Gamma(\mathbb{Z}_n))| = \begin{cases} 2^{\frac{p-1}{2}} & \text{if } n = p, \text{ where } p \text{ is a prime number,} \\ (\frac{n}{p_1})^{p_1} & \text{otherwise.} \end{cases}$$
(ii) *For any composite integer n, $deg(v) = (\frac{n}{p_1})^{p_1-1} + (\frac{n}{p_1} - 1)(\frac{n}{p_1})^{p_1-2} + (\frac{n}{p_1} - 1)^2(\frac{n}{p_1})^{p_1-3} + \cdots + (\frac{n}{p_1} - 1)^{p_1-1} - 1$ for all $v \in V(I_\Gamma(\mathbb{Z}_n))$. In particular if n is even, then $deg(v) = n - 2$ for all v.*
(iii) *For any prime integer p, the graph $I_\Gamma(\mathbb{Z}_p) = K_{2^{\frac{p-1}{2}}}$.*

Proof. (i) Suppose $n = p$ for some prime p. By Theorem 3 and Lemma 4, each element $x_i \neq 0$ in any γ-set of $T_\Gamma(\mathbb{Z}_p)$ has 2 choices and hence the number of γ-sets in $T_\Gamma(\mathbb{Z}_n)$ is equal to $2^{\frac{p-1}{2}}$. If $n \neq p$ for any prime p, then any γ-set of $T_\Gamma(\mathbb{Z}_n)$ contains p_1 elements. Note that by Theorem 2, each element x_i in any γ-set of $T_\Gamma(\mathbb{Z}_n)$ has $\frac{n}{p_1}$ choices and so the number of γ-sets in $T_\Gamma(\mathbb{Z}_n)$ is $(\frac{n}{p_1})^{p_1}$.

(ii) Let $v = \{x_1, x_2, \ldots, x_{p_1}\} \in V(I_\Gamma(\mathbb{Z}_n))$. By the definition of the intersection graph, v is adjacent to all the vertices in $I_\Gamma(\mathbb{Z}_n)$ containing x_i for some

$i = 1, \ldots, p_1$. By part (i), there are $(\frac{n}{p_1} - 1)^{p_1 - 1}$ vertices in $I_\Gamma(\mathbb{Z}_n)$ containing x_1. Since there are $(\frac{n}{p_1} - 1)^{p_1 - 2}$ vertices in $I_\Gamma(\mathbb{Z}_n)$ containing both x_1 and x_2, v is adjacent to $(\frac{n}{p_1} - 1)^{p_1 - 1} - (\frac{n}{p_1} - 1)^{p_1 - 2} = (\frac{n}{p_1} - 1)(\frac{n}{p_1})^{p_1 - 2}$ new vertices in $I_\Gamma(\mathbb{Z}_n)$. Continuing in this way and since $I_\Gamma(\mathbb{Z}_n)$ is simple, we get required result. If n is even, then $p_1 = 2$ and so $deg(v) = n - 2$.

(iii) By Lemma 4, 0 is an element in every γ-set of $T_\Gamma(\mathbb{Z}_p)$ and so from (i), the result follows. ∎

Remark 6. Let n be any composite integer, $G = I_\Gamma(\mathbb{Z}_n)$ and $v \in V(G)$. Then $2 \le deg(v) \le (\frac{n}{p_1})^{p_1} - 2$.

Proof. Since each vertex $v = \{x_1, \ldots, x_{p_1}\} \in V(G)$ is adjacent to distinct vertices $\{x_1, \ldots, x_{p_1 - 1}, x_{p_1} + p_1\}, \{x_1 + p_1, x_2, \ldots, x_{p_1}\}$, $deg(v) \ge 2$. Also any vertex $v = \{x_1, \ldots, x_{p_1}\} \in V(G)$ is not adjacent to a vertex $u = \{x_1 + p_1, \ldots, x_{p_1} + p_1\} \in V(G)$ and so $deg(v) \le |V(I_\Gamma(\mathbb{Z}_n))| - 2 = (\frac{n}{p_1})^{p_1} - 2$. ∎

Note that the bounds obtained in the above remark are sharp. For example, the lower and upper bounds are same in case of $n = 4$. In the next proposition we show that, this is the only case where lower and upper bounds are equal.

Proposition 7. *For any composite integer $n > 2$, $deg(v)$ attains either lower or upper bound of* Remark 6 *if and only if $n = 4$.*

Proof. When $n = 4$, $2 = deg(v) = (\frac{n}{p_1})^{p_1} - 2$ and hence only if part is trivial.

Conversely, assume that $deg(v)$ attains either lower or upper bound of text Remark 6.

Case 1. Let $deg(v) = 2$. If n is even, by Lemma 5(ii), $deg(v) = n - 2 = 2$ and so $n = 4$. If n is odd, then $n \ge 3p_1$ where p_1 is the smallest odd prime divisor of n. Since $deg(v) = 2$, $(\frac{n}{p_1})^{p_1 - 1} + (\frac{n}{p_1} - 1)(\frac{n}{p_1})^{p_1 - 2} + (\frac{n}{p_1} - 1)^2(\frac{n}{p_1})^{p_1 - 3} + \cdots + (\frac{n}{p_1} - 1)^{p_1 - 1} = 3$. Note that each term in this expression is positive and $(\frac{n}{p_1})^{p_1 - 1} \ge 9$. Thus there exists no odd positive integer n satisfying this equation.

