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A σ_3 TYPE CONDITION FOR HEAVY CYCLES IN WEIGHTED GRAPHS

Shenggui Zhang* and Xueliang Li

Department of Applied Mathematics
Northwestern Polytechnical University
Xi'an, Shaanxi 710072, P.R. China

and

Hajo Broersma

Faculty of Mathematical Sciences
University of Twente
P.O. Box 217
7500 AE Enschede, The Netherlands

Abstract

A weighted graph is a graph in which each edge e is assigned a non-negative number $w(e)$, called the weight of e. The weight of a cycle is the sum of the weights of its edges. The weighted degree $d^w(v)$ of a vertex v is the sum of the weights of the edges incident with v. In this paper, we prove the following result: Suppose G is a 2-connected weighted graph which satisfies the following conditions: 1. The weighted degree sum of any three independent vertices is at least m; 2. $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x,y) = 2$; 3. In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight. Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$. This generalizes a theorem of Fournier and Fraisse on the existence of long cycles in k-connected unweighted graphs in the case $k = 2$. Our proof of the above result also suggests a new proof to the theorem of Fournier and Fraisse in the case $k = 2$.

Keywords: weighted graph, (long, heavy, Hamilton) cycle, weighted degree, (weighted) degree sum.

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1. Terminology and Notation

We use Bondy and Murty [4] for terminology and notation not defined here and consider finite simple graphs only.

Let $G = (V, E)$ be a simple graph. G is called a *weighted graph* if each edge e is assigned a non-negative number $w(e)$, called the *weight* of e. For any subgraph H of G, $V(H)$ and $E(H)$ denote the sets of vertices and edges of H, respectively. The *weight* of H is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

An *optimal cycle* is one with maximum weight. For each vertex $v \in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices in H that are adjacent to v. We define the *weighted degree* of v in H by

$$d_H^w(v) = \sum_{h \in N_H(v)} w(vh).$$

When no confusion occurs, we will denote $N_G(v), d_G(v)$ and $d_G^w(v)$ by $N(v)$, $d(v)$ and $d^w(v)$, respectively. An (x, y)-*path* is a path connecting the two vertices x and y. The distance between two vertices x and y, denoted by $d(x, y)$, is the length of a shortest (x, y)-path. If u and v are two vertices on a path P, $P[u, v]$ denotes the segment of P from u to v. The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$. For a positive integer $k \leq \alpha(G)$ we denote by $\sigma_k(G)$ the minimum value of the degree sum of any k independent vertices, and by $\sigma_k^w(G)$ the minimum value of the weighted degree sum of any k independent vertices. Instead of $\sigma_1(G)$ and $\sigma_1^w(G)$, we use the notations $\delta(G)$ and $\delta^w(G)$, respectively.

2. Results

There have been many results on the existence of long cycles in graphs. The following three theorems are well-known.

Theorem A (Dirac [5]). *Let G be a 2-connected graph such that $\delta(G) \geq r$. Then G contains either a Hamilton cycle or a cycle of length at least $2r$.*

Theorem B (Pósa [7]). *Let G be a 2-connected graph such that $\sigma_2(G) \geq s$. Then G contains either a Hamilton cycle or a cycle of length at least s.*

Theorem C (Fournier and Fraisse [6]). *Let G be a k-connected graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}(G) \geq m$. Then G contains either a Hamilton cycle or a cycle of length at least $2m/(k+1)$.*

It is easy to see that Theorem B generalizes Theorem A, and Theorem C in turn generalizes Theorem B.

An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex v, and the weight of a cycle is simply the length of the cycle.

Theorem A and Theorem B were generalized to weighted graphs by the following two theorems, respectively.

Theorem 1 (Bondy and Fan [3]). *Let G be 2-connected weighted graph such that $\delta^w(G) \geq r$. Then either G contains a cycle of weight at least $2r$ or every optimal cycle is a Hamilton cycle.*

Theorem 2 (Bondy et al. [2]). *Let G be 2-connected weighted graph such that $\sigma_2^w(G) \geq s$. Then G contains either a Hamilton cycle or a cycle of weight at least s.*

A natural question is whether Theorem C also admits an analogous generalization for weighted graphs. This leads to the following problem.

Problem 1. *Let G be a k-connected weighted graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}^w(G) \geq m$. Is it true that G contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$?*

If the answer to the question of this problem is positive, then the result would be best possible and it would generalize Theorem C and Theorem 2.

