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TOTAL VERTEX IRREGULARITY STRENGTH OF DISJOINT UNION OF HELM GRAPHS

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Abstract

A total vertex irregular k-labeling ϕ of a graph G is a labeling of the vertices and edges of G with labels from the set $\{1, 2, \ldots, k\}$ in such a way that for any two different vertices x and y their weights $wt(x)$ and $wt(y)$ are distinct. Here, the weight of a vertex x in G is the sum of the label of x and the labels of all edges incident with the vertex x. The minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G. We have determined an exact value of the total vertex irregularity strength of disjoint union of Helm graphs.

Keywords: vertex irregular total k-labeling, Helm graphs, total vertex irregularity strength.

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1. Introduction

Let us consider a simple (without loops and multiple edges) undirected graph $G = (V, E)$. For a graph G we define a labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ to be a total vertex irregular k-labeling of the graph G if for every two different vertices x and y of G one has $wt(x) \neq wt(y)$ where the weight of a vertex x in the labeling ϕ is $wt(x) = \phi(x) + \sum_{y \in N(x)} \phi(xy)$, where $N(x)$ is the set of neighbors of x. In [4] Bača, Jendrol', Miller and Ryan defined a new graph invariant, called the *total vertex irregularity strength* of G, $tvs(G)$, that is the minimum k for which the graph G has a vertex irregular total k-labeling.

The original motivation for the definition of the total vertex irregularity strength came from irregular assignments and the irregularity strength of graphs introduced in [6] by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba, and studied by numerous authors [5, 7, 8, 9, 10].

An *irregular assignment* is a k-labeling of the edges $f : E \to \{1, 2, \dots, k\}$ such that the vertex weights $w(x) = \sum_{y \in N(x)} f(xy)$ are different for all vertices of G, and the smallest k for which there is an irregular assignment is the *irregularity strength*, $s(G)$. The lower bound on the $s(G)$ is given by the inequality

$$s(G) \geq \max_{1 \leq i \leq \Delta} \frac{n_i + i - 1}{i}.$$

The first upper bounds including the vertex degrees in the denominator were given in [8]. The best upper bound known so far can be found in [11]. Namely, the authors have proved that $s(G) \leq \left\lceil \frac{6n}{\delta} \right\rceil$.

The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

It is easy to see that irregularity strength $s(G)$ of a graph G is defined only for graphs containing at most one isolated vertex and no connected component of order 2. On the other hand, the total vertex irregularity strength $tvs(G)$ is defined for every graph G.

If an edge labeling $f : E \to \{1, 2, \dots, s(G)\}$ provides the irregularity strength $s(G)$, then we extend this labeling to total labeling ϕ in such a way

$$\begin{aligned} \phi(xy) &= f(xy), && \text{for every } xy \in E(G), \\ \phi(x) &= 1, && \text{for every } x \in V(G). \end{aligned}$$

Thus, the total labeling ϕ is a vertex irregular total labeling and for graphs with no component of order ≤ 2, $tvs(G) \leq s(G)$.

Nierhoff [12] proved that for all (p, q)-graphs G with no component of order at most 2 and $G \neq K_3$, the irregularity strength $s(G) \leq p - 1$. From this result it follows that $tvs(G) \leq p - 1$.

In [4] several bounds and exact values of $tvs(G)$ were determined for different types of graphs (in particular for stars, cliques and prisms). Among others, the authors proved the following theorem

Theorem 1. *Let G be a (p, q)-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then* $\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1$.

In the case of r-regular graphs we therefore obtain $\left\lceil \frac{p+r}{r+1} \right\rceil \leq tvs(G) \leq p - r + 1$.

For graphs with no component of order ≤ 2, Bača *et al.* in [4] strengthened also these upper bounds, proving that $tvs(G) \leq p - 1 - \left\lceil \frac{p-2}{\Delta+1} \right\rceil$.

These results were then improved by Przybyło in [14] for sparse graphs and for graphs with large minimum degree. In the latter case the bounds $tvs(G) < 32\,\frac{p}{\delta} + 8$ in general and $tvs(G) < 8\,\frac{p}{r} + 3$ for r-regular (p, q)-graphs were proved to hold.