Case 2. Let $deg(v) = (\frac{n}{p_1})^{p_1} - 2$. If n is even, then $deg(v) = n - 2 = (\frac{n}{2})^2 - 2$ and so $n = 4$. If n is odd, then $n \ge 3p_1$. Since $deg(v) = (\frac{n}{p_1})^{p_1} - 2$,

$$(\frac{n}{p_1})^{p_1 - 1} + (\frac{n}{p_1} - 1)(\frac{n}{p_1})^{p_1 - 2} + (\frac{n}{p_1} - 1)^2(\frac{n}{p_1})^{p_1 - 3} + \cdots + (\frac{n}{p_1} - 1)^{p_1 - 1} + 1 = (\frac{n}{p_1})^{p_1}.$$

Note that $(\frac{n}{p_1} - 1)(\frac{n}{p_1})^{p_1 - 2} < (\frac{n}{p_1})^{p_1 - 1} - 1$, $(\frac{n}{p_1} - 1)^{p_1 - 1} < (\frac{n}{p_1})^{p_1 - 1} - 1$ and each of the other terms of left hand side is $\le (\frac{n}{p_1})^{p_1 - 1}$. Thus $(\frac{n}{p_1})^{p_1} < p_1(\frac{n}{p_1})^{p_1 - 1} - 1 \le (\frac{n}{p_1})^{p_1} - 1$, a contradiction. ∎

Lemma 8. *Let $n \ge 4$ be any composite integer. Then*
(i) *$I_\Gamma(\mathbb{Z}_n)$ is a regular graph.*

(ii) $I_\Gamma(\mathbb{Z}_n)$ is not a complete graph. In particular, $I_\Gamma(\mathbb{Z}_n)$ has no vertex of degree $|V(I_\Gamma(\mathbb{Z}_n))| - 1$.

(iii) $I_\Gamma(\mathbb{Z}_n)$ is a bipartite graph if and only if $n = 4$.

(iv) $I_\Gamma(\mathbb{Z}_n)$ is a cycle if and only if $n = 4$.

Proof. (i) Follows from the fact ascertained in Lemma 5(ii).

(ii) Follows from Remark 6.

(iii) If $n = 4$, then $I_\Gamma(\mathbb{Z}_n) = K_{2,2}$.

Conversely, assume that $I_\Gamma(\mathbb{Z}_n)$ is complete bipartite and suppose $n > 4$. Then $n \geq 3p_1$ where p_1 is the smallest prime divisor of n and so the distinct vertices $v_1 = \{0, 1, \ldots, p_1 - 1\}$, $v_2 = \{0, 1, \ldots, p_1 - 2, 2p_1 - 1\}$ and $v_3 = \{0, 1, \ldots, p_1 - 2, 3p_1 - 1\}$ form an odd cycle, a contradiction. Therefore $n = 4$.

(iv) If $n = 4$, then $I_\Gamma(\mathbb{Z}_n) = C_4$.

Conversely, if $I_\Gamma(\mathbb{Z}_n)$ is a cycle, then $deg(v) = 2$ for all v. Now $n = 4$ follows from Proposition 7. ∎

Corollary 9. $I_\Gamma(\mathbb{Z}_n)$ is a complete graph if and only if $n = p$ for some prime.

Theorem 10. $I_\Gamma(\mathbb{Z}_n)$ is a chordal graph if and only if $n = p$ for some prime.

Proof. If $n = p$ for some prime, then $I_\Gamma(\mathbb{Z}_n)$ is complete and so $I_\Gamma(\mathbb{Z}_n)$ is a chordal graph.

Conversely, $I_\Gamma(\mathbb{Z}_n)$ is a chordal graph. Suppose n is a composite integer and p_1 is the smallest prime divisor of n. Then $n \geq 2p_1$. Clearly the subgraph induced by the set $\{\{0, 1, \ldots, p_1 - 1\}, \{1, 2, \ldots, p_1\}, \{p_1, p_1 + 1, \ldots, 2p_1 - 1\}, \{0, p_1 + 1, p_1 + 2 \ldots, 2p_1 - 1\}\}$ is C_4, which is a contradiction. ∎

The following result characterizes the values of n for which $I_\Gamma(\mathbb{Z}_n)$ is planar. Let S_k denote the sphere with k handles, where k is a non-negative integer, that is, S_k is an oriented surface of genus k. The *genus* of any graph G, denoted $g(G)$, is the minimal integer ℓ such that the graph can be embedded in S_ℓ. A genus 0 graph is called a planar graph and a genus 1 graph is called a toroidal graph. For details on embedding a graph in a surface, see [15].

Theorem 11. $I_\Gamma(\mathbb{Z}_n)$ is planar if and only if $n \leq 5$.

Proof. If $n = 2, 3$ or 5, then by Lemma 5(iii), $I_\Gamma(\mathbb{Z}_n)$ is planar. If $n = 4$, then $I_\Gamma(\mathbb{Z}_n) = K_{2,2}$ and so planar.

Conversely, assume that $I_\Gamma(\mathbb{Z}_n)$ is planar. Suppose $n > 5$. If n is prime, then clearly $I_\Gamma(\mathbb{Z}_n)$ is not planar. Suppose n is a composite integer and p_1 is the smallest prime divisor of n. If n is odd, then $n \geq 3p_1$ and so $(\frac{n}{p_1})^{p_1 - 1} \geq 3^2 = 9$. Now, by Theorem 20 (next section), $K_5 \subseteq I_\Gamma(\mathbb{Z}_n)$ and so, by Kuratowski's Theorem, $I_\Gamma(\mathbb{Z}_n)$ is non-planar. If n is even and $n \geq 10$, then $(\frac{n}{p_1})^{p_1 - 1} \geq 5$ and so by Theorem 20, $I_\Gamma(\mathbb{Z}_n)$ is non-planar. Therefore n is either 6 or 8. In both of these cases, one can check using the Remark 12 that, $I_\Gamma(\mathbb{Z}_n)$ is non-planar. ∎

Remark 12. If $n = 6$, then we can draw $I_\Gamma(\mathbb{Z}_n)$ on the surface of a torus and the same is given in Figure 1. Therefore $g(I_\Gamma(\mathbb{Z}_n)) = 1$.
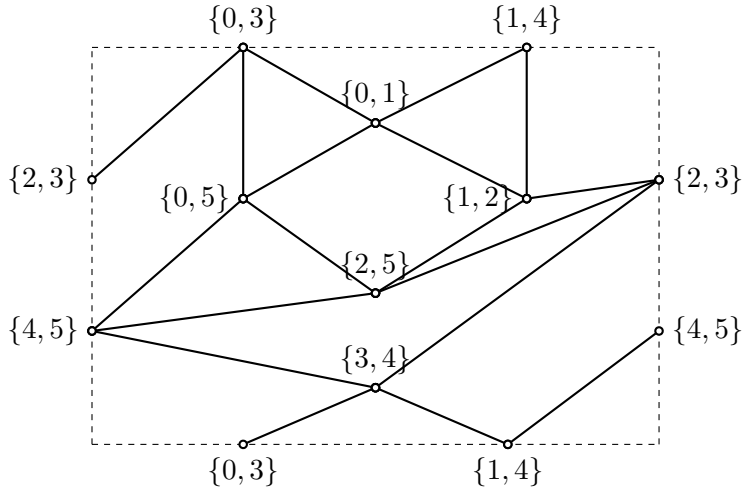

