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## MINIMAL NON-SELFCENTRIC RADIALY-MAXIMAL GRAPHS OF RADIUS 4

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### Abstract

There is a hypothesis that a non-selfcentric radially-maximal graph of radius  $r$  has at least  $3r - 1$  vertices. Using some recent results we prove this hypothesis for  $r = 4$ .

**Keywords:** non-selfcentric, radially-maximal, critical, center, radius, planar graph.

**2000 Mathematics Subject Classification:** 05C12.

### 1. INTRODUCTION AND RESULT

Let  $G$  be a graph. By  $E(G)$  we denote the edge set of  $G$ , and by  $\overline{G}$  we denote the complement of  $G$ . The radius of  $G$  is denoted by  $r(G)$  and the diameter

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of  $G$  is denoted by  $d(G)$ . We say that the graph  $G$  is *radially-maximal* if  $r(G + e) < r(G)$  for every edge  $e \in E(\overline{G})$ .

Obviously, for every  $r$  there is a radially-maximal graph of radius  $r$ , as can be shown by complete graphs (in the case  $r = 1$ ) and even cycles (in the case  $r > 1$ ). However, both complete graphs and cycles are selfcentric graphs. Here we recall that a graph  $G$  is *selfcentric* if  $r(G) = d(G)$ . If  $r(G) < d(G)$  then  $G$  is a *non-selfcentric* graph. One may expect that a graph is radially-maximal if it is either a very dense graph or a balanced (highly symmetric) one. Therefore, it is interesting that for  $r \geq 3$  there are non-selfcentric radially-maximal graphs of radius  $r$  which are planar (such graphs are neither symmetric nor dense). In fact, in [1] we have the following conjecture:

**Conjecture A.** Let  $G$  be a non-selfcentric radially-maximal graph with radius  $r \geq 3$  on the minimum number of vertices. Then

- (a)  $G$  has exactly  $3r - 1$  vertices;
- (b)  $G$  is planar;
- (c) the maximum degree of  $G$  is 3 and the minimum degree of  $G$  is 1.

Conjecture A was verified for the case  $r = 3$ . By an exhaustive computer search it was shown that there are just two non-selfcentric radially-maximal graphs of radius 3 on at most 8 vertices. These graphs are depicted in Figure 1. As one can see, they are planar, their minimum degree is 1, the maximum degree is 3, and each of them has exactly 8 vertices.

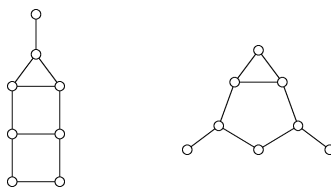


Figure 1

For higher values of  $r$  the conjecture was open, although by an extensive computer search we found that there are exactly 8 graphs of radius 4 fulfilling all the conclusions of Conjecture A. These graphs are depicted in Figure 2.

Although we are not able to prove the (a) part of Conjecture A in general, we have:

**Assertion 1.** *For every  $r \geq 3$  there exists a non-selfcentric radially-maximal graph with radius  $r$  on  $3r - 1$  vertices.*

Hence, the (a) part of Conjecture A will be true if we prove that there are no non-selfcentric radially-maximal graphs with radius  $r$  on less than  $3r - 1$  vertices.

Let  $C$  be a cycle in a graph  $G$ . We say that  $C$  is a *geodesic cycle*, if for any two vertices of  $C$ , their distance on  $C$  equals their distance in  $G$ .

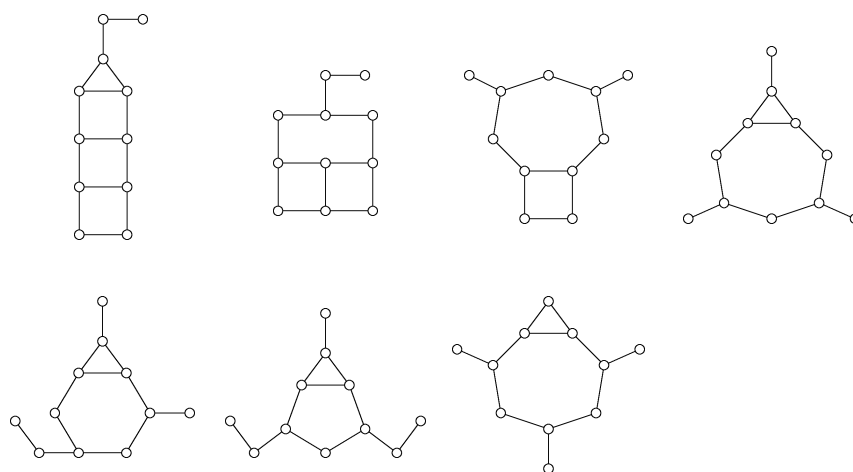


Figure 2

Recently, in [2] Haviar, Hrnčiar and Monoszová proved the following beautiful statement:

**Theorem B.** *Let  $G$  be a graph with  $r(G) = r$ ,  $d(G) \leq 2r - 2$ , on at most  $3r - 2$  vertices. Then  $G$  contains a geodesic cycle of length either  $2r$  or  $2r + 1$ .*

Using this statement we are able to prove the (a) part of Conjecture A for  $r = 4$ :

**Theorem 2.** *Let  $G$  be a non-selfcentric radially-maximal graph with radius 4 on the minimum number of vertices. Then  $G$  has exactly 11 vertices.*

The proofs of Assertion 1 and Theorem 2 are presented in the next section.

## 2. PROOFS

**Proof of Assertion 1.** Let  $G_r$  be a graph obtained from the first graph in Figure 1 by extending the path on the top by  $r-3$  vertices, and by extending the ladder at the bottom by  $r-3$  new spokes. Then  $G_3$  is the first graph in Figure 1,  $G_4$  is the first graph in Figure 2, while a general version of  $G_r$  (in horizontal position) is depicted in Figure 3. The central vertices of  $G_r$  are denoted by  $c_1, c_2, c_3, c_4$  and  $c_5$ , the vertices of the path at the top are  $v_3, v_4, \dots, v_r$ , the vertices of one “leg” of the ladder are  $w_3, w_4, \dots, w_r$ , and the vertices of the other “leg” are  $z_3, z_4, \dots, z_r$ , see Figure 3.

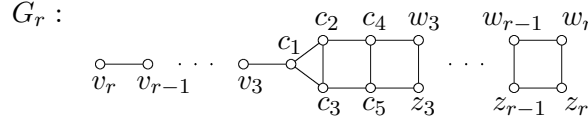


Figure 3

In the following, let us denote by  $N_t^G(v)$  the set of vertices of  $G$ , which are at distance  $t$  from  $v$ , while the eccentricity of a vertex  $v$  is denoted by  $e_G(v)$ . Observe that  $N_3^{G_3}(c_1) = \{w_3, z_3\}$ , so that  $N_r^{G_r}(c_1) = \{w_r, z_r\}$ . Analogously,  $N_r^{G_r}(c_2) = \{z_r\}$ ,  $N_r^{G_r}(c_3) = \{w_r\}$ ,  $N_r^{G_r}(c_4) = \{v_r\}$  and  $N_r^{G_r}(c_5) = \{v_r\}$ . Since the distance from  $v_r$  to  $w_r$  is  $2r-2$ ,  $G_r$  is a non-selfcentric graph with radius  $r$  on  $3r-1$  vertices. We show that  $G_r$  is radially-maximal.

First consider adding an edge  $c_i c_j$  to  $G_r$ . Since the distance from  $c_4$  to  $v_r$  is less than  $r$  in  $G_r + c_1 c_4$ , we have  $e_{G_r + c_1 c_4}(c_4) < r$ . Analogously,  $e_{G_r + c_1 c_5}(c_5) < r$ ,  $e_{G_r + c_2 c_5}(c_2) < r$  and  $e_{G_r + c_3 c_4}(c_3) < r$ .

Further,  $e_{G_r + c_i v_j}(c_4) < r$  if  $i \in \{1, 2, 4\}$  and  $e_{G_r + c_i v_j}(c_5) < r$  if  $i \in \{3, 5\}$ . Moreover,  $e_{G_r + c_i w_j}(c_3) < r$  and  $e_{G_r + c_i z_j}(c_2) < r$ .