In [3] Anholcer, Kalkowski and Przybyło established a new upper bound of the form $tvs(G) \leq 3\frac{p}{\delta} + 1$.

Wijaya and Slamin [15] found the exact values of the total vertex irregularity strength of wheels, fans, suns and friendship graphs. Wijaya, Slamin, Surahmat and Jendrol' [16] determined an exact value for complete bipartite graphs. Ahmad and Bača [1] found the exact value of the total vertex irregularity strength for Jahangir graphs $J_{n,2}$ for $n \geq 4$ and for 4-regular circulant graphs $C_n(1, 2)$ for $n \geq 5$ namely, $tvs(J_{n,2}) = \lceil \frac{n+1}{2} \rceil$ and $tvs(C_n(1, 2)) = \lceil \frac{n+4}{5} \rceil$.

The main aim of this paper is determined an exact value of the total vertex irregularity strength of disjoint union of Helm graphs.

2. Main Results

Helm graphs are obtained from wheels by attaching a pendant edge to each vertex of the n-cycle. It follows that the Helm graph denoted H_n has $2n + 1$ vertices (n vertices of degree 4, n vertices of degree one and one vertex of degree n) and $3n$ edges. In [13], Nurdin *et al.* determined the lower bound of total vertex irregularity strength of connected graphs. In the next theorem, we showed the lower bound of total vertex irregularity strength of any graph.

Theorem 2. *Let G be a graph with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, then* $tvs(G) \geq \max_{i=0}^{\Delta(G)} \left\{ \left\lceil \frac{(\sum_{p=1}^{i} n_p) + \delta(G)}{i+1} \right\rceil \right\}$, *where n_i represents number of vertices of degree i in G.*

Proof. Let G be any graph with minimum degree $\delta(G)$, maximum degree $\Delta(G)$ and n_i, $i = \delta(G), \delta(G) + 1, \ldots, \Delta(G)$ represents number of vertices of degree i

in G. Let $s = \max_{i=0}^{\Delta(G)} \left\{ \left\lceil \frac{(\sum_{p=1}^{i} n_p)+\delta(G)}{i+1} \right\rceil \right\}$. Assume that $s = \left\lceil \frac{(\sum_{i=0}^{j} n_i)+\delta(G)}{j+1} \right\rceil$, for some j. In any vertex irregular total $k-$labeling on G the smallest weight among all vertices of degree $\delta(G), \delta(G)+1, \ldots,$ and j is at least $\delta(G)+1$ and the largest of them is at least $(\sum_{i=0}^{j} n_i)+\delta(G)$. Thus the value of k will be minimum if the largest weight is at the vertex of degree j. Since the weight of any vertex of degree j is the sum of $j+1$ positive labels, so at least one label is at least $\left\lceil \frac{(\sum_{i=0}^{j} n_i)+\delta(G)}{j+1} \right\rceil$. Therefore the minimum value of the k is at least s. This gives $\max_{i=0}^{\Delta(G)} \left\{ \left\lceil \frac{(\sum_{p=1}^{i} n_p)+\delta(G)}{i+1} \right\rceil \right\} \leq tvs(G)$ and we are done. ∎

In [2], Ahmad *et al.* determined the total vertex irregularity strength of Helm graph. In the next Theorem, we determined the total vertex irregularity strength of the union of isomorphic Helm graph H_3 and H_4.

Theorem 3. *The total vertex irregularity strength of the union of isomorphic Helm graph H_n is $tvs(mH_n) = \lceil \frac{nm+1}{2} \rceil$, for $m \geq 2, n = 3, 4$.*

Proof. The vertex set and edge set of G are $V(mH_n) = \{c^j : 1 \leq j \leq m\} \cup \{u_i^j, v_i^j : 1 \leq j \leq m, 1 \leq i \leq n\}$, $E(mH_n) = \{c^j v_i^j : 1 \leq j \leq m, 1 \leq i \leq n\} \cup \{v_{i+1}^j v_i^j : 1 \leq j \leq m, 1 \leq i \leq n\} \cup \{u_i^j v_i^j : 1 \leq j \leq m, 1 \leq i \leq n\}$, respectively. The disjoint union of Helm graph H_n contains nm vertices of degree one, nm vertices of degree four and m vertices of degree n, where m is the number of components of Helm graph H_n. The lower bound of mH_n follows from Theorem 2. Put $k = \lceil \frac{nm+1}{2} \rceil$. To show that k is an upper bound for total vertex irregularity strength of mH_n, we describe a total vertex k-labeling $\phi : V(mH_n) \cup E(mH_n) \to \{1, 2, \ldots, k\}$ for $m \geq 2,\ n = 3, 4$ as follows.

