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A CLASSIFICATION FOR MAXIMAL NONHAMILTONIAN BURKARD-HAMMER GRAPHS

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Abstract

A graph $G = (V, E)$ is called a split graph if there exists a partition $V = I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of G induced by I and K are empty and complete graphs, respectively. In 1980, Burkard and Hammer gave a necessary condition for a split graph G with $|I| < |K|$ to be hamiltonian. We will call a split graph G with $|I| < |K|$ satisfying this condition a Burkard-Hammer graph. Further, a split graph G is called a maximal nonhamiltonian split graph if G is nonhamiltonian but $G + uv$ is hamiltonian for every $uv \notin E$ where $u \in I$ and $v \in K$. Recently, Ngo Dac Tan and Le Xuan Hung have classified maximal nonhamiltonian Burkard-Hammer graphs G with minimum degree $\delta(G) \geq |I| - 3$. In this paper, we classify maximal nonhamiltonian Burkard-Hammer graphs G with $|I| \neq 6, 7$ and $\delta(G) = |I| - 4$.

Keywords: split graph, Burkard-Hammer condition, Burkard-Hammer graph, hamiltonian graph, maximal nonhamiltonian split graph.

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1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ (or V and E for short) will denote its vertex-set and its edge-set, respectively. For a subset $W \subseteq V(G)$, the set of all neighbours of W is denoted by $N_G(W)$ or $N(W)$ for short. For a vertex $v \in V(G)$, the degree of v, denoted by $\deg(v)$, is the number $|N(v)|$. The minimum degree of G, denoted by $\delta(G)$, is the number $\min\{\deg(v) \mid v \in V(G)\}$. By $N_{G,W}(v)$ or $N_W(v)$ for short we denote the set $W \cap N_G(v)$. The subgraph of G induced by W is denoted by $G[W]$. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

A graph $G = (V, E)$ is called a *split graph* if there exists a partition $V = I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of G induced by I and K are empty and complete graphs, respectively. We will denote such a graph by $S(I \cup K, E)$. Further, a split graph $G = S(I \cup K, E)$ is called a *complete split graph* if every $u \in I$ is adjacent to every $v \in K$. The notion of split graphs was introduced in 1977 by Földes and Hammer [4]. These graphs are interesting because they are related to many problems in combinatorics (see [3, 5, 10]) and in computer science (see [6, 7]).

In 1980, Burkard and Hammer gave a necessary condition for a split graph $G = S(I \cup K, E)$ with $|I| < |K|$ to be hamiltonian [2] (see Section 2 for more detail). We will call this condition the *Burkard-Hammer condition*. Also, we will call a split graph $G = S(I \cup K, E)$ with $|I| < |K|$, which satisfies the Burkard-Hammer condition, a *Burkard-Hammer graph*.

Thus, by [2] any hamiltonian split graph $G = S(I \cup K, E)$ with $|I| < |K|$ is a Burkard-Hammer graph. In general, the converse is not true. The first nonhamiltonian Burkard-Hammer graph has been indicated in [2]. Further infinite families of nonhamiltonian Burkard-Hammer graphs have been constructed recently in [13].

A split graph $G = S(I \cup K, E)$ is called a *maximal nonhamiltonian split graph* if G is nonhamiltonian but the graph $G + uv$ is hamiltonian for every $uv \notin E$ where $u \in I$ and $v \in K$. It is known from a result in [12] that any nonhamiltonian Burkard-Hammer graph is contained in a maximal nonhamiltonian Burkard-Hammer graph. So knowledge about maximal nonhamiltonian Burkard-Hammer graphs provides us certain information about nonhamiltonian Burkard-Hammer graphs.

It has been shown in [12] that there are no nonhamiltonian Burkard-Hammer graphs and therefore no maximal nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) \geq |I| - 2$. In the same paper [12], Ngo Dac Tan and Le Xuan Hung have classified maximal nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - 3$. Namely, they have proved in [12] that for every integer $n > 5$ there exists up to isomorphisms exactly one maximal nonhamiltonian Burkard-Hammer graph $G = S(I \cup K, E)$ with $|K| = n$ and $\delta(G) = |I| - 3$ which is the graph $H^{4,n}$ in their notation there (see the definition of $H^{4,n}$ in Section 2). Recently, Ngo Dac Tan and Iamjaroen have constructed in [14] a family of maximal nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - 4$. In this paper, we will show that if a maximal nonhamiltonian Burkard-Hammer graph $G = S(I \cup K, E)$ with $\delta(G) = |I| - 4$ has $|I| \neq 6, 7$, then G must be a graph in the family constructed by Ngo Dac Tan and Iamjaroen in [14]. Namely, we will prove the following main result of the paper.

Theorem 1. *Let $G = S(I \cup K, E)$ be a split graph with $|I| \neq 6, 7$ and $\delta(G) = |I| - 4$. Then G is a maximal nonhamiltonian Burkard-Hammer graph if and only if G is isomorphic to the expansion $H^{4,t}[G_2, v_2^*]$ where $t = |K| - |I| + 5$ and $G_2 = S(I_2 \cup K_2, E_2)$ is a complete split graph with $|K_2| - 1 = |I_2| = |I| - 5 \geq 3$.*

The expansion graph $H^{4,t}[G_2, v_2^*]$ will be defined in Section 2.

Thus, we will get the classification of maximal nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - 4$ for the case $|I| \neq 6, 7$.

We would like to note that there is an interesting discussion about the Burkard-Hammer condition in [9]. Concerning the hamiltonian problem for split graphs, readers can see also [8] and [11].

2. Preliminaries

Let $G = S(I \cup K, E)$ be a split graph and $I' \subseteq I, K' \subseteq K$. Denote by $B_G(I' \cup K', E')$ the graph $G[I' \cup K'] - E(G[K'])$. It is clear that $G' = B_G(I' \cup K', E')$ is a bipartite graph with the bipartition subsets I' and K'. So we will call $B_G(I' \cup K', E')$ the *bipartite subgraph of G induced by I' and K'*. For a component $G'_j = B_G(I'_j \cup K'_j, E'_j)$ of $G' = B_G(I' \cup K', E')$ we define

$$k_G(G'_j) = k_G(I'_j, K'_j) = \begin{cases} |I'_j| - |K'_j| & \text{if } |I'_j| > |K'_j|, \\ 0 & \text{otherwise.} \end{cases}$$

If $G' = B_G(I' \cup K', E')$ has r components $G'_1 = B_G(I'_1 \cup K'_1, E'_1), \ldots, G'_r = B_G(I'_r \cup K'_r, E'_r)$ then we define

$$k_G(G') = k_G(I', K') = \sum_{j=1}^{r} k_G(G'_j).$$

A component $G'_j = B_G(I'_j \cup K'_j, E'_j)$ of $G' = B_G(I' \cup K', E')$ is called a *T-component* (resp., *H-component, L-component*) if $|I'_j| > |K'_j|$ (resp., $|I'_j| = |K'_j|, |I'_j| < |K'_j|$). Let $h_G(G') = h_G(I', K')$ denote the number of H-components of G'.