It seems very difficult to settle this problem, even for the case $k = 2$. In the next section, we prove that the answer to the case $k = 2$ of Problem 1 is positive if we add some extra conditions. These extra conditions were motivated by a recent generalization of a theorem of Fan to weighted graphs (cf. [8]). Our result is an analogue and also a generalization of Theorem C to weighted graphs in the case $k = 2$.

Theorem 3. *Let G be a 2-connected weighted graph which satisfies the following conditions:*

1. *The weighted degree sum of any three independent vertices is at least m;*

2. *$w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x,y) = 2$;*
3. *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.

3. Proof of Theorem 3

Let G be a 2-connected weighted graph satisfying the conditions of Theorem 3. Suppose that G does not contain a Hamilton cycle. Then it suffices to prove that G contains a cycle of weight at least $2m/3$.

Choose a path $P = v_1 v_2 \cdots v_p$ in G such that

(a) P is as long as possible;

(b) $w(P)$ is as large as possible, subject to (a);

(c) $d^w(v_1) + d^w(v_p)$ is as large as possible, subject to (a) and (b).

From the choice of P, we can immediately see that $N(v_1) \cup N(v_p) \subseteq V(P)$.

Claim 1. *There exists no cycle of length p.*

Proof. Suppose there exists a cycle C of length p. Since G contains no Hamilton cycle and G is connected, we can find a vertex $u \in V(G) \backslash V(C)$ and a path Q from u to a vertex $v \in V(C)$, such that Q is internally disjoint from C. The subgraph $C \cup Q$ of G contains a path longer than P, contradicting the choice of P in (a). ∎

Claim 2. *$v_1 v_p \notin E(G)$.*

Proof. If $v_1 v_p \in E(G)$, then we can find a cycle $C = v_1 v_2 \cdots v_p v_1$ of length p, contradicting Claim 1. ∎

Claim 3. *If $v_i \in N(v_1)$, then $v_{i-1} \notin N(v_p)$.*

Proof. Suppose $v_i \in N(v_1)$ and $v_{i-1} \in N(v_p)$. Then we can form a cycle $C = v_1 v_i v_{i+1} \cdots v_p v_{i-1} v_{i-2} \cdots v_1$ with length p, again contradicting Claim 1. ∎

Claim 4. *If $v_i \in N(v_1)$, then $w(v_{i-1} v_i) \geq w(v_1 v_i)$. If $v_j \in N(v_p)$, then $w(v_j v_{j+1}) \geq w(v_j v_p)$.*

Proof. If $v_i \in N(v_1)$, the path $P' = v_{i-1}v_{i-2}\cdots v_1 v_i \cdots v_p$ has the same length as P. So, because of (b), we must have $w(P) \geq w(P')$, hence $w(v_{i-1}v_i) \geq w(v_1 v_i)$. The second assertion can be proved similarly. ■

Since G is 2-connected, by Lemma 1 of [1], there is a sequence of internally disjoint paths $P_1, P_2, \ldots, P_m$ such that

(1) P_k has end vertices x_k and y_k, and $V(P_k) \cap V(P) = \{x_k, y_k\}$ for $k = 1, 2, \ldots, m$;

(2) $v_1 = x_1 < x_2 < y_1 \leq x_3 < y_2 \leq x_4 < \cdots < y_{m-2} \leq x_m < y_{m-1} < y_m = v_p$, where the inequalities denote the order of the vertices on P.
By Claim 2, we have $m \geq 2$. It is not difficult to see that we can choose these paths such that

(3) if $v_i \in N(v_1)$, then $v_i \in P[v_2, x_2] \cup P[y_1, x_3]$ for $m \geq 3$, or $v_i \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$;

(4) if $v_j \in N(v_p)$, then $v_j \in P[y_{m-2}, x_m] \cup P[y_{m-1}, v_{p-1}]$ for $m \geq 3$, or $v_j \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$.