Figure 1. Embedding of $I_\Gamma(\mathbb{Z}_6)$ in torus

Theorem 13. *For any integer $n > 2$, the following holds.*

(i) $gr(I_\Gamma(\mathbb{Z}_n)) = \begin{cases} 4 & \text{if } n = 4, \\ 3 & \text{otherwise.} \end{cases}$

(ii) $diam(I_\Gamma(\mathbb{Z}_n)) \leq 2$. *In particular, $I_\Gamma(\mathbb{Z}_n)$ is connected.*

(iii) $I_\Gamma(\mathbb{Z}_n)$ *is self-centered.*

Proof. Let p be a prime integer.

(i) Follows from the facts that, $I_\Gamma(\mathbb{Z}_p)$ is complete and by the proof of (iii) in Lemma 8.

(ii) If $n = p$ for some prime p, then $diam(I_\Gamma(\mathbb{Z}_n)) = 1$. On the other hand, let $u = \{x_1, \ldots, x_{p_1}\}$ and $v = \{y_1, \ldots, y_{p_1}\}$ be two vertices in $I_\Gamma(\mathbb{Z}_n)$. If $x_i = y_j$ for some $1 \leq i, j \leq p_1$, then u and v are adjacent. If $x_i \neq y_j$ for all $1 \leq i, j \leq p_1$. Assume that $y_1 \equiv k \pmod{p_1}$. By Theorem 3, there is a x_i for $1 \leq i \leq p_1$ such that $x_i \equiv k \pmod{p_1}$. Then $w = \{x_1, \ldots, x_{i-1}, y_1, x_{i+1}, \ldots, x_{p_1}\} \in V(I_\Gamma(\mathbb{Z}_n))$ and so $u - w - v$ is a path in $I_\Gamma(\mathbb{Z}_n)$. Hence $diam(G) = 2$.

(iii) From the proof of (ii), we have for all $v \in V(I_\Gamma(\mathbb{Z}_n))$,

$$e(v) = \begin{cases} 1 & \text{if } n \text{ is a prime number,} \\ 2 & \text{otherwise.} \end{cases}$$

Hence $I_\Gamma(\mathbb{Z}_n)$ is self-centered.   ∎

3.   Eulerian and Hamiltonian Nature of $I_\Gamma(\mathbb{Z}_n)$

In this section, we are interested in the Eulerian and Hamiltonian nature of $I_\Gamma(\mathbb{Z}_n)$. We begin this section with a lemma, which is used frequently in next two sections. Unless otherwise specified, we assume that every vertex in $I_\Gamma(\mathbb{Z}_n)$ is of the form $\{x_1, \ldots, x_{p_1}\}$ with $x_1 < \cdots < x_{p_1}$.

Lemma 14. *Let n be a composite integer, p_1 be the smallest prime divisor of n and $A_i = \{\{i, x_2, \ldots, x_{p_1}\} \in V(I_\Gamma(\mathbb{Z}_n))\}$ for $0 \le i \le n - p_1$. Then $\langle A_i \rangle \subseteq I_\Gamma(\mathbb{Z}_n)$ is complete and $\langle A_i \cup A_j \rangle$ is connected for all $i \ne j$.*

Proof. Clearly $\langle A_i \rangle \subseteq I_\Gamma(\mathbb{Z}_n)$ is complete. Let $u = \{i, x_2, \ldots, x_{p_1}\} \in A_i$, $i < j$ and $j \equiv k \pmod{p_1}$. By Theorem 3, there is an element $x \in u$ such that $x \equiv k \pmod{p_1}$. If $x > i$, replace x by j in u and rearrange elements in ascending order. Let the new vertex be u'. Note that $u' \in A_i$ and $j \in u'$ and so u' is adjacent to all the vertices of A_j. If $x = i$, then the set $\{i, j+1, j+2, \ldots, j+(p_1-1)\} \in A_i$ and is adjacent to $\{j, j+1, \ldots, j+(p_1-1)\} \in A_j$. Thus $\langle A_i \cup A_j \rangle$ is connected. ∎

Theorem 15. *For any positive integer n, $I_\Gamma(\mathbb{Z}_n)$ is Eulerian if and only if n is a composite integer.*

Proof. If n is prime, then $I_\Gamma(\mathbb{Z}_n)$ is a complete graph with even number of vertices and so is not Eulerian. If n is even, then $deg(v) = n - 2$ for all $v \in V(I_\Gamma(\mathbb{Z}_n))$ and so every vertex is an even vertex. When n is odd, $(\frac{n}{p_1})^{p_1-1}$ is odd and $(\frac{n}{p_1} - 1)$ is even. From this, in view of Lemma 5(ii), we have $deg(v)$ is even for all $v \in V(I_\Gamma(\mathbb{Z}_n))$. Hence in both the cases $I_\Gamma(\mathbb{Z}_n)$ is Eulerian. ∎

Theorem 16. *For any positive integer n, $I_\Gamma(\mathbb{Z}_n)$ is Hamiltonian.*

Proof. When n is a prime, $I_\Gamma(\mathbb{Z}_n)$ is complete and so trivially Hamiltonian. Let n be a composite integer. Arrange every vertex in $I_\Gamma(\mathbb{Z}_n)$ in the form $\{x_1, \ldots, x_{p_1}\}$ with $x_1 < \cdots < x_{p_1}$ and $A_i = \{v \in V(I_\Gamma(\mathbb{Z}_n)) : x_1 = i\}$, $0 \le i \le n - p_1$. By Theorem 14, $\langle A_i \rangle$ is complete and $\langle A_i \cup A_j \rangle$ is connected in $I_\Gamma(\mathbb{Z}_n)$ for all $i \ne j$. Also note that the vertex $u = \{i, i+1, \ldots, i+p_1-1\} \in A_i$ is adjacent to all the vertices in A_{i+1}. Start with the vertex $\{0, n-p_1+1, \ldots, n-1\} \in A_0$, traverse the vertices in $\langle A_0 \rangle$ through a spanning path in $\langle A_0 \rangle$, pass on to $\langle A_1 \rangle$. Continue this through $\langle A_2 \rangle$, $\langle A_3 \rangle, \ldots, \langle A_{n-p_1} \rangle$ to get a Hamiltonian path ending at $\{n - p_1, \ldots, n-1\} \in A_{n-p_1}$. This Hamiltonian path together with the edge joining $\{n-p_1, n-p_1+1, \ldots, n-1\}$ and $\{0, n-p_1+1, \ldots, n-1\}$ gives a required Hamiltonian cycle in $I_\Gamma(\mathbb{Z}_n)$. ∎