In 1980, Burkard and Hammer proved the following necessary but not sufficient condition for hamiltonian split graphs [2].

Theorem 2 [2]**.** *Let $G = S(I \cup K, E)$ be a split graph with $|I| < |K|$. If G is hamiltonian, then*

$$k_G(I', K') + \max\left\{1, \frac{h_G(I', K')}{2}\right\} \leq |N_G(I')| - |K'|$$

holds for all $\emptyset \neq I' \subseteq I, K' \subseteq N_G(I')$ with $(k_G(I', K'), h_G(I', K')) \neq (0, 0)$.

We will shortly call the condition in Theorem 2 the *Burkard-Hammer condition*. We also call a split graph $G = S(I \cup K, E)$ with $|I| < |K|$, which satisfies the Burkard-Hammer condition, a *Burkard-Hammer graph*. Thus, by Theorem 2 any hamiltonian split graph $G = S(I \cup K, E)$ with $|I| < |K|$ is a Burkard-Hammer graph. For split graphs $G = S(I \cup K, E)$ with $|I| < |K|$ and $\delta(G) \geq |I| - 2$ the converse is true [12]. But it is not true in general. The first example of a nonhamiltonian Burkard-Hammer graph has been indicated in [2]. Recently, Ngo Dac Tan and Le Xuan Hung have classified nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - 3$. Namely, they have proved the following result.

Theorem 3 [12]**.** *Let $G = S(I \cup K, E)$ be a split graph with $|I| < |K|$ and the minimum degree $\delta(G) \geq |I| - 3$. Then*

(i) *If $|I| \neq 5$, then G has a Hamilton cycle if and only if G satisfies the Burkard-Hammer condition;*

(ii) *If $|I| = 5$ and G satisfies the Burkard-Hammer condition, then G has no Hamilton cycles if and only if G is isomorphic to one of the graphs $H^{1,n}$, $H^{2,n}$, $H^{3,n}$ or $H^{4,n}$ listed in Table* 1.

Table 1. The graphs $H^{1,n}$, $H^{2,n}$, $H^{3,n}$ and $H^{4,n}$.

The graph G	The vertex-set $V(G) = I^* \cup K^*$	The edge-set $E(G) = E_1^* \cup \ldots \cup E_5^* \cup E_{K^*}^*$
$H^{1,n}$ $(n > 5)$	$I^* = \{u_1^*, u_2^*, u_3^*, u_4^*, u_5^*\}$, $K^* = \{v_1^*, v_2^*, \ldots, v_n^*\}$.	$E_1^* = \{u_1^*v_1^*, u_1^*v_2^*\}$, $E_2^* = \{u_2^*v_2^*, u_2^*v_4^*\}$, $E_3^* = \{u_3^*v_2^*, u_3^*v_3^*, u_3^*v_6^*\}$, $E_4^* = \{u_4^*v_1^*, u_4^*v_4^*, u_4^*v_6^*\}$, $E_5^* = \{u_5^*v_5^*, u_5^*v_6^*\}$, $E_{K^*}^* = \{v_i^*v_j^* \| i \neq j; i, j = 1, \ldots, n\}$.
$H^{2,n}$	$V(H^{2,n}) = V(H^{1,n})$	$E(H^{2,n}) = E(H^{1,n}) \cup \{u_4^*v_2^*\}$
$H^{3,n}$	$V(H^{3,n}) = V(H^{1,n})$	$E(H^{3,n}) = E(H^{1,n}) \cup \{u_5^*v_2^*\}$
$H^{4,n}$	$V(H^{4,n}) = V(H^{1,n})$	$E(H^{4,n}) = E(H^{1,n}) \cup \{u_4^*v_2^*, u_5^*v_2^*\}$

Theorem 3 shows that there are up to isomorphisms only four nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $K = N(I)$ and $\delta(G) = |I| - 3$, namely, the graphs $H^{1,6}, H^{2,6}, H^{3,6}$ and $H^{4,6}$. In contrast with this result, the number of nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $K = N(I)$ and $\delta(G) = |I| - 4$ is infinite. This is a recent result of Ngo Dac Tan and Iamjaroen [13]. We remind now one of the constructions in this work, which is needed here.

Let $G_1 = S(I_1 \cup K_1, E_1)$ and $G_2 = S(I_2 \cup K_2, E_2)$ be split graphs with

$$V(G_1) \cap V(G_2) = \emptyset$$

and v be a vertex of K_1. We say that a graph G is an *expansion of* G_1 *by* G_2 *at* v if G is the graph obtained from $(G_1 - v) \cup G_2$ by adding the set of edges

$$E_0 = \{x_i v_j \mid x_i \in V(G_1) \setminus \{v\}, v_j \in K_2 \text{ and } x_i v \in E_1\}.$$

It is clear that such a graph G is a split graph $S(I \cup K, E)$ with $I = I_1 \cup I_2$, $K = (K_1 \setminus \{v\}) \cup K_2$ and is uniquely determined by G_1, G_2 and $v \in K_1$.

Because of this, we will denote this graph G by $G_1[G_2, v]$. Further, a graph G is called an *expansion of* G_1 *by* G_2 if it is an expansion of G_1 by G_2 at some vertex $v \in K_1$.

As an example, we show in Figure 1 the expansion of the graph $H^{4,n}$ by the complete split graph $G_2 = S(I_2 \cup K_2, E_2)$ with $I_2 = \{u_1, u_2\}$ and $K_2 = \{v_1, v_2, v_3\}$ at the vertex v_2^* of $H^{4,n}$.

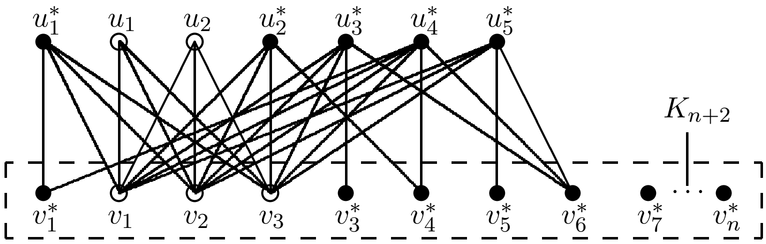

Figure 1. The expansion $H^{4,n}[G_2, v_2^*]$.

The following results are needed later.