For $n = 3$

$$
\begin{array}{ll}
\phi(u_i^j) = \lceil \frac{3(j-1)}{2} \rceil + \lceil \frac{i}{2} \rceil, & \text{for } 1 \leq i \leq 3,\ 1 \leq j \leq m \text{ odd},\\
\phi(u_i^j) = \lfloor \frac{3(j-1)}{2} \rfloor + \lceil \frac{i+1}{2} \rceil, & \text{for } 1 \leq i \leq 3,\ 1 \leq j \leq m \text{ even},\\
\phi(c^j) = k-1, & \text{for } 1 \leq j \leq m,\\
\phi(v_i^j) = m-1, & \text{for } 1 \leq i \leq 2,\ 1 \leq j \leq m,\\
\phi(v_3^j) = k, & \text{for } 1 \leq j \leq m,\\
\phi(c^j v_3^j) = \phi(v_i^j v_{i+1}^j) = k, & \text{for } 1 \leq i \leq 3,\ 1 \leq j \leq m,\\
\phi(c^j v_i^j) = \lceil \frac{i+j-1}{2} \rceil, & \text{for } 1 \leq i \leq 2,\ 1 \leq j \leq m,\\
\phi(v_i^j u_i^j) = \lceil \frac{3(j-1)}{2} \rceil + \lceil \frac{i+1}{2} \rceil, & \text{for } 1 \leq i \leq 3,\ 1 \leq j \leq m \text{ odd},\\
\phi(v_i^j u_i^j) = \lceil \frac{3(j-1)}{2} \rceil + \lceil \frac{i}{2} \rceil, & \text{for } 1 \leq i \leq 3,\ 1 \leq j \leq m \text{ even}.
\end{array}
$$

This labeling gives the weight of the vertices as follows.

$$\begin{array}{ll} wt(u_i^j) = 3(j-1)+i+1, & \text{for } 1 \le i \le 3,\ 1 \le j \le m, \\ wt(v_i^j) = 2k+m+i+2(j-1), & \text{for } 1 \le i \le 2,\ 1 \le j \le m, \\ wt(v_3^j) = 4k+\lceil \frac{3(j-1)}{2} \rceil + 2, & \text{for } 1 \le j \le m, \\ wt(c^j) = 2k+j, & \text{for } 1 \le j \le m. \end{array}$$

For $n = 4$

$$\begin{array}{ll} \phi(v_i^j) = \phi(u_i^j) = 2(j-1)+\lceil \frac{i}{2} \rceil, & \text{for } 1 \le i \le 4,\ 1 \le j \le m, \\ \phi(c^j) = k+1-j, & \text{for } 1 \le j \le m, \\ \phi(c^j v_i^j) = k, & \text{for } 1 \le i \le 4, 1 \le j \le m, \\ \phi(v_i^j v_{i+1}^j) = \lfloor \frac{k}{2} \rfloor, & \text{for } 1 \le i \le 4,\ 1 \le j \le m, \\ \phi(v_i^j u_i^j) = 2(j-1)+\lceil \frac{i+1}{2} \rceil, & \text{for } 1 \le i \le 4,\ 1 \le j \le m. \end{array}$$

This labeling gives the weight of the vertices as follows.

$$\begin{array}{ll} wt(u_i^j) = 4(j-1)+i+1, & \text{for } 1 \le i \le 4,\ 1 \le j \le m, \\ wt(v_i^j) = k+2\lfloor \frac{k}{2} \rfloor + 4(j-1)+i+1, & \text{for } 1 \le i \le 4,\ 1 \le j \le m, \\ wt(c^j) = 5k+1-j, & \text{for } 1 \le j \le m. \end{array}$$

It is easy to check that the weight of the vertices are distinct. The above constructions show that $tvs(mH_n) \le \lceil \frac{nm+1}{2} \rceil$.