Now denote by C_k the cycle $P_k \cup P[x_k, y_k]$ for $k = 1, 2, \ldots, m$, and let C be the cycle whose edge set is the symmetric difference of the edge sets of these cycles C_k. By (3), (4) and Claim 3 we have for all $v_i \in N(v_1)\backslash\{y_1\}$ and $v_j \in N(v_p)\backslash\{x_m\}$ that $v_{i-1}v_i,\ v_j v_{j+1} \in E(C)$ and $v_{i-1}v_i \neq v_j v_{j+1}$. Also note that since $N(v_1) \cup N(v_p) \subseteq V(P)$, we must have $P_1 = v_1 y_1$ and $P_m = x_m v_p$. Using Claim 4, this shows that

$$\begin{aligned} w(C) &\geq \sum_{v_i \in N(v_1)\backslash\{y_1\}} w(v_{i-1}v_i) + \sum_{v_j \in N(v_p)\backslash\{x_m\}} w(v_j v_{j+1}) \\ &\quad + w(v_1 y_1) + w(x_m y_p) \\ &\geq \sum_{v_i \in N(v_1)} w(v_1 v_i) + \sum_{v_j \in N(v_p)} w(v_j v_p) \\ &= d^w(v_1) + d^w(v_p). \end{aligned}$$

Without loss of generality, we can assume that $d^w(v_1) \leq w(C)/2$.

Since G is 2-connected, v_1 is adjacent to at least one vertex on P other than v_2. Choose $v_k \in N(v_1) \cap V(P)$ such that k is as large as possible. By Claim 2 it is clear that $3 \leq k \leq p-1$.

Now we consider two cases.

Case 1. There exists a vertex $v_i \in V(P)$ such that $v_1v_i \in E(G)$ but $v_1v_{i-1} \notin E(G)$ for some i with $3 \le i \le k$.
By Claim 3 we know that $v_{i-1}v_p \notin E(G)$, so the three vertices v_1, v_{i-1} and v_p are independent. From Condition 2 of the theorem and the fact $d(v_1, v_{i-1}) = 2$ we know that $v_{i-1}v_{i-2}\cdots v_1v_i\cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), we have $d^w(v_{i-1}) \le d^w(v_1) \le w(C)/2$. With $d^w(v_1)+d^w(v_p) \le w(C)$, we have $d^w(v_1)+d^w(v_{i-1})+d^w(v_p) \le 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle C is at least $2m/3$.

Case 2. $v_1v_i \in E(G)$ for all i with $3 \le i \le k$.

Case 2.1. $w(v_1v_{i-1}) = w(v_1v_i) = w(v_{i-1}v_i) = w^*$ for all i with $3 \le i \le k$.
For every i with $2 \le i \le k-1$, v_i can not be adjacent to any vertex outside P. Otherwise, there will be a path of length p, contradicting the choice of P in (a). Since G is 2-connected, there must be an edge $v_jv_s \in E(G)$ with $j < k < s$. Choose $v_jv_s \in E(G)$ such that $j < k < s$ and s is as large as possible. From Claim 3 we have $s < p$.

Case 2.1.1. $s \ge k+2$.
By the choice of v_k we know that $v_1v_{s-1} \notin E(G)$. If $v_{s-1}v_p \in E(G)$, then we can form a cycle $v_1v_{j+1}\cdots v_{s-1}v_p\cdots v_sv_j\cdots v_1$ of length p, contradicting Claim 1. So, the three vertices v_1, v_{s-1} and v_p are independent. By the choice of v_k, we have $d(v_1, v_s) = 2$. If $v_jv_{s-1} \in E(G)$, then $d(v_1, v_{s-1}) = 2$. Then it follows from Condition 2 of the theorem that $w(v_jv_{s-1}) = w(v_jv_s) = w(v_1v_j) = w^*$, and from Condition 3 of the theorem we get $w(v_{s-1}v_s) = w^*$. If $v_jv_{s-1} \notin E(G)$, then $d(v_jv_{s-1}) = 2$. This implies that $w(v_{s-1}v_s) = w(v_jv_s) = w^*$. Thus, in both cases the path $v_{s-1}v_{s-2}\cdots v_{j+1}v_1\cdots v_jv_s\cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), we know that $d^w(v_{s-1}) \le d^w(v_1) \le w(C)/2$. With $d^w(v_1) + d^w(v_p) \le w(C)$, we have $d^w(v_1) + d^w(v_{s-1}) + d^w(v_p) \le 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle C is at least $2m/3$.

Case 2.1.2. $s = k+1$.
By Claim 3 we may assume that $k+1 < p$. From the 2-connectedness of G and the choice of v_s, there must be an edge $v_kv_t \in E(G)$ such that $t \ge k+2$. By the choice of v_k, we know that $v_1v_{t-1} \notin E(G)$. On the other hand, if $v_{t-1}v_p \in E(G)$, then we can form a cycle $v_1v_{j+1}\cdots v_kv_t\cdots v_pv_{t-1}\cdots v_{k+1}$

$v_j \cdots v_1$ of length p, contradicting Claim 1. So, the three vertices v_1, v_{t-1} and v_p are independent.