It remains to check adding an edge joining two non-central vertices. Obviously,  $e_{G_r + v_i v_j}(c_4) < r$ ,  $e_{G_r + w_i w_j}(c_3) < r$  and  $e_{G_r + z_i z_j}(c_2) < r$ . If  $i < j$  then  $e_{G_r + w_i z_j}(c_2) < r$  and if  $i > j$  then  $e_{G_r + w_i z_j}(c_3) < r$ . Finally, if  $i \geq j$  then  $e_{G_r + v_i w_j}(c_4) < r$  and if  $i < j$  then  $e_{G_r + v_i w_j}(c_1) < r$ . By symmetry, the case of adding  $v_i z_j$  can be reduced to the previous one. ■

For two vertices, say  $u$  and  $v$ , by  $d_G(u, v)$  we denote their distance in  $G$ . In the proof of Theorem 2 we use the following statement:

**Lemma 3.** *Let  $G$  be a radially-maximal graph of radius  $r$  and diameter  $d$ . Then  $d \leq 2r - 2$ .*

**Proof.** Let  $P = (v_0, v_1, \dots, v_d)$  be a path on which the diameter of  $G$  is achieved. Denote by  $G'$  the graph obtained from  $G$  by adding the edge  $v_0v_d$ , and denote by  $P'$  the cycle  $(v_0, v_1, \dots, v_d, v_0)$  of  $G'$ .

Since  $G$  is a radially-maximal graph, we have  $r(G') \leq r - 1$ . Let  $v$  be a vertex of  $G'$  such that  $e_{G'}(v) = r(G') \leq r - 1$ . As  $e_G(v) \geq r$ , it holds  $d_G(v, v_0) \neq d_G(v, v_d)$ . Without loss of generality we may assume that  $d_G(v, v_0) < d_G(v, v_d)$ . Then  $d_{G'}(v, v_0) = d_{G'}(v, v_d) - 1$ . Denote  $d_{v_0} = d_{G'}(v, v_0) = d_G(v, v_0)$ . Further, denote by  $P_i$  ( $P'_i$ ) the set of vertices of  $P$  ( $P'$ ) which distance to  $v$  in  $G$  ( $G'$ ) is at most  $i$ .

If  $i \leq d_{v_0}$  then  $|P_i| \leq 2i + 1$ , since otherwise  $P$  would not be a diametrical path. (There would be a short-cut via  $v$ .) As  $i \leq d_{v_0}$ ,  $|P'_i| \leq 2i + 1$  as well.

If  $i > d_{v_0}$  then  $|P_i| \leq 2d_{v_0} + 1 + (i - d_{v_0})$ , since otherwise  $P$  would not be a diametrical path. (Observe that  $d_G(v, v_0) = d_{v_0}$ .) However, the presence of the edge  $v_0v_d$  in  $G'$  causes that  $P'_i$  contains also some vertices which are not in  $P_i$ . Since  $P$  is a diametrical path, there are only  $i - d_{v_0}$  vertices in  $P'_i - P_i$ , namely  $v_d, v_{d-1}, \dots, v_{d-(i-d_{v_0})+1}$ . Hence,  $|P'_i| = |P_i| + (i - d_{v_0}) \leq 2d_{v_0} + 1 + (i - d_{v_0}) + (i - d_{v_0}) = 2i + 1$ .

Thus, for every  $i$  we have  $|P'_i| \leq 2i + 1$ . Since  $e_{G'}(v) \leq r - 1$ , the set  $P'_{r-1}$  contains all the vertices of  $P'$ . Hence,  $|P'| = |P| \leq 2(r - 1) + 1 = 2r - 1$ , so that the length of  $P$  is at most  $2r - 2$ . ■

Every graph of radius  $r$  can be completed into a radially-maximal graph of radius  $r$  by adding edges which do not decrease the radius. In the following proof, the edges  $e \in E(\overline{G})$  such that  $r(G) = r(G + e)$  are called *r-admissible*. However, we may not add any admissible edge to  $G$ , since we like to remain in the class of non-selfcentric graphs. Hence, the edges  $e \in E(\overline{G})$  such that  $G + e$  is not selfcentric, are called *s-admissible*. Finally, we like to preserve a special substructure in  $G$ . There will be a cycle  $C$  in  $G$  which has to be geodesic. So that the edges  $e \in E(\overline{G})$  for which  $C$  remains a geodesic cycle in  $G + e$ , are called *g-admissible*. We may join these notions. For example, if an edge  $e \in E(\overline{G})$  is both *s-admissible* and *g-admissible*, then we say that  $e$  is *sg-admissible*. Finally, an edge is *admissible* if it is *rsg-admissible*.

Observe that if there are two edges, say  $e$  and  $f$ , which are admissible in  $G$ , then  $f$  is not necessarily an admissible edge in  $G + e$ . Hence, there may be more radially-maximal graphs obtained by adding edges to  $G$ .