Combining with the lower bounds, we conclude that $tvs(mH_n) = \lceil \frac{nm+1}{2} \rceil$. ∎

In the next theorem, we determined the total vertex irregularity strength of a disjoint union of not necessarily isomorphic Helm graphs.

Theorem 4. *For $n_j > 4, m \ge 2$, let $G \cong \bigcup_{j=1}^m H_{n_j}$ then $tvs(G) = \left\lceil \frac{(\sum_{j=1}^m n_j)+1}{2} \right\rceil$.*

Proof. The disjoint union of a Helm graphs has $\sum_{j=1}^m n_j$ vertices of degree 1 and 4, and m vertices of degree between $[4, \Delta]$. From Theorem 2,

$$tvs(\bigcup_{j=1}^m H_{n_j}) \ge \left\lceil \frac{(\sum_{j=1}^m n_j)+1}{2} \right\rceil.$$

For our convenience, let t_1 be the number of H_{n_j}'s with even n_j. We arrange H_{n_j}'s such that all even n_j appear in the first t_1 places. The vertex set and edge set of disjoint union of Helm graphs are $V(G) = \{c^j : 1 \le j \le m\} \cup \{u_i^j, v_i^j : 1 \le j \le m, 1 \le i \le n_j\}$, $E(G) = \{c^j v_i^j : 1 \le j \le m, 1 \le i \le n_j\} \cup \{v_{i+1}^j v_i^j : 1 \le j \le m, 1 \le i \le n_j\} \cup \{u_i^j v_i^j : 1 \le j \le m, 1 \le i \le n_j\}$, respectively. Put $k = \left\lceil \frac{(\sum_{j=1}^m n_j)+1}{2} \right\rceil$. To show that k is an upper bound for total vertex irregularity strength of $\bigcup_{j=1}^m H_{n_j}$, we describe a total vertex k-labeling $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ as follows:

$$\begin{array}{ll}
\phi(v_i^j) = \phi(u_i^j) = \frac{\sum_{p=1}^{j} n_{p-1}}{2} + \lceil \frac{i}{2} \rceil, & \text{for } 1 \le i \le n_j,\ 1 \le j \le t_1,\\
\phi(v_i^j u_i^j) = \frac{\sum_{p=1}^{j} n_{p-1}}{2} + \lceil \frac{i+1}{2} \rceil, & \text{for } 1 \le i \le n_j,\ 1 \le j \le t_1,\\
\phi(c^j v_i^j) = \phi(v_i^j v_{i+1}^j) = k, & \text{for } 1 \le i \le n_j,\ 1 \le j \le m,\\
\phi(c^j) = j, & \text{for } 1 \le j \le m.
\end{array}$$

$t_1 \equiv 1 \pmod 2$

$$\begin{array}{ll}
\phi(v_i^j) = \phi(u_i^j) = \frac{\sum_{p=1}^{j} n_{p-1}}{2} + \lceil \frac{i}{2} \rceil, & \text{for } 1 \le i \le n_j,\ t_1+1 \le j \le m \text{ even},\\
\phi(v_i^j) = \phi(u_i^j) = \left\lfloor \frac{\sum_{p=1}^{j} n_{p-1}}{2} \right\rfloor + \lceil \frac{i+1}{2} \rceil, & \text{for } 1 \le i \le n_j,\ t_1+1 \le j \le m \text{ odd},\\
\phi(v_i^j u_i^j) = \frac{\sum_{p=1}^{j} n_{p-1}}{2} + \lceil \frac{i+1}{2} \rceil, & \text{for } 1 \le i \le n_j,\ t_1+1 \le j \le m \text{ even},\\
\phi(v_i^j u_i^j) = \left\lceil \frac{\sum_{p=1}^{j} n_{p-1}}{2} \right\rceil + \lceil \frac{i}{2} \rceil, & \text{for } 1 \le i \le n_j,\ t_1+1 \le j \le m \text{ odd}.
\end{array}$$