Lemma 4 [11]**.** *Let $G = S(I \cup K, E)$ be a split graph with $|I| < |K|$. Then G has a Hamilton cycle if and only if $|N(I)| > |I|$ and the subgraph $G' = G[I \cup N(I)]$ has a Hamilton cycle.*

Lemma 5 [12]**.** *Let $G = S(I \cup K, E)$ be a Burkard-Hammer graph. Then for any $u \in I$ and $v \in K$ with $uv \notin E$, the graph $G + uv$ also is a Burkard-Hammer graph.*

Lemma 6 [12]**.** *Let $G = S(I \cup K, E)$ be a Burkard-Hammer graph. Then for any $\emptyset \neq I' \subseteq I$, we have $|N(I')| > |I'|$.*

Theorem 7 [13]**.** *Let $G_1 = S(I_1 \cup K_1, E_1)$ be a Burkard-Hammer graph and $G_2 = S(I_2 \cup K_2, E_2)$ be a complete split graph with $|I_2| < |K_2|$. Then any expansion of G_1 by G_2 is a Burkard-Hammer graph.*

Theorem 8 [13]**.** *Let $G_1 = S(I_1 \cup K_1, E_1)$ be an arbitrary split graph and $G_2 = S(I_2 \cup K_2, E_2)$ be a split graph with $|K_2| = |I_2| + 1$. Then an expansion of G_1 by G_2 is a hamiltonian graph if and only if both G_1 and G_2 are hamiltonian graphs.*

Let $G = S(I \cup K, E)$ be a split graph. Set

$$B_i(G) = \{v \in K \mid |N_{G,I}(v)| = i\}.$$

If the graph G is clear from the context then we also write B_i instead of $B_i(G)$.

Theorem 9 [14]**.** *Let $G = S(I \cup K, E)$ be a maximal nonhamiltonian Burkard-Hammer graph with $|I| \geq 7$ and $\delta(G) = |I| - 4$. Then $B_4 = B_5 = \cdots = B_{|I|-1} = \emptyset$ but $B_3 \neq \emptyset$.*

Theorem 10 [14]**.** *Any expansion of the graph $H^{4,n}$ by a complete split graph $G_2 = S(I_2 \cup K_2, E_2)$ with $|I_2| = |K_2| - 1 \geq 1$ at the vertex v_2^* of $H^{4,n}$ is a maximal nonhamiltonian Burkard-Hammer graph $G = S(I \cup K, E)$ with $\delta(G) = |I| - 4$.*

Let C be a cycle in a graph $G = (V, E)$. By $\overrightarrow{C}$ we denote the cycle C with a given orientation and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $w_1, w_2 \in V(C)$, then $w_1 \overrightarrow{C} w_2$ denotes the consecutive vertices of C from w_1 to w_2 in the direction specified by $\overrightarrow{C}$. The same vertices in the reverse order are given by $w_2 \overleftarrow{C} w_1$. We will consider $w_1 \overrightarrow{C} w_2$ and $w_2 \overleftarrow{C} w_1$ both as paths and as vertex sets. If $w \in V(C)$, then w^+ denotes the successor of w on $\overrightarrow{C}$, and w^- denotes its predecessor. The vertices $(w^+)^+$ and $(w^-)^-$ are written briefly by w^{++} and w^{--}, respectively. Similar notation as described above for a cycle is also used for a path.

We prove now the following lemma.

Lemma 11. *Let $G = S(I \cup K, E)$ be a Burkard-Hammer graph with $|I| \geq 7$ and $\delta(G) = |I| - 4$. Then G is a maximal nonhamiltonian split graph if and only if $G' = G[I \cup N_G(I)]$ is a maximal nonhamiltonian split graph.*

Proof. Let $G = S(I \cup K, E)$ be a Burkard-Hammer graph with $|I| \geq 7$ and $\delta(G) = |I| - 4$. Then by Lemma 6 $|N_G(I)| > |I|$.

First suppose that G is a maximal nonhamiltonian split graph. Then by Lemma 4 it is not difficult to see that $G' = G[I \cup N_G(I)]$ is a maximal nonhamiltonian split graph.

Conversely, suppose that $G' = G[I \cup N_G(I)] = S(I' \cup K', E')$ where $I' = I$ and $K' = N_G(I)$ is a maximal nonhamiltonian split graph. By Lemma 4, G is nonhamiltonian. So it remains to prove that for any $u \in I$

and any $v \in K$ with $uv \notin E$ the graph $H = G + uv$ is hamiltonian. We consider separately two cases.

Case 1. $v \in N_G(I)$.
Then $u \in I'$, $v \in K'$ and $uv \notin E'$. Therefore, $H' = G' + uv$ has a Hamilton cycle because G' is a maximal nonhamiltonian split graph. Since $H' = H[I \cup N_H(I)]$, by Lemma 4 H also has a Hamilton cycle.

Case 2. $v \in K \setminus N_G(I)$.
First assume that u is adjacent in G to all vertices of $N_G(I)$. Then we consider the graph $G - u$ which is a Burkard-Hammer graph $S(I_u \cup K, E_u)$ with $I_u = I \setminus \{u\}$ and $E_u = E \setminus \{uw \mid w \in N_G(I)\}$. Since $|I_u| \neq 5$ and $\delta(G - u) \geq |I_u| - 3$, by Theorem 3, $G - u$ has a Hamilton cycle C_u. We fix an orientation for C_u. Since $v \in K \setminus N_G(I)$ both v^- and v^+ are in K. By going from v along C_u in the direction specified by $\overrightarrow{C_u}$ we can find a vertex w such that $w \in N_G(I)$ but $w^- \in K \setminus N_G(I)$. Then w is adjacent in G to u by our assumption. Therefore, $C = vuw\overrightarrow{C_u}v$ is a Hamilton cycle of $G + uv = H$ if $w = v^+$ and $C = vuw\overrightarrow{C_u}v^-w^-\overleftarrow{C_u}v$ is a Hamilton cycle of $G + uv = H$ if $w \neq v^+$.

Now assume that there is a vertex $v_1 \in N_G(I)$ such that u is not adjacent in G to v_1. By Case 1, $G + uv_1$ has a Hamilton cycle C' that must contain the edge uv_1 because G is nonhamiltonian. We fix an orientation for C' so that $u^+ = v_1$. Since $v \in K \setminus N_G(I)$, we have $v^+ \in K$. Therefore, $C = uv\overleftarrow{C'}v_1v^+\overrightarrow{C'}u$ is a Hamilton cycle of $G + uv = H$.

The proof of the lemma is complete. ■

3. Classification for Case $|I| \neq 6, 7$

First of all, we prove the following Lemmas 12 and 13.

Lemma 12. *Let $G = S(I \cup K, E)$ be a maximal nonhamiltonian Burkard-Hammer graph with $m = |I| \neq 6$, $n = |K|$ and $\delta(G) = |I| - 4$. Then $|I| \geq 7$ and G possesses a Hamilton path P with the endvertices u_1 and v_n such that $u_1 \in I, v_n \in B_3$ and if $\overrightarrow{P} = u_1 \ldots v_n$ is the path P with the orientation from u_1 to v_n, then $v_n^- \in I$.*

Proof. Let $G = S(I \cup K, E)$ be a maximal nonhamiltonian Burkard-Hammer graph with $m = |I| \neq 6, n = |K|$ and $\delta(G) = |I| - 4$. By Lemma 6, for any vertex $u \in I$ we have $|N(u)| > |\{u\}| = 1$. So, $\delta(G) = |I| - 4 \geq 2$ and therefore we must have $|I| \geq 6$. This implies that $|I| \geq 7$ because we assume that $|I| \neq 6$. Now by Theorem 9, $B_4 = B_5 = \cdots = B_{m-1} = \emptyset$ but $B_3 \neq \emptyset$. Choose a vertex $v_n \in B_3$. Since $m = |I| \geq 7$ we can find a vertex $u_1 \in I \setminus N_I(v_n)$.