In the following proof we use the notation introduced above. However, if we examine a specific edge, say  $h$ , and we realize that our graph cannot contain this edge, then in the sequel  $h$  is not an  $a$ -admissible edge for any  $a \in \{r, s, g\}$ .

**Proof of Theorem 2.** Suppose that there is a non-selfcentric radially-maximal graph with radius  $r = 4$  on at most  $3r - 2 = 10$  vertices. As there is no radially-maximal graph of radius  $r$  with diameter  $d > 2r - 2$  by Lemma 3, our graph contains a geodesic cycle of length either 9 or 8, by Theorem B.

First suppose that a non-selfcentric radially-maximal graph with radius 4 on at most 10 vertices contains a geodesic cycle  $(v_0, v_1, \dots, v_8, v_0)$  of length 9. Since the cycle  $C$  is a selfcentric graph, there must be another vertex, say  $v_9$ , adjacent to a vertex of  $C$ . Without loss of generality we may assume that  $v_2v_9$  is an edge in our graph. Denote  $G = C + v_2v_9$ . Obviously,  $G$  has 10 vertices.

Observe that  $G$  is not a radially-maximal graph. There are four  $rg$ -admissible edges in  $G$ , namely  $v_0v_9$ ,  $v_1v_9$ ,  $v_3v_9$  and  $v_4v_9$ . But the edges  $v_0v_9$  and  $v_4v_9$  are not  $s$ -admissible. Since the graphs  $G + v_1v_9$  and  $G + v_3v_9$  are isomorphic, without loss of generality we may complete the graph  $G$  by adding the edge  $v_1v_9$ . Denote  $G' = G + v_1v_9$ . But  $G'$  is not a radially-maximal graph, as the edge  $v_3v_9$  is  $r$ -admissible in  $G'$ . However, there are no  $sg$ -admissible edges in  $G'$ , which contradicts our assumption that a non-selfcentric radially-maximal graph with radius 4 on at most 10 vertices contains a geodesic cycle of length 9.

Thus, suppose that there is a non-selfcentric radially-maximal graph with radius 4 on at most 10 vertices which contains a geodesic cycle  $(v_0, v_1, \dots, v_7, v_0)$  of length 8. Since a cycle is a selfcentric graph, there must be another vertex, say  $v_8$ , adjacent to  $C$  by an edge, say  $v_2v_8$ . Observe that  $v_1v_8$  is an  $r$ -admissible edge in  $C + v_2v_8$ , so that  $C + v_2v_8$  is not a radially-maximal graph. But there are no  $sg$ -admissible edges in  $C + v_2v_8$ . Thus, there must be another vertex, say  $v_9$ . Since now we attained the upper bound for the number of vertices, there are no other vertices in our graph. In what follows we distinguish two cases with several subcases each.

*Case 1.* Our graph contains the edge  $v_8v_9$ . Denote  $G = C + v_2v_8 + v_8v_9$ . As  $v_1v_8$  is an  $r$ -admissible edge in  $G$ , the graph  $G$  is not radially-maximal. However, there are only seven  $rg$ -admissible edges in  $G$ , namely  $v_0v_8$ ,  $v_1v_8$ ,

$v_3v_8$ ,  $v_4v_8$ ,  $v_1v_9$ ,  $v_2v_9$  and  $v_3v_9$ . In fact, there are no other  $sg$ -admissible edges in  $G$ .

First consider the graph  $G' = G + v_1v_9$ . Since the edge  $v_2v_9$  is  $r$ -admissible in  $G'$ , this graph is not radially-maximal. But there are only four admissible edges in  $G'$ , namely  $v_1v_8$ ,  $v_2v_9$ ,  $v_0v_8$  and  $v_3v_9$ . However, the unique admissible edge in  $G' + v_1v_8$  is  $v_0v_8$ . Since  $v_2v_9$  is  $r$ -admissible in  $G' + v_1v_8 + v_0v_8$ , adding the edge  $v_1v_8$  to  $G'$  will not create the required radially-maximal graph. Hence, we can exclude the edge  $v_1v_8$ , and by symmetry also the edge  $v_2v_9$ , from the list of admissible edges for  $G'$ . But then there is no admissible edge in  $G' + v_0v_8$  although  $r(G' + v_0v_8 + v_1v_8) = 4$ . Hence, also the edges  $v_0v_8$  and  $v_3v_9$  can be excluded from the list of admissible edges for  $G'$ . This shows that  $G'$  cannot be completed to a required radially-maximal graph, so that we can exclude the edge  $v_1v_9$ , and by symmetry also the edge  $v_3v_9$ , from the list of admissible edges for  $G$ .