If $v_k v_{t-1} \in E(G)$, then from Condition 2 of the theorem we have $w(v_k v_{t-1}) = w(v_k v_t) = w(v_1 v_k) = w^*$, and from Condition 3 of the theorem, the edge $v_{t-1} v_t$ has weight w^*. If $v_k v_{t-1} \notin E(G)$, then from Condition 2 of the theorem we also get $w(v_{t-1} v_t) = w^*$. Thus, in both cases the path $v_{t-1} v_{t-2} \cdots v_{k+1} v_j \cdots v_1\ v_{j+1} \cdots v_k v_t \cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), $d(v_{t-1}) \leq d^w(v_1) \leq w(C)/2$. With $d^w(v_1) + d^w(v_p) \leq w(C)$, we have $d^w(v_1) + d^w(v_{t-1}) + d^w(v_p) \leq 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle C is at least $2m/3$.

This completes the proof of Case 2.1.

Case 2.2. There is some vertex v_i with $3 \leq i \leq k$ such that $w(v_1 v_{i-1})$, $w(v_1 v_i)$ and $w(v_{i-1} v_i)$ are all different.
In this case, choose vertex v_j such that $w(v_1 v_{j-1})$, $w(v_1 v_j)$ and $w(v_{j-1} v_j)$ are all different, and j is as large as possible. Denote the weight of $v_1 v_j$, $v_{j-1} v_j$ and $v_1 v_{j-1}$ by w_1, w_2 and w_3, respectively. It follows from Condition 3 (or Condition 2 if $j = k$) that $w(v_{j-1} v_j) = w_2 \neq w_1 = w(v_j v_{j+1})$, and from Condition 2 of the theorem that $v_{j-1} v_{j+1} \in E(G)$. If $j < k$, then the weight of the edge $v_{j-1} v_{j+1}$ is different from the weight w_1 of the edge $v_{j+1} v_{j+2}$ since there is a triangle $v_1 v_{j-1} v_{j+1} v_1$ and $w(v_1 v_{j-1}) = w_3 \neq w_1 = w(v_1 v_{j+1})$. With the same argument, we can prove that $v_{j-1} v_i \in E(G)$ for all i with $j \leq i \leq k+1$. By the choice of v_k, we have that $w(v_{j-1} v_{k+1}) = w_3$.

Suppose first that $v_k v_{k+2} \in E(G)$. Then $d(v_1, v_{k+2}) = 2$. This shows that $w(v_k v_{k+2}) = w(v_1 v_k) = w_1$. From $w(v_k v_{k+1}) = w(v_k v_{k+2}) = w_1$ and Condition 3 of the theorem we know that $w(v_{k+1} v_{k+2}) = w_1$. Therefore, there must be an edge $v_{j-1} v_{k+2} \in E(G)$ since the two edges $v_{j-1} v_{k+1}$ and $v_{k+1} v_{k+2}$ have different weights. Again, by the fact $d(v_1, v_{k+2}) = 2$, we obtain that $w(v_{j-1} v_{k+2}) = w(v_1 v_{j-1}) = w_3$. This leads to a triangle $v_{j-1} v_{k+1} v_{k+2} v_{j-1}$ in which $w(v_{j-1} v_{k+1}) = w(v_{j-1} v_{k+2}) = w_3$ and $w(v_{k+1} v_{k+2}) = w_1$, contradicting Condition 3 of the theorem. Hence $v_k v_{k+2} \notin E(G)$. Thus $d(v_k, v_{k+2}) = 2$. This implies that $w(v_{k+1} v_{k+2}) = w(v_k v_{k+1}) = w_1$. Therefore, there must be an edge $v_{j-1} v_{k+2} \in E(G)$ and $w(v_{j-1} v_{k+2}) = w_3$. This also leads to a triangle $v_{j-1} v_{k+1} v_{k+2} v_{j-1}$ which is impossible by Condition 3 of the theorem.

The proof of the theorem is complete. ∎

4. Remarks

The proof of Theorem C in [6] is very complicated. It is clear that our proof of Theorem 3 provides a simpler proof for Theorem C in the case $k = 2$. We do not know whether the extra conditions in Theorem 3 are necessary. The results in [8] indicate that for some generalizations of long cycle results to weighted graphs one cannot avoid such additional conditions. We do not believe that there is an analogous generalization of Theorem C for the case $k \neq 2$.

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