Since $u_1 v_n \notin E$ and G is a maximal nonhamiltonian split graph, $G + u_1 v_n$ has a Hamilton cycle D which must contain the edge $u_1 v_n$. So $P = D - u_1 v_n$ is a Hamilton path in G with u_1 and v_n its endvertices.

Let $\overrightarrow{P} = u_1 \ldots v_n$ be the path P with the orientation from u_1 to v_n. If $v_n^- \in I$, then P already is a Hamilton path required in the lemma. So we suppose that in $\overrightarrow{P}$ the vertex v_n^- is in K. Since $|N_I(v_n)| = 3$, there exists $u \in N_I(v_n)$. Then $\overrightarrow{P'} = u_1 \overrightarrow{P} u^- v_n^- \overleftarrow{P} u v_n$ is also a Hamilton path of G with the endvertices u_1 and v_n. But in $\overrightarrow{P'}$ the predecessor of v_n is u which is in I. Thus, the path P' is a Hamilton path required in the lemma. The proof of Lemma 12 is complete. ■

Let $G = S(I \cup K, E)$ be a maximal nonhamiltonian Burkard-Hammer graph with $m = |I| \neq 6, n = |K|$ and $\delta(G) = |I| - 4$ and let P, u_1 and v_n be as in Lemma 12. Set $\overline{N_I(v_n)} = I \setminus N_I(v_n)$. Then we have $|\overline{N_I(v_n)}| = |I| - |N_I(v_n)| = m - 3$. Let

$$u_1, u_2, \ldots, u_{m-3}$$

be the vertices of $\overline{N_I(v_n)}$ occurring on $\overrightarrow{P}$ in the order of their indices. Set

$$P_1 = u_1 \overrightarrow{P} u_2^-, P_2 = u_2 \overrightarrow{P} u_3^-, \ldots, P_{m-4} = u_{m-4} \overrightarrow{P} u_{m-3}^-, P_{m-3} = u_{m-3} \overrightarrow{P} v_n.$$

Then these subpaths of P appear on $\overrightarrow{P}$ in the order of their indices. Because of this we will call the subpath P_j, $j = 1, \ldots, m-3$, the *j-th subpath of P*. If v is a neighbour of u_1 and v^- is adjacent to v_n, then $C = u_1 \overrightarrow{P} v^- v_n \overleftarrow{P} v u_1$ is a Hamilton cycle of G, a contradiction. Thus, v^- is not adjacent to v_n, i.e., $v^- \in \overline{N_I(v_n)} = \{u_1, \ldots, u_{m-3}\}$. Hence, $v \in \{u_1^+, \ldots, u_{m-3}^+\}$. We have proved the following lemma.

Lemma 13. $N(u_1) \subseteq \{u_1^+, \ldots, u_{m-3}^+\}$. ■

The following Lemmas 14, 15 and 16 help us to know the structure of G in more detail.

Lemma 14. (i) *If $v \in N(u_j) \cap V(P_i)$ with $j \leq i$ where $i \in \{1, 2, \ldots, m-3\}$, $j \in \{2, \ldots, m-3\}$ and $u_j^+ \in N(u_1)$, then $v^- \notin N(v_n)$;*

(ii) *If $v \in N(u_j) \cap V(P_i)$ with $j > i$ where $i \in \{1, 2, \ldots, m-3\}$, $j \in \{2, \ldots, m-3\}$ and $u_j^+ \in N(u_1)$, then $v^+ \notin N(v_n)$.*

Proof. First assume that $v \in N(u_j) \cap V(P_i)$ with $j \leq i$ where $i \in \{1, 2, \ldots, m-3\}$, $j \in \{2, \ldots, m-3\}$ and $u_j^+ \in N(u_1)$. If $v = u_j^+$ then $v^- = u_j \notin N(v_n)$. If $v \neq u_j^+$ and $v^- \in N(v_n)$, then $C = u_j \overleftarrow{P} u_1 u_j^+ \overrightarrow{P} v^- v_n \overleftarrow{P} v u_j$ is a Hamilton cycle of G, a contradiction.

Now assume that $v \in N(u_j) \cap V(P_i)$ with $j > i$ where $i \in \{1, 2, \ldots, m-3\}$, $j \in \{2, \ldots, m-3\}$ and $u_j^+ \in N(u_1)$. If $v^+ \in N(v_n)$, then $C = u_j v \overleftarrow{P} u_1 u_j^+ \overrightarrow{P} v_n v^+ \overrightarrow{P} u_j$ is a Hamilton cycle of G, contradicting the nonhamiltonicity of G again. This completes the proof of Lemma 14. ■

By Lemma 14, $\overline{N_I(v_n)} = \{u_1, u_2, \ldots, u_{m-3}\}$ and $\delta(G) = |I| - 4$, we have immediately the following Lemma 15.

Lemma 15. *If $u_j^+ \in N(u_1)$ for $j \in \{2, \ldots, m-3\}$, then $N(u_j) \subseteq \{u_2^-, u_3^-, \ldots, u_j^-, u_j^+, u_{j+1}^+, \ldots, u_{m-3}^+\}$.* ■

Lemma 16. *If integers a and b with $2 \leq a < b \leq m-3$ are such that $u_a^+ \in N(u_1)$, u_a^- is adjacent to u_b and u_a is adjacent to u_b^+, then both $u_{a-1}^+ = u_a^-$ and $u_a^+ = u_{a+1}^-$ hold.*

Proof. Suppose, on the contrary, that $u_{a-1}^+ \neq u_a^-$. Then $u_a^{--} \notin \overline{N_I(v_n)}$ and therefore it is adjacent to v_n. Further, since $m \geq 7$, we have $\deg(u_j) \geq m-4 \geq 3$ for every $j \in \{1, 2, \ldots, m-3\}$. Therefore, since u_1 is adjacent to u_a^+, $C = u_1 u_a^+ \overrightarrow{P} u_b u_a^- u_a u_b^+ \overrightarrow{P} v_n u_a^{--} \overleftarrow{P} u_1$ is a Hamilton cycle of G, a contradiction. Similarly, if $u_a^+ \neq u_{a+1}^-$, then u_a^{++} is adjacent to v_n. So, since $\deg(u_j) \geq m-4 \geq 3$ for every $j \in \{1, 2, \ldots, m-3\}$ and u_1 is adjacent to u_a^+, $C = u_1 u_a^+ u_a u_b^+ \overrightarrow{P} v_n u_a^{++} \overrightarrow{P} u_b u_a^- \overleftarrow{P} u_1$ is a Hamilton cycle of G, contradicting the nonhamiltonicity of G again. The proof of Lemma 16 is complete. ■

Now we prove the following two Lemmas 17 and 18 which are crucial for the classification.