Now consider the graph  $G' = G + v_2v_9$ . As the edge  $v_1v_9$  is  $r$ -admissible in  $G'$ , this graph is not radially-maximal. If we add an edge  $e \in \{v_1v_8, v_1v_9, v_3v_8, v_3v_9\}$  to  $G'$ , then the new graph contains a subgraph isomorphic to  $G + v_1v_9$ . This is a subcase already solved above. The remaining edges  $v_0v_8$  and  $v_4v_8$  are not  $s$ -admissible in  $G'$ , so that also the edge  $v_2v_9$  can be excluded from our list. As a consequence,  $v_9$  is a vertex of degree one.

Now consider the graph  $G' = G + v_0v_8$ . As  $v_2v_9$  is  $r$ -admissible edge in  $G'$ , this graph is not radially-maximal. Since  $v_9$  is a vertex of degree one, the edge  $v_1v_8$  is the unique admissible edge in  $G'$ . However,  $G' + v_1v_8$  is not a radially-maximal graph. Although out of consideration,  $v_2v_9$  is an  $r$ -admissible edge in  $G' + v_1v_8$ . Hence, we can exclude the edges  $v_0v_8$  and  $v_4v_8$  from our list.

Finally, consider the graph  $G' = G + v_1v_8$ . As  $v_2v_9$  is an  $r$ -admissible edge in  $G'$ , this graph is not radially-maximal. By the previous cases, there is a unique admissible edge in  $G'$  which is not out of consideration, namely  $v_3v_8$ . But  $v_2v_9$  is an  $r$ -admissible edge also in  $G' + v_3v_8$ , so that also the edges  $v_1v_8$  and  $v_3v_8$  can be excluded from our list. This completes the analysis of Case 1.

*Case 2.* Our graph does not contain the edge  $v_8v_9$ . But then  $v_9$  is adjacent to some  $v_q$ ,  $0 \leq q \leq 7$ . Denote  $G = C + v_2v_8 + v_qv_9$ . Since  $v_1v_8$  is an  $r$ -admissible edge in  $G$ , this graph is not radially-maximal. There are only eight admissible edges in  $G$ , namely  $v_0v_8$ ,  $v_1v_8$ ,  $v_3v_8$ ,  $v_4v_8$ ,  $v_{q-2}v_9$ ,  $v_{q-1}v_9$ ,  $v_{q+1}v_9$  and  $v_{q+2}v_9$  (the addition is modulo 8).

Consider the graph  $G' = G + v_0v_8$ . Since the edge  $v_{q-1}v_9$  is  $r$ -admissible in  $G'$ , this graph is not radially-maximal. However,  $v_1v_8$  is the unique admissible edge in  $G'$ . (In fact,  $v_1v_8$  is the unique  $sg$ -admissible edge in  $G'$ .) Since  $v_{q-1}v_9$  is  $r$ -admissible even in  $G' + v_1v_8$ , we can exclude the edge  $v_0v_8$  from our list. And by symmetry, we can exclude also the edges  $v_4v_8$ ,  $v_{q-2}v_9$  and  $v_{q+2}v_9$ .

Thus, consider the graph  $G' = G + v_1v_8$ . As  $v_{q-1}v_9$  is an  $r$ -admissible edge in  $G'$ , this graph is not radially-maximal. Since the case  $G' + v_3v_8$  reduces to the previous subcase, by symmetry, there is only one edge addable to  $G'$ , namely  $v_{q-1}v_9$ . The graph  $G' + v_{q-1}v_9$  is not selfcentric only if  $q = 6$ . But even in that case  $G' + v_5v_9$  is not a radially-maximal graph, as  $v_0v_8$  is an  $r$ -admissible edge in this graph. However, there are no  $s$ -admissible edges in  $G' + v_5v_9$ , so that also the edge  $v_1v_8$  can be excluded from our list. And by symmetry, we can exclude also the remaining edges  $v_3v_8$ ,  $v_{q-1}v_9$  and  $v_{q+1}v_9$ , which completes the proof. ■

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