Corollary 17. *For any positive integer n, $I_\Gamma(\mathbb{Z}_n)$ is pancyclic if and only if $n \ne 4$.*

Proof. Let $I_\Gamma(\mathbb{Z}_n)$ be pancyclic. Suppose $n = 4$, then $G = K_{2,2}$ does not contain C_3, which is contradiction to our assumption.

Conversely, let $n \neq 4$. Let $|V(I_\Gamma(\mathbb{Z}_n))| = m$ and p_1 be the smallest prime divisor of n. By Theorem 16, $I_\Gamma(\mathbb{Z}_n)$ is Hamiltonian and so $I_\Gamma(\mathbb{Z}_n)$ contains C_m. Remove the vertex $\{n - p_1, \ldots, n - 1\}$ from the cycle C_m and note that $\{n - p_1, \ldots, n - 1\} \in A_{n-p_1}$. Observe that the vertex $\{n - p_1 - 1, \ldots, n - 2\} \in A_{n-p_1-1}$ is adjacent to $\{0, n - p_1 - 1, n - p_1, \ldots, n - 1\} \in A_0$ and so we have C_{m-1} as a subgraph of $I_\Gamma(\mathbb{Z}_n)$. Let $0 \leq i < j < k \leq n - p_1$. If $i \not\equiv j \pmod{p_1}$, then by Theorem 3, there exists a vertex $u \in A_i$ which contains j and so u is adjacent to all the vertices in A_j. From this, leaving the vertices in C_{m-1} one by one from $A_1, A_2, \ldots, A_{p_1-1}, A_{p_1+1}, \ldots, A_{2p_1-1}, \ldots, A_{n-p_1-1}$. Now the remaining cycle contains vertices from $A_0, A_{p_1}, \ldots, A_{n-2p_1}$. Note that if $i \equiv j \equiv k \pmod{p_1}$, then the subgraph induced by the vertices $\{i, j + 1, k + 2, k + 3, \ldots, k + p_1 - 1\} \in A_i, \{j, j + 1, \ldots, j + p_1 - 1\} \in A_j$ and $\{k, k + 1, \ldots, j + p_1 - 1\} \in A_k$ forms K_3. From this and $|A_0| = (\frac{n}{p_1})^{p_1-1} \geq 3$, we get cycles of all lengths as subgraphs of $I_\Gamma(\mathbb{Z}_n)$. Hence $I_\Gamma(\mathbb{Z}_n)$ is pancyclic.   ∎

4.   Independent and Clique numbers of $I_\Gamma(\mathbb{Z}_n)$

In this section, we obtain the values of independent number and clique number of $I_\Gamma(\mathbb{Z}_n)$. First we start with vertex and edge independent numbers of $I_\Gamma(\mathbb{Z}_n)$.

Lemma 18. *Let n be a composite integer and p_1 be the smallest prime divisor of n. Then the independence number $\beta_0(I_\Gamma(\mathbb{Z}_n)) = \frac{n}{p_1}$.*

Proof. Since $\langle S \rangle = \langle \{0, 1, \ldots, p_1 - 1\}, \{p_1, \ldots, 2p_1 - 1\}, \ldots, \{n - p_1, \ldots, n - 1\} \rangle$ includes all elements of $\mathbb{Z}_n$ only once, S is a maximal independent set in $I_\Gamma(\mathbb{Z}_n)$.   ∎

Lemma 19. *Let n be a composite integer and p_1 be the smallest prime divisor of n. Then the edge independent number $\alpha'(I_\Gamma(\mathbb{Z}_n)) = \left\lfloor \frac{1}{2}(\frac{n}{p_1})^{p_1} \right\rfloor$. Moreover $I_\Gamma(\mathbb{Z}_n)$ has a perfect matching if and only if $n = 4k$ for some $k \in \mathbb{Z}^+$.*

Proof. Let $|V(I_\Gamma(\mathbb{Z}_n))|$ be even, every vertex in $I_\Gamma(\mathbb{Z}_n)$ is of the form $\{x_1, \ldots, x_{p_1}\}$ with $x_1 < \cdots < x_{p_1}$ and $A_i = \{\{i, X_2, \ldots, x_{p_1}\} \in V(I_\Gamma(\mathbb{Z}_n))\}$. If $|A_i|$ is even for all $i = 0, 1, \ldots, n - p_1$, then $\langle A_i \rangle$ has a perfect matching. If $|A_i|$ and $|A_j|$ are odd for some $j \neq i$, by Lemma 14, $\langle A_i \cup A_j \rangle$ is connected for all $i \neq j$ and so there exists $u \in A_i$ and $v \in A_j$ such that $uv \in E(I_\Gamma(\mathbb{Z}_n))$. Consider a maximum matching M_i of A_i not containing u and a maximum matching M_j of $\langle A_j \rangle$ not containing v, then $M_i \cup M_j \cup \{uv\}$ is a perfect matching of $\langle A_i \cup A_j \rangle$. Proceeding in this way, one can get a perfect matching of $I_\Gamma(\mathbb{Z}_n)$ and so $\alpha'(I_\Gamma(\mathbb{Z}_n)) = \frac{|V(I_\Gamma(\mathbb{Z}_n))|}{2}$.

If $|V(I_\Gamma(\mathbb{Z}_n))|$ is odd, then as proved above, we have $\alpha'(I_\Gamma(\mathbb{Z}_n)) = \frac{|V(I_\Gamma(\mathbb{Z}_n))|-1}{2}$. From these, $I_\Gamma(\mathbb{Z}_n)$ has a perfect matching if and only if $|V(I_\Gamma(\mathbb{Z}_n))|$ is even. Since n is composite, by Lemma 5, $|V(I_\Gamma(\mathbb{Z}_n))| = (\frac{n}{p_1})^{p_1}$ is even if and only if $n = 4k$ for some $k \in \mathbb{Z}^+$. ∎

Theorem 20. *Let n be a composite integer and p_1 be the smallest prime divisor of n. Then the clique number*

$$\omega(I_\Gamma(\mathbb{Z}_n)) = \begin{cases} 2^{\frac{p-1}{2}} & \text{if } n = p, \text{ where } p \text{ is a prime number}, \\ (\frac{n}{p_1})^{p_1-1} & \text{otherwise}. \end{cases}$$

Proof. First case follows from Lemma 5(iii). Hence assume that n is a composite number. Let $B_i = \{v \in V(I_\Gamma(\mathbb{Z}_n)) : v \text{ contains } i\}$. Note that $\langle B_i \rangle$ is complete. Next we claim that no vertex in $V(I_\Gamma(\mathbb{Z}_n)) - B_i$ is adjacent to all the vertices in B_i. For, let $u = \{y_1, \ldots, y_{p_1}\} \in V(I_\Gamma(\mathbb{Z}_n)) - B_i$ and so $y_j \neq i$ for all j. Then, by Theorem 3, there exists j such that $y_j \equiv i \pmod{p_1}$. From this $w = \{y_1 + p_1, \ldots, y_{j-1} + p_1, i, y_{j+1} + p_1, \ldots, y_{p_1} + p_1\} \in B_i$ and w is not adjacent to u. Hence $\omega(I_\Gamma(\mathbb{Z}_n)) = |\{v = \{x_1, \ldots, x_{p_1}\} \in V(I_\Gamma(\mathbb{Z}_n)) : v \text{ contains } i, \text{ for some } i, 1 \leq i \leq p_1\}| = (\frac{n}{p_1})^{p_1-1}$. ∎

For any t with $1 \leq t \leq \lfloor \frac{\omega(G)}{2} \rfloor$, $K_{t,t}$ is a subgraph of $I_\Gamma(\mathbb{Z}_n)$. However, for some t out of this range, $K_{t,t}$ may be a subgraph of $I_\Gamma(\mathbb{Z}_n)$. The theorem proved below identifies an upper bound for such a number t, when n is even.

Theorem 21. *Let $n > 2$ be even.*
 (i) *If $n > 6$, then $K_{t,t}$ is a subgraph of $I_\Gamma(\mathbb{Z}_n)$ if and only if $1 \leq t \leq \lfloor \frac{n}{4} \rfloor$.*
 (ii) *If $n = 4$ or 6, then $K_{t,t}$ is a subgraph of $I_\Gamma(\mathbb{Z}_6)$ if and only if $t = 1$ or 2.*

Proof. (i) Let $n \neq 6$ be even. Then by Theorem 20, $K_{\lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{4} \rfloor}$ is a subgraph of $I_\Gamma(\mathbb{Z}_n)$.
Let $n \neq 4k$ for any $k \in \mathbb{Z}^+$. Suppose $K_{\lceil \frac{n}{4} \rceil, \lceil \frac{n}{4} \rceil}$ is a subgraph of $I_\Gamma(\mathbb{Z}_n)$. Assume that $X = \{u_1, \ldots, u_{\lceil \frac{n}{4} \rceil}\}$ and $Y = \{v_1, \ldots, v_{\lceil \frac{n}{4} \rceil}\}$ are partition sets of $K_{\lceil \frac{n}{4} \rceil, \lceil \frac{n}{4} \rceil}$. Since $n \neq 6$ and $n \neq 4k$, $\lceil \frac{n}{4} \rceil \geq 3$. Note that each vertex in X as well as Y are two elements subsets of $\mathbb{Z}_n$. Suppose there is a $x \in u_1 \cap \cdots \cap u_{\lceil \frac{n}{4} \rceil}$. Since $|Y| \geq 3$, x must be in all the vertices of Y. Since each γ-set contains one odd and one even element in $\mathbb{Z}_n$, there are only $\frac{n}{2}$ γ-sets containing x and so vertices in $I_\Gamma(\mathbb{Z}_n)$. With such $\frac{n}{2}$ vertices, one cannot have $K_{\lceil \frac{n}{4} \rceil, \lceil \frac{n}{4} \rceil}$ as a subgraph. Suppose there is $x \in \mathbb{Z}_n$ is common to some of the vertices in X and $y \in \mathbb{Z}_n$ is common to some vertices in X. Now, there is at most only one vertex $\{x, y\} \in V(I_\Gamma(\mathbb{Z}_n))$ which is adjacent to all the vertices of X. That is, $\{x, y\}$ is the only vertex in Y, a contradiction to $|Y| \geq 3$. Therefore $K_{\lceil \frac{n}{4} \rceil, \lceil \frac{n}{4} \rceil}$ is not a subgraph of $I_\Gamma(\mathbb{Z}_n)$.

Similarly if $n = 4k$, then one can prove that $K_{\lceil \frac{n}{4} \rceil +1, \lceil \frac{n}{4} \rceil +1}$ is not a subgraph of $I_\Gamma(\mathbb{Z}_n)$.

(ii) If $n = 4$, then $I_\Gamma(\mathbb{Z}_n) = K_{2,2}$ and if $n = 6$, then the subgraph induced by $\{0,1\}$, $\{2,3\}$, $\{0,3\}$ and $\{1,2\}$ is a maximal complete bi-partite graph of $I_\Gamma(\mathbb{Z}_n)$. ∎

5. COLORING AND CONNECTIVITY OF $I_\Gamma(\mathbb{Z}_n)$

In this section, we study the connectivity and coloring of the intersection graph of gamma sets in $T_\Gamma(\mathbb{Z}_n)$. In particular, we give a necessary and sufficient condition for $I_\Gamma(\mathbb{Z}_n)$ to be Class one. First we obtain the chromatic number of $I_\Gamma(\mathbb{Z}_n)$. Note that the chromatic number of the intersection graph G is the minimum number of sets into which the elements of $V(G)$ can be partitioned so that in each set, every two elements of $V(G)$ are disjoint.

Theorem 22. *For any integer n, $\chi(I_\Gamma(\mathbb{Z}_n)) = \omega(I_\Gamma(\mathbb{Z}_n))$.*

Proof. Clearly if n is prime, then $\chi(I_\Gamma(\mathbb{Z}_n)) = \omega(I_\Gamma(\mathbb{Z}_n))$. Let n be a composite integer, p_1 be the smallest prime divisor of n and $v = \{x_1, \ldots, x_{p_1}\} \in V(I_\Gamma(\mathbb{Z}_n))$. Clearly the set $X = \{\{x_1, \ldots, x_{p_1}\}, \{x_1 + p_1, \ldots, x_{p_1} + p_1\}, \ldots, \{x_1 + (n - p_1), \ldots, x_{p_1} + (n - p_1)\}\}$ is an independent set and so we can assign a single color to all the vertices of X. Since $v \in V(I_\Gamma(\mathbb{Z}_n))$ is arbitrary and $|X| = \frac{n}{p_1}$, $\chi(I_\Gamma(\mathbb{Z}_n)) \leq \frac{|V(I_\Gamma(\mathbb{Z}_n))|}{n/p_1} = (\frac{n}{p_1})^{p_1 - 1}$. Since $\chi(I_\Gamma(\mathbb{Z}_n)) \geq \omega(I_\Gamma(\mathbb{Z}_n))$, $\chi(I_\Gamma(\mathbb{Z}_n)) = (\frac{n}{p_1})^{p_1 - 1}$. ∎

Theorem 23. *Let n be an integer. Then $I_\Gamma(\mathbb{Z}_n)$ is perfect if and only if $n = p$ for some prime p or $n = 4$ or 6.*

Proof. If $n = p$ for some prime p, then $I_\Gamma(\mathbb{Z}_n)$ is complete and so is perfect. If $n = 4$ or 6, then one can verify that $I_\Gamma(\mathbb{Z}_n)$ is perfect.

Conversely, suppose $n \geq 8$ is a composite integer, then $n \geq 3p_1$ where p_1 is the smallest prime divisor of n. Note that the subgraph induced by $\{\{0, 1, \ldots, p_1 - 1\}, \{1, 2, \ldots, p_1\}, \{0, p_1 + 1, p_1 + 2, \ldots, 2p_1 - 1\}, \{p_1, 2p_1 + 1, 2p_1 + 2, \ldots, 3p_1 - 1\}, \{p_1 + 1, 2p_1 + 1, 2p_1 + 2, \ldots, 3p_1 - 1\}\}$ is C_5 and is a subgraph of both $I_\Gamma(\mathbb{Z}_n)$ and $\overline{I_\Gamma(\mathbb{Z}_n)}$. Thus, by the strong perfect graph theorem, $I_\Gamma(\mathbb{Z}_n)$ is not a perfect graph. ∎

By Vizing's theorem, for any graph G, $\Delta \leq \chi'(G) \leq \Delta + 1$, where $\chi'(G)$ is the edge chromatic number of G. A graph G is said to be of Class one if $\chi'(G) = \Delta$, where as G is said to be of Class two if $\chi'(G) = \Delta + 1$. In this regard, one can note the following results.

Result 1 (Theorem 10.6, [9]). *A regular graph G is of Class one if and only if G is 1-factorable.*

Result 2 (Corollary 10.7, [9]). *Every regular graph of odd order is of Class two.*

Theorem 24. *For any integer $n > 2$, $I_\Gamma(\mathbb{Z}_n)$ is of Class one if and only if $n = p$ for some prime p or $n = 4k$ for some $k \in \mathbb{Z}^+$.*

Proof. Let $G = I_\Gamma(\mathbb{Z}_n)$. If $n = p$ for some prime p, then by Theroem 5(i), G is an even order complete graph and so G is of Class one. Assume that n is a composite integer. If n is odd or $n = 2k$ for some odd integer k, then by Lemma 5, $|V(G)|$ is odd and so by Result 2, G is of Class two. Hence, if G is of Class one, then $n = 4k$.

Conversely, let $n = 4k$ for some integer k and $G_i = \langle \{i, i+j\} \in V(G) : j \in \mathbb{Z}_n \ j \text{ is odd} \rangle \subset G$. If $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are adjacent in G, then either $x_1 = x_2$ or $y_1 = y_2$. Suppose $x_1 = x_2$, then $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are adjacent in G_{x_1} and so $G = \bigcup_{i=0}^{n-1} G_i$. Since $V(G_i) \cap V(G_j) = \{i, j\}$ for all $i \neq j$, G can be written as union of edge disjoint $K_{\frac{n}{2}}$. From this and by the fact that $\frac{n}{2}$ is even, $I_\Gamma(\mathbb{Z}_n)$ is $(n-2)$-factorable and hence G is of Class one. ∎

Remark 25. Theorem 10.5 in [9], tells that almost every graph is of Class one, whereas in the domain of intersection graphs of gamma sets in $T_\Gamma(\mathbb{Z}_n)$ corresponding to various n, Class two graphs are more than Class one. In fact, for various n, if $\mathcal{G}_i$ denotes the set of all $I_\Gamma(\mathbb{Z}_n)$ and of Class i, then $\lim_{n \to \infty} \frac{|\mathcal{G}_2|}{|\mathcal{G}_1|} \geq 2$.

Next, we obtain the vertex connectivity of the intersection graph of gamma sets in the total graph on $\mathbb{Z}_n$.

Theorem 26. *For any composite integer n, $k(I_\Gamma(\mathbb{Z}_n)) = deg(v)$ for any $v \in V(I_\Gamma(\mathbb{Z}_n))$.*

Proof. Let $G = I_\Gamma(\mathbb{Z}_n)$, p_1 is the smallest prime divisor of n and $v \in V(G)$. If $n = 4$, then $G = K_{2,2}$ and so $k(G) = 2 = deg(v)$. Hence we assume that $n \geq 6$. Suppose $S = \{v_1, \ldots, v_{deg(v)-1}\} \in V(G)$ is a cut-set of $I_\Gamma(\mathbb{Z}_n)$.

Case 1. Assume that $S \subset N(u)$ for some $u \in V(G) - S$. Since $|S| = deg(v) - 1$ and $I_\Gamma(\mathbb{Z}_n)$ is regular, there exists exactly one $u_1 \in V(G) - S$ such that u_1 is adjacent to u. Let $w_1 = \{y_1, \ldots, y_{p_1}\}, w_2 = \{z_1, \ldots, z_{p_1}\} \in V(G) - N[u]$. Let $y_1 \equiv k \pmod{p_1}$. By Theorem 3, there exists $z_i \in w_2$ such that $z_i \equiv k \pmod{p_1}$ and so $w_3 = \{z_i, y_2, y_3, \ldots, y_{p_1}\} \in V(G) - N[v]$ is adjacent to both w_1 and w_2. From this $\langle V(G) - N[u] \rangle$ is connected. Note that $N_G[u] = N_G[v]$ if and only if $u = v$. From this, u_1 is adjacent to at least one vertex in $V(G) - N[u]$ and so $\langle V(G) - S \rangle$ is connected, which is a contradiction to S is a cut-set.

Case 2. Suppose that $S \subset N(u) \cup N(v)$ for some $u, v \in V(G) - S$. As argued above, one can see that $\langle V(G) - \{N[u] \cup N[v]\} \rangle$ is connected. Note that $n \geq 3p_1$.

Let $u_1 = \{z_1, \ldots, z_{p_1}\} \in \{N(u) \cup N(v)\} - S$ and $z_i \equiv h_i \pmod{p_1}$. Without loss of generality one can assume that, there exists a k such that $z_i \in u \cup v$ for $i \geq k$ and $z_i \notin u \cup v$ for $i < k$. Suppose $k \geq 2$. Since $n \geq 3p_1$, $|\mathbb{Z}_n - \{u \cup v\}| \geq p_1$ and by Theorem 3, there exists $a_i \in \mathbb{Z}_n - \{u \cup v\}$ such that $a_i \equiv h_i \pmod{p_1}$. Now $\{z_1, \ldots, z_{k-1}, a_k, \ldots, a_{p_1}\} \in V(G) - \{N[u] \cup N[v]\}$ is adjacent to u_1. Hence, there exists at least two vertices $u_2 \in N[u] - S$ and $u_3 \in N[v] - S$ such that u_2 and u_3 are adjacent to some vertex in $V(G) - \{N[u] \cup N[v]\}$ and so $\langle V(G) - S \rangle$ is connected. If $k = 1$, then u_1 is adjacent to both u and v. Let $A = \{w = \{y_1, \ldots, y_{p_1}\} \in N(u) \cup N(v) : w \subseteq u \cup v\}$. Now assume that for each $w \in A$, there exists an integer r such that $y_j \in u$ for $j \leq r$ and $y_j \in v$ for $j > r$. Let $\ell_r = \max\{\binom{p_1}{r}, \binom{p_1}{p_1 - r}\}$. Note that $|A| \leq \sum_{i=1}^{p_1} \ell_i$ and for $1 \leq r \leq p_1 - 1$, $\ell_r \leq (\frac{n}{p_1})^r$. Since $n \geq 6$, $\ell_{p_1} = 1 \leq (\frac{n}{p_1} - 1)^{p_1 - 1} - 1$. From this we have, $|A| \leq \sum_{i=1}^{p_1} \ell_i \leq (\frac{n}{p_1})(\frac{n}{p_1})^{p_1 - 2} + (\frac{n}{p_1})^2 (\frac{n}{p_1} - 1)^{p_1 - 3} + \cdots + (\frac{n}{p_1})^{p_1 - 1} + (\frac{n}{p_1} - 1)^{p_1 - 1} - 1 = \delta(G)$. Since $|N(u) \cup N(v) - S| = \delta(G) + 1$, there exists a vertex $u_4 \in \{N(u) \cup N(v) - S\}$ such that $u_4 \notin A$ and so by discussed above $\langle V(G) - S \rangle$ is connected, a contradiction.

Similarly, one can prove the fact in the remaining cases also. Since $k(G) \leq \delta(G)$, we have $k(I_\Gamma(\mathbb{Z}_n)) = deg(v)$. ∎

If G is a graph of diameter 2, then edge connectivity $k'(G) = \delta(G)$ [8, p.77]. Since $I_\Gamma(\mathbb{Z}_n)$ is of diameter 2, we have the following:

Remark 27. For any composite integer n and $v \in V(I_\Gamma(\mathbb{Z}_n))$, $k'(I_\Gamma(\mathbb{Z}_n)) = deg(v) = k(I_\Gamma(\mathbb{Z}_n))$.

6. DOMINATION PARAMETERS OF $I_\Gamma(\mathbb{Z}_n)$

In this section, we are interested in certain domination related properties. Note that, for any prime number p, $I_\Gamma(\mathbb{Z}_n) = K_{2^{\frac{p-1}{2}}}$. Therefore, throughout this section, we assume that n is a composite integer.

Theorem 28. *Let n be a composite integer and p_1 be the smallest prime divisor of n. Then $\gamma(I_\Gamma(\mathbb{Z}_n)) = \frac{n}{p_1}$.*

Proof. Let $G = I_\Gamma(\mathbb{Z}_n)$. Clearly $\langle \{0, 1, \ldots, p_1 - 1\}, \{p_1, \ldots, 2p_1 - 1\}, \ldots, \{n - p_1, \ldots, n - 1\}\rangle$ is a dominating set in G and so $\gamma(G) \leq \frac{n}{p_1}$. Suppose $S = \{v_1, \ldots, v_{\frac{n}{p_1} - 1}\} \in V(G)$ is a dominating set of G. But S not contains at least p_1 elements of $\mathbb{Z}_n$ say, $x_1, \ldots, x_k$ where $k \geq p_1$. By Theorem 3, from these k elements, there exists at least p_1 elements form a γ-set of $T_\Gamma(\mathbb{Z}_n)$ and so S is not dominating at least a vertex of G. Thus $\gamma(G) = \frac{n}{p_1}$. ∎

As discussed above one can verify the following results.

Corollary 29. *Let n be a composite integer and p_1 be the smallest prime divisor of n. Then*

(i) $i(I_\Gamma(\mathbb{Z}_n)) = \frac{n}{p_1}$.

(ii) $\gamma_c(I_\Gamma(\mathbb{Z}_n)) = \frac{n}{p_1}$.

(iii) $\gamma_t(I_\Gamma(\mathbb{Z}_n)) = \gamma_{cl}(I_\Gamma(\mathbb{Z}_n)) = \frac{n}{p_1}$.

(iv) $I_\Gamma(\mathbb{Z}_n)$ *is well-covered and excellent.*

(v) $d(I_\Gamma(\mathbb{Z}_n)) = d_i(I_\Gamma(\mathbb{Z}_n)) = d_t(I_\Gamma(\mathbb{Z}_n)) = (\frac{n}{p_1})^{p_1}$ *and so $I_\Gamma(\mathbb{Z}_n)$ is not domatically full.*

Proof. (ii) Let $S = \{\{0, 1, \ldots, p_1 - 1\}, \{0, p_1 + 1, \ldots, 2p_1 - 1\}, \ldots, \{0, n - p_1 + 1, \ldots, n-1\}\}$. Then all subsets in S cover all the elements of $\mathbb{Z}_n - \{p_1, 2p_1, \ldots, n-p_1\}$. Let $Y = \{p_1, 2p_1, \ldots, n-p_1\}$. By Theorem 3, any vertex of $I_\Gamma(\mathbb{Z}_n)$ contains an element of Y must contains at least an element from $\mathbb{Z}_n - Y$ and so S is a γ_c-set of $I_\Gamma(\mathbb{Z}_n)$. ∎

Theorem 30. *Let n be a composite integer and p_1 be the smallest prime divisor of n. Then $\gamma_p(I_\Gamma(\mathbb{Z}_n))$ exists if and only if n is even. Moreover if n is even, then $\gamma_p(I_\Gamma(\mathbb{Z}_n)) = \frac{n}{p_1}$.*

Proof. If n is even, then $S = \{\{0, 1\}, \{0, 3\}, \ldots, \{0, n - 1\}\}$ is a γ_p-set and so $\gamma_p(I_\Gamma(\mathbb{Z}_n)) = \frac{n}{p_1}$.

Conversely assume that $\gamma_p(I_\Gamma(\mathbb{Z}_n))$ exists and n is odd. Let $\gamma_p(I_\Gamma(\mathbb{Z}_n)) = k$ and $S = \{v_1, \ldots, v_k\} \subset V(I_\Gamma(\mathbb{Z}_n))$ be a γ_p-set of $I_\Gamma(\mathbb{Z}_n)$.

Case 1. Assume that $v_i \cap v_j = \emptyset$ for all $1 \le i < j \le k$. Since $\gamma_p \ge \gamma = \frac{n}{p_1}$, all subsets in S cover all the elements of $\mathbb{Z}_n$. Thus every vertex $v \in V(I_\Gamma(\mathbb{Z}_n)) - S$ is adjacent to at least two vertices of S, otherwise $v = v_\ell$ for some $\ell = 1, \ldots, k$. Therefore S is not a γ_p-set of $I_\Gamma(\mathbb{Z}_n)$.

Case 2. Suppose there exists an $x \in \mathbb{Z}_n$ such that $x \in v_i \cap v_j$ for some $i, j \in \{1, \ldots, k\}$. Then all vertices containing x must be in S in order to have S as a perfect dominating set. Thus, by Theorem 3, all subsets in S must covers all the elements of $\mathbb{Z}_n - \{x + p_1, x + 2p_1, \ldots, x + (n - p_1)\}$. Let $X = \{x + p_1, x + 2p_1, \ldots, x + (n - p_1)\}$. Since $p_1 \ge 3$ and by Theorem 3, every element of $\mathbb{Z}_n - X$ must belong to at least two vertices of S. Note that elements of X alone cannot form a γ-set of $T_\Gamma(\mathbb{Z}_n)$ and so S is not γ_p-set of $I_\Gamma(\mathbb{Z}_n)$. ∎

Acknowledgment

References

[1] S. Akbari, D. Kiani, F. Mohammadi and S. Moradi, *The total graph and regular graph of a commutative ring*, J. Pure Appl. Algebra **213** (2009) 2224–2228.
doi:10.1016/j.jpaa.2009.03.013

[2] D.F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra **320** (2008) 2706–2719.
doi:10.1016/j.jalgebra.2008.06.028

[3] D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999) 434–447.
doi:10.1006/jabr.1998.7840

[4] N. Ashrafia, H.R. Maimanibc, M.R. Pournakicd and S. Yassemie, *Unit graphs associated with rings*, Comm. Algebra **38** (2010) 2851-2871.
doi:10.1080/00927870903095574

[5] R. Balakrishnan and K. Ranganathan, A text book of Graph Theory, (Springer, 2000).

[6] I. Chakrabarty, S. Ghosh, T.K. Mukherjee and M.K. Sen, *Intersection graphs of ideals of rings*, Electronic Notes in Discrete Math. **23** (2005) 23–32.
doi:10.1016/j.endm.2005.06.104

[7] I. Chakrabarty, S. Ghosh, T.K. Mukherjee and M.K. Sen, *Intersection graphs of ideals of rings*, Discrete Math. **309** (2009) 5381-5392.
{doi10.1016/j.disc.2008.11.034

[8] G. Chartrand and L. Lesniak, Graphs and Digraphs, (Chapman & Hall/CRC., 2000).

[9] G. Chartrand and P. Zhang, Chromatic Graph Theory, (CRC Press, 2009).

[10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamental of Domination in Graphs, (Marcel Dekker Inc., 1998).

[11] H.R. Maimani, M. Salimi, A. Sattari and S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra **319** (2008) 1801–1808.
doi:10.1016/j.jalgebra.2007.02.003

[12] T.A. McKee and F.R. McMorris, Topics in Intersection Graph Theory, (SIAM Monographs on Discrete Math. Applications., 1999).
doi:10.1137/1.9780898719802

[13] T. Tamizh Chelvam and T. Asir, *A note on total graph of $\mathbb{Z}_n$*, J. Discrete Math. Sci. Cryptography **14** (2011) 1–7.

[14] T. Tamizh Chelvam and T. Asir, *Domination in the total graph on $\mathbb{Z}_n$*, J. Combin. Math. Combin. Comput., submitted.

[15] A.T. White, Graphs, Groups and Surfaces, (North-Holland, Amsterdam., 1973).