Lemma 17. *Let $G = S(I \cup K, E)$ be a maximal nonhamiltonian Burkard-Hammer graph with $|I| \neq 6, 7$ and $\delta(G) = |I| - 4$. Then $|I| \geq 8$ and G possesses a vertex $v \in B_3$ such that some vertex $u \in \overline{N_I(v)} = I \setminus N_I(v)$ has $\deg(u) \geq |I| - 3$.*

Proof. Let $G = S(I \cup K, E)$ be a maximal nonhamiltonian Burkard-Hammer graph with $m = |I| \notin \{6, 7\}, n = |K|$ and $\delta(G) = |I| - 4$. By Lemma 12, $m = |I| \geq 8$ and G possesses a Hamilton path P with the end-vertices u_1 and v_n such that $u_1 \in I, v_n \in B_3$ and if $\overrightarrow{P} = u_1 \dots v_n$ be the path P with the orientation from u_1 to v_n then $v_n^- \in I$.

Suppose, on the contrary, that G does not satisfy the last conclusion of the lemma. This means that for any vertex $v \in B_3$ and for any vertex $u \in \overline{N_I(v)} = I \setminus N_I(v)$, we have $\deg(u) \leq m - 4$. But $\delta(G) = m - 4$. So for any vertex $u \in \overline{N_I(v)}$, $\deg(u) = m - 4$.

We already noticed before Lemma 13 that $|\overline{N_I(v_n)}| = m - 3$. There we also denoted the vertices of $\overline{N_I(v_n)}$ in the order of their appearing on $\overrightarrow{P}$ by $u_1, u_2, \dots, u_{m-3}$ and defined the subpaths $P_1, P_2, \dots, P_{m-3}$ of P.

By Lemma 13, $N(u_1) \subseteq \{u_1^+, \dots, u_{m-3}^+\}$.
From this and $\deg(u_1) = m - 4$ it follows that there exists $r_0 \in \{2, 3, \dots, m-3\}$ such that vertices u_j^+ with $j \in \{1, 2, \dots, m-3\} \setminus \{r_0\}$ and only these vertices are neighbours of u_1.

By Lemma 15 we have

$$N(u_j) \subseteq \{u_2^-, u_3^-, \dots, u_j^-, u_j^+, u_{j+1}^+, \dots, u_{m-3}^+\}$$

for any $j \in \{2, \dots, m-3\} \setminus \{r_0\}$.

Claim 3.1. (i) If $3 \leq r_0 \leq m - 4$, then either some of $u_2, \dots, u_{r_0-1}$ is adjacent to $u_{r_0}^+$ or some of $u_{r_0+1}, \dots, u_{m-3}$ is adjacent to $u_{r_0}^-$.

(ii) If $r_0 = 2$, then some of $u_3, \dots, u_{m-3}$ is adjacent to u_2^-.

(iii) If $r_0 = m - 3$, then some of $u_2, \dots, u_{m-4}$ is adjacent to u_{m-3}^+.

Proof. First we prove the assertion (i). So we assume now that $3 \leq r_0 \leq m - 4$. Suppose, on the contrary, that $u_{r_0}^+ \notin N(u_j)$ for every $j \in \{2, \dots, r_0 - 1\}$ and $u_{r_0}^- \notin N(u_j)$ for every $j \in \{r_0 + 1, \dots, m - 3\}$. Then by Lemma 13, Lemma 15 and $\deg(u_j) = m - 4$, we have

$N(u_1) = \{u_1^+, \dots, u_{r_0-1}^+, u_{r_0+1}^+, \dots, u_{m-3}^+\}$,

$N(u_j) = \{u_2^-, \dots, u_j^-, u_j^+, \dots, u_{r_0-1}^+, u_{r_0+1}^+, \dots, u_{m-3}^+\}$ for $2 \leq j \leq r_0 - 1$
and

$N(u_j) = \{u_2^-, \dots, u_{r_0-1}^-, u_{r_0+1}^-, \dots, u_j^-, u_j^+, \dots, u_{m-3}^+\}$ for $r_0 + 1 \leq j \leq m - 3$.

If all equalities $u_1^+ = u_2^-, \dots, u_{r_0-2}^+ = u_{r_0-1}^-, u_{r_0+1}^+ = u_{r_0+2}^-, \dots, u_{m-4}^+ = u_{m-3}^-$ hold, then all vertices $u_1, \dots, u_{r_0-1}, u_{r_0+1}, \dots, u_{m-3}$ are adjacent to each of $u_1^+, \ldots, u_{r_0-2}^+$, $u_{r_0+1}^+, \ldots, u_{m-3}^+$. This implies that each of vertices $u_1^+, \ldots, u_{r_0-2}^+, u_{r_0+1}^+, \ldots, u_{m-3}^+$ has at least 4 neighbours in I because $m \geq 8$. Therefore, by Theorem 9, each of these vertices is adjacent to all vertices in I. In particular, they are adjacent to u_{r_0}. But $u_{r_0}^-$ and $u_{r_0}^+$ are also neighbours of u_{r_0}. So $\deg(u_{r_0}) \geq m-3$, contradicting the assumption about G.

Thus, there exists the number $j_0 \in \{1, \ldots, m-4\} \setminus \{r_0-1, r_0\}$ such that $u_{j_0}^+ \neq u_{j_0+1}^-$. So both $u_{j_0}^{++}$ and $u_{j_0+1}^{--}$ are adjacent to v_n. If $j_0 \neq 1$, then $C = u_{j_0+1}u_{j_0}^-\overleftarrow{P}u_1u_{j_0}^+u_{j_0}u_{j_0+1}^+\overrightarrow{P}v_nu_{j_0}^{++}\overrightarrow{P}u_{j_0+1}$ is a Hamilton cycle of G. If $j_0 = 1$, then $C = u_1u_2^+\overrightarrow{P}u_{m-3}u_2^-u_2\, u_{m-3}^+\overrightarrow{P}v_nu_2^{--}\overleftarrow{P}u_1$ is a Hamilton cycle of G. We have got a contradiction in all possible situations. So the assertion (i) of the claim must be true.

Assertions (ii) and (iii) can be proved by similar arguments. We leave it to the reader to carry out the proofs of (ii) and (iii) in detail.

The proof of Claim 3.1 is complete. ■

Now if $r_0 = m-3$, then u_{m-3}^+ must be adjacent to some vertex u_j with $j \in \{2, \ldots, m-4\}$ by Claim 3.1. Let $P' = u_j\overleftarrow{P}u_1u_j^+\overrightarrow{P}v_n$. Then P' is a Hamilton path of G with the endvertices u_j and v_n and all vertices of $\overline{N_I(v_n)}$ are in $u_j\overrightarrow{P'}u_{m-3}^+$. Moreover, in P' the vertex u_j is adjacent to u_{m-3}^+. So by considering u_j instead of u_1 and P' instead of P, if necessary, we may assume that

$$2 \leq r_0 \leq m-4.$$

Claim 3.2. There exists $j_0 \in \{1, 2, \ldots, m-4\}$ such that $u_{j_0}^+ \neq u_{j_0+1}^-$.

Proof. Suppose, on the contrary, that $u_j^+ = u_{j+1}^-$ for every $j \in \{1, 2, \ldots, m-4\}$. If $N(u_{r_0}) \cap u_{m-3}^{++}\overrightarrow{P}v_n = \emptyset$, then by Lemmas 13 and 15 we have $N(\{u_1, u_2, \ldots, u_{m-3}\}) = \{u_1^+, u_2^+, \ldots, u_{m-3}^+\}$. It follows that $|N(\{u_1, u_2, \ldots, u_{m-3}\})| = |\{u_1^+, u_2^+, \ldots, u_{m-3}^+\}| = |\{u_1, u_2, \ldots, u_{m-3}\}|$, contradicting Lemma 6. Thus, $N(u_{r_0}) \cap u_{m-3}^{++}\overrightarrow{P}v_n \neq \emptyset$. Let w be a vertex of $N(u_{r_0}) \cap u_{m-3}^{++}\overrightarrow{P}v_n$. Then $w^- \notin \overline{N_I(v_n)}$ and therefore w^- is adjacent to v_n.

By Claim 3.1, if $3 \leq r_0 \leq m-4$ then either some of $u_2, \ldots, u_{r_0-1}$ is adjacent to $u_{r_0}^+$ or some of $u_{r_0+1}, \ldots, u_{m-3}$ is adjacent to $u_{r_0}^-$ and if $r_0 = 2$ then some of $u_3, \ldots, u_{m-3}$ is adjacent to u_2^-. If some of $u_2, \ldots, u_{r_0-1}$ is adjacent to $u_{r_0}^+$, say u_{i_0}, then $C = u_1\overrightarrow{P}u_{i_0}u_{r_0}^+\overrightarrow{P}w^-v_n\overleftarrow{P}wu_{r_0}\overleftarrow{P}u_{i_0}^+u_1$ is a Hamilton cycle of G, a contradiction. If some of $u_{r_0+1}, \ldots, u_{m-3}$ is adjacent to $u_{r_0}^-$, say u_{j_0}, then $C = u_1\overrightarrow{P}u_{r_0}^-u_{j_0}\overleftarrow{P}u_{r_0}w\overrightarrow{P}v_nw^-\overleftarrow{P}u_{j_0}^+u_1$ is a Hamilton cycle of G, a contradiction again.

Thus, there must exist a subscript $j_0 \in \{1, 2, \ldots, m-4\}$ such that $u_{j_0}^+ \neq u_{j_0+1}^-$. ■

Claim 3.3. $u_{m-3}^{++} \in I$.

Proof. By Claim 3.2 there exists $j_0 \in \{1, 2, \ldots, m-4\}$ such that $u_{j_0}^+ \neq u_{j_0+1}^-$. Then $u_{j_0+1}^{--}$ is adjacent to v_n. Therefore, if $u_{m-3}^{++} \in K$, then $C = u_1u_{m-3}^+\overleftarrow{P}u_{j_0+1}^-u_{m-3}^{++}\overrightarrow{P}v_nu_{j_0+1}^{--}\overleftarrow{P}u_1$ is a Hamilton cycle of G, a contradiction. ■

Claim 3.4. u_{m-3}^+ is adjacent to all vertices of G.

Proof. Assume that u_{m-3}^+ is not adjacent to u_j for each $j \in \{2, 3, \ldots, m-4\}$. Then by Lemma 15 and $\deg(u_j) = m-4$ we have

$$N(u_j) = \{u_2^-, u_3^- \ldots, u_j^-, u_j^+, \ldots, u_{m-4}^+\}$$

for every $j \in \{2, 3, \ldots, m-4\} \setminus \{r_0\}$.

If $r_0 = m-4$, then by applying Lemma 16 for $a = 2, \ldots, m-6$ and $b = m-5$ we get $u_1^+ = u_2^-, \ldots, u_{m-6}^+ = u_{m-5}^-$. In particular, since $m \geq 8$, we always have $u_1^+ = u_2^-$ and $u_2^+ = u_3^-$. Suppose that $u_{m-5}^+ \neq u_{m-4}^-$. Then u_{m-5}^{++} is adjacent to v_n. Now if u_{m-3} is adjacent to $u_2^- = u_1^+$, then $C = u_1u_{m-5}^+\overleftarrow{P}u_2^-u_{m-3}\overleftarrow{P}u_{m-5}^{++}v_n\overleftarrow{P}u_{m-3}^+u_1$ is a Hamilton cycle of G, a contradiction. Thus, u_{m-3} is not adjacent to u_2^-. Together with Lemma 15 and $\deg(u_{m-3}) = m-4$, this implies that u_{m-3} is adjacent to $u_3^- = u_2^+$. Therefore, $C = u_1u_1^+u_2u_{m-5}^+\overleftarrow{P}u_3^-u_{m-3}\overleftarrow{P}u_{m-5}^{++}v_n\overleftarrow{P}u_{m-3}^+u_1$ is a Hamilton cycle of G, a contradiction again. Thus, we also have $u_{m-5}^+ = u_{m-4}^-$ if $r_0 = m-4$.

If $r_0 = m-5$, then by applying Lemma 16 for $a = 2, \ldots, m-6$ and $b = m-4$ we get $u_1^+ = u_2^-, \ldots, u_{m-6}^+ = u_{m-5}^-$. In particular, we have $u_1^+ = u_2^-$. Since $m \geq 8$, we have $r_0 = m-5 \geq 3$. So u_1 is adjacent to u_2^+. Now if $u_{m-5}^+ \neq u_{m-4}^-$, then u_{m-5}^{++} is adjacent to v_n and therefore $C = u_1u_2^+\overrightarrow{P}u_{m-5}^+u_2u_2^-u_{m-4}\overleftarrow{P}u_{m-5}^{++}v_n\overleftarrow{P}u_{m-4}^+u_1$ is a Hamilton cycle of G, a contradiction. Thus, we also have $u_{m-5}^+ = u_{m-4}^-$ if $r_0 = m-5$.

If $r_0 = 2$, then by applying Lemma 16 for $a = 3, \ldots, m-5$ and $b = m-4$ we get $u_2^+ = u_3^-, \ldots, u_{m-5}^+ = u_{m-4}^-$. In particular, we have $u_3^+ = u_4^-$. Since $m \geq 8$, we have $m-4 \geq 4$. Hence, u_4 is adjacent to u_2^-. Now if $u_1^+ \neq u_2^-$, then u_2^{--} is adjacent to v_n and therefore $C = u_1u_3^+u_3u_2^+u_2u_2^-u_4\overrightarrow{P}v_nu_2^{--}\overleftarrow{P}u_1$ is a Hamilton cycle of G, a contradiction. Thus, we also have $u_1^+ = u_2^-$ if $r_0 = 2$.

If $2 < r_0 < m-5$, then by applying Lemma 16 for $a = 2, \dots, r_0 - 1$ and $b = m-4$ we get $u_1^+ = u_2^-, \dots, u_{r_0-1}^+ = u_{r_0}^-$ and by applying Lemma 16 for $a = r_0+1, \dots, m-5$ and $b = m-4$ we also get $u_{r_0}^+ = u_{r_0+1}^-, \dots, u_{m-5}^+ = u_{m-4}^-$.

Thus, we always have $u_1^+ = u_2^-, \dots, u_{m-5}^+ = u_{m-4}^-$ for any value of r_0. By Claim 3.2, we must have $u_{m-4}^+ \neq u_{m-3}^-$. Hence, u_{m-4}^{++} is adjacent to v_n. Since $m \geq 8$, $\deg(u_{m-3}) = m-4 \geq 4$. It follows that there exists $j_0 \in \{3, \dots, m-4\}$ such that u_{m-3} is adjacent to $u_{j_0}^-$ because by Lemma 15 $N(u_{m-3}) \subseteq \{u_2^-, \dots, u_{m-3}^-, u_{m-3}^+\}$. Therefore, $C = u_1 u_{m-3}^+ \overrightarrow{P} v_n u_{m-4}^{++} \overrightarrow{P} u_{m-3} u_{j_0}^- \overrightarrow{P} u_{m-4}^+ u_{j_0-1} \overleftarrow{P} u_1$ is a Hamilton cycle of G. This final contradiction shows the assumption that u_{m-3}^+ is not adjacent to u_j for each $j \in \{2, 3, \dots, m-4\}$ is false.

So u_{m-3}^+ must be adjacent to a vertex u_j with $j \in \{2, 3, \dots, m-4\}$. By Claim 3.3, u_{m-3}^{++} is in I. Hence, $|N_I(u_{m-3}^+)| \geq 4$ because u_1, u_j, u_{m-3} and u_{m-3}^{++} are in $N_I(u_{m-3}^+)$. By Theorem 9, u_{m-3}^+ must be adjacent to all vertices of G.

The proof of Claim 3.4 is complete. ■

Claim 3.5. $u_{m-4}^+ = u_{m-3}^-$.

Proof. Suppose, on the contrary, that $u_{m-4}^+ \neq u_{m-3}^-$. Then $u_{m-4}^{++} \notin \overline{N_I(v_n)}$ and therefore it is adjacent to v_n. Further, since $m \geq 8$, $\deg(u_{m-3}) = m-4 \geq 4$. Together with $N(u_{m-3}) \subseteq \{u_2^-, \dots, u_{m-3}^-, u_{m-3}^+\}$, it follows that there exists $s_0 \in \{2, \dots, m-4\}$ such that $N(u_{m-3}) = \{u_2^-, \dots, u_{m-3}^-, u_{m-3}^+\} \setminus \{u_{s_0}^-\}$. Now if $r_0 < m-4$, then by taking $x \in \{2, \dots, m-4\}$ with $x \neq s_0$ we have $C = u_1 u_{m-4}^+ \overleftarrow{P} u_x u_{m-3}^+ \overrightarrow{P} v_n u_{m-4}^{++} \overrightarrow{P} u_{m-3} u_x^- \overleftarrow{P} u_1$ is a Hamilton cycle of G, a contradiction.

Thus, $r_0 = m-4$ must hold. Now we consider separately the following cases.

Case 1. There exists a vertex $u_t \in \{u_2, \dots, u_{m-5}\}$ adjacent to u_{m-4}^+. We have $N(u_{m-3}) = \{u_2^-, \dots, u_{m-3}^-, u_{m-3}^+\} \setminus \{u_{s_0}^-\}$. Since $m \geq 8$ there exists $x \in \{2, \dots, m-4\} \setminus \{t, s_0\}$. Then u_x^- is adjacent to u_{m-3} because $x \neq s_0$. If $2 \leq x \leq t-1$, then $C = u_t u_{m-4}^+ \overleftarrow{P} u_t^+ u_1 \overrightarrow{P} u_x^- u_{m-3} \overleftarrow{P} u_{m-4}^{++} v_n \overleftarrow{P} u_{m-3}^+ u_x \overrightarrow{P} u_t$ is a Hamilton cycle of G. If $t+1 \leq x \leq m-4$, then $C = u_t u_{m-4}^+ \overleftarrow{P} u_x u_{m-3}^+ \overrightarrow{P} v_n u_{m-4}^{++} \overrightarrow{P} u_{m-3} u_x^- \overleftarrow{P} u_t^+ u_1 \overrightarrow{P} u_t$ is a Hamilton cycle of G. We have got a contradiction in all possible situations. Thus, this case cannot occur.

Case 2. No vertices $u_j \in \{u_2, \ldots, u_{m-5}\}$ are adjacent to u^+_{m-4}.
In this case, for $j \in \{2, \ldots, m-5\}$, since $\deg(u_j) = m-4$ and $N(u_j) \subseteq \{u^-_2, \ldots, u^-_j, u^+_j, \ldots, u^+_{m-3}\}$ by Lemma 15, we have

$$N(u_j) = \{u^-_2, \ldots, u^-_j, u^+_j, \ldots, u^+_{m-5}, u^+_{m-3}\}.$$

By applying Lemma 16 for $a = 2, \ldots, m-6$ and $b = m-5$ we get $u^+_1 = u^-_2, \ldots, u^+_{m-6} = u^-_{m-5}$.

We show now that $u^+_{m-5} = u^-_{m-4}$ also holds. We have $N(u_{m-3}) = \{u^-_2, \ldots, u^-_{m-3}, u^+_{m-3}\} \setminus \{u^-_{s_0}\}$. If $s_0 \neq m-5$, then by applying Lemma 16 for $a = m-5$ and $b = m-3$ we get $u^+_{m-5} = u^-_{m-4}$. So we may assume now that $s_0 = m-5$. With this assumption we have u_{m-3} is adjacent to $u^-_2 = u^+_1$. Therefore, if $u^+_{m-5} \neq u^-_{m-4}$, then u^{++}_{m-5} is adjacent to v_n and therefore $C = u_1 u^+_{m-5} \overleftarrow{P} u^-_2 u_{m-3} \overleftarrow{P} u^{++}_{m-5} v_n \overleftarrow{P} u^+_{m-3} u_1$ is a Hamilton cycle of G, a contradiction. So, $u^+_{m-5} = u^-_{m-4}$ always holds.

Thus, $N(u_j) = \{u^-_2, \ldots, u^-_j, u^+_j, \ldots, u^+_{m-5}, u^+_{m-3}\} = \{u^+_1, \ldots, u^+_{m-5}, u^+_{m-3}\}$ for $j \in \{2, \ldots, m-5\}$. Hence, each of the vertices $u_1, u_2, \ldots, u_{m-5}$ is adjacent to each of the vertices $u^+_1, \ldots, u^+_{m-5}$. Since $N(u_{m-3}) = \{u^-_2, \ldots, u^-_{m-3}, u^+_{m-3}\} \setminus \{u^-_{s_0}\}$, the vertex u_{m-3} is adjacent to each of the vertices $u^+_1, \ldots, u^+_{s_0-2}, u^+_{s_0}, \ldots, u^+_{m-5}$. It follows that $|N_I(u^+_1)| \geq 4, \ldots, |N_I(u^+_{s_0-2})| \geq 4, |N_I(u^+_{s_0})| \geq 4, \ldots, |N_I(u^+_{m-5})| \geq 4$. By Theorem 9, $u^+_1, \ldots, u^+_{s_0-2}, u^+_{s_0}, \ldots, u^+_{m-5}$ are adjacent to all vertices of G. In particular, they are adjacent to u_{m-4}. Further, since $m \geq 8$ and all $u_1, \ldots, u_{m-5}$ are adjacent to $u^+_{s_0-1}$, we have $|N_I(u^+_{s_0-1})| \geq 3$. Now if $|N_I(u^+_{s_0-1})| > 3$, then again by Theorem 9 the vertex $u^+_{s_0-1}$ is adjacent to all vertices of G. In particular, it is adjacent to u_{m-4}. So $\{u^+_1, \ldots, u^+_{m-3}\} \subseteq N(u_{m-4})$. (We recall that u^+_{m-3} is adjacent to all vertices of G by Claim 3.4.) Therefore $\deg(u_{m-4}) \geq m-3$, contradicting our assumption about G. Thus, $|N_I(u^+_{s_0-1})| = 3$. Since $\{u_1, \ldots, u_{m-5}\} \subseteq N_I(u^+_{s_0-1})$, this can happen only if $m = 8$ and $N_I(u^+_{s_0-1}) = \{u_1, u_2, u_3\}$. Hence, $u^+_{s_0-1} \in B_3$. Let $u \in N_I(v_n)$ be such that $u \neq u^{++}_{m-3}$. Then $u \in \overline{N_I(u^+_{s_0-1})}$ and therefore $\deg(u) = m-4$ by our assumption about G. On the other hand, since $u^+_1 = u^-_2, \ldots, u^+_{m-5} = u^-_{m-4}$ as we have shown above, u is either in $u^+_{m-4} \overrightarrow{P} u^-_{m-3}$ or, if $u^{+++}_{m-3} \neq v_n$, in $u^{+++}_{m-3} \overrightarrow{P} v^-_n$. So, both u^- and u^+ are different from $u^+_1, \ldots, u^+_{s_0-2}, u^+_{s_0}, \ldots, u^+_{m-5}$ and u^+_{m-3}. But all the vertices $u^+_1, \ldots, u^+_{s_0-2}, u^+_{s_0}, \ldots, u^+_{m-5}, u^+_{m-3}, u^-$ and u^+ are in $N(u)$. (Recall that $u^+_1, \ldots, u^+_{s_0-2}, u^+_{s_0}, \ldots, u^+_{m-5}$ and u^+_{m-3} are adjacent to all vertices of G.) Hence, $\deg(u) \geq m-3$, contradicting $\deg(u) = m-4$ obtained before.

Thus, Case 2 also cannot occur. This means our assumption that $u^+_{m-4} \neq u^-_{m-3}$ is false.

The proof of Claim 3.5 is complete. ■

Claim 3.6. $u^+_1 = u^-_2$.

Proof. Let $Q = u_{m-3}\overleftarrow{P}u_1u^+_{m-3}\overrightarrow{P}v_n$. Then Q is a Hamilton path in G with the endvertices u_{m-3} and v_n. In $\overrightarrow{Q} = u_{m-3}\dots v_n$, the vertices $u_1, \dots, u_{m-3}$ of $\overline{N_I(v_n)}$ appear in the reverse order of their indices. So, if v^+ and v^- are still the successor and the predecessor of a vertex v, respectively, with respect to $\overrightarrow{P}$, then the first subpath Q_1 of Q is $u_{m-3}\overleftarrow{P}u^+_{m-4}$, …, the $(m-4)$-th subpath Q_{m-4} of Q is $u_2\overleftarrow{P}u^+_1$ and the $(m-3)$-th subpath Q_{m-3} of Q is $u_1u^+_{m-3}\overrightarrow{P}v_n$. Since u_{m-3} is adjacent to u^+_{m-3}, the path Q can play the role of P in the discussion of Claim 3.5. Therefore, by exchanging the roles of u_j and $u_{(m-3)-(j-1)}$, u^+_j and $u^-_{(m-3)-(j-1)}$, u^{++}_j and $u^{--}_{(m-3)-(j-1)}$, r_0 and s_0, respectively, we can repeat arguments in Claim 3.5 to show that $u^-_2 = u^+_1$. ■

Now we complete the proof of Lemma 17. By Claims 3.5 and 3.6, $u^+_{m-4} = u^-_{m-3}$ and $u^+_1 = u^-_2$. Therefore, by Claim 3.2 there is a subscript j_0 such that $2 \leq j_0 \leq m-5$ and $u^+_{j_0} \neq u^-_{j_0+1}$. Then both $u^{++}_{j_0}$ and $u^{--}_{j_0+1}$ are adjacent to v_n.

Assume that u_1 is adjacent to u^+_{m-4}. If $u^-_{j_0+1}$ is adjacent to u_{m-3}, then $C = u_1u^+_{m-4}\overleftarrow{P}u^-_{j_0+1}u_{m-3}\overrightarrow{P}v_nu^{--}_{j_0+1}\overleftarrow{P}u_1$ is a Hamilton cycle of G, a contradiction. Thus, $u^-_{j_0+1}$ cannot be adjacent to u_{m-3}. Therefore, since $\deg(u_{m-3}) = m-4$ and $N(u_{m-3}) \subseteq \{u^-_2, \dots, u^-_{m-3}, u^+_{m-3}\}$, the vertex u_{m-3} must be adjacent to $u^-_{j_0}$. Now $Q = u_1\overrightarrow{P}u^-_{j_0}u_{m-3}\overleftarrow{P}u_{j_0}u^+_{m-3}\ \overrightarrow{P}v_n$ can play the role of P in Claim 3.5. So we can get a contradiction as