$t_1 \equiv 0 \pmod 2$

$$\begin{array}{ll}
\phi(v_i^j) = \phi(u_i^j) = \frac{\sum_{p=1}^{j} n_{p-1}}{2} + \lceil \frac{i}{2} \rceil, & \text{for } 1 \le i \le n_j,\ t_1+1 \le j \le m \text{ odd},\\
\phi(v_i^j) = \phi(u_i^j) = \left\lfloor \frac{\sum_{p=1}^{j} n_{p-1}}{2} \right\rfloor + \lceil \frac{i+1}{2} \rceil, & \text{for } 1 \le i \le n_j,\ t_1+1 \le j \le m \text{ even},\\
\phi(v_i^j u_i^j) = \frac{\sum_{p=1}^{j} n_{p-1}}{2} + \lceil \frac{i+1}{2} \rceil, & \text{for } 1 \le i \le n_j,\ t_1+1 \le j \le m \text{ odd},\\
\phi(v_i^j u_i^j) = \left\lceil \frac{\sum_{p=1}^{j} n_{p-1}}{2} \right\rceil + \lceil \frac{i}{2} \rceil, & \text{for } 1 \le i \le n_j,\ t_1+1 \le j \le m \text{ even}.
\end{array}$$

This labeling gives the weight of the vertices as follows:

$$\begin{array}{ll}
wt(u_i^j) = (\sum_{p=1}^{j} n_{p-1}) + i + 1, & \text{for } 1 \le i \le n_j,\ 1 \le j \le m,\\
wt(v_i^j) = (\sum_{p=1}^{j} n_{p-1}) + 3k + i + 1, & \text{for } 1 \le i \le n_j,\ 1 \le j \le m,\\
wt(c^j) = kn_j + j, & \text{for } 1 \le j \le m.
\end{array}$$

It is easy to check that the weight of the vertices are distinct. This labeling construction shows that $tvs(\bigcup_{j=1}^{m} H_{n_j}) \le \left\lceil \frac{(\sum_{j=1}^{m} n_j)+1}{2} \right\rceil$. Combining with the lower bounds, we conclude that $tvs(\bigcup_{j=1}^{m} H_{n_j}) = \left\lceil \frac{(\sum_{j=1}^{m} n_j)+1}{2} \right\rceil$. ■

Since by deleting the central vertex of a Helm graph, we obtain a sun graph then we have the following Corollary.

Corollary 5. *For $n_j \ge 3,\ 1 \le j \le m,\ tvs(\bigcup_{j=1}^{m} S_{n_j}) = \left\lceil \frac{(\sum_{j=1}^{m} n_j)+1}{2} \right\rceil$.*

Proof. The disjoint union of a sun graphs has $\sum_{j=1}^{m} n_j$ vertices of degree 1 and 3. From Theorem 2, we have $tvs(\bigcup_{j=1}^{m} S_{n_j}) \geq \left\lceil \frac{(\sum_{j=1}^{m} n_j)+1}{2} \right\rceil$. We label the graph $\bigcup_{j=1}^{m} H_{n_j}$ like in the proof of Theorem 4. Then we remove the central vertices together with all incident edges. As this operation does not change the weights of the vertices u_i^j and the weight of each v_i^j decreases by $k = \left\lceil \frac{(\sum_{j=1}^{m} n_j)+1}{2} \right\rceil$, it implies the existence of vertex-irregular total $\left\lceil \frac{(\sum_{j=1}^{m} n_j)+1}{2} \right\rceil$−labeling of graph $\bigcup_{j=1}^{m} S_{n_j}$. ∎

We believe that the lower bound of Theorem 2 is tight, so we propose the following Conjecture.

Conjecture 6. *Let G be a graph with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, then $tvs(G) = \max_{i=0}^{\Delta(G)} \left\{ \left\lceil \frac{(\sum_{p=1}^{i} n_p)+\delta(G)}{i+1} \right\rceil \right\}$, where n_i represents number of vertices of degree i in G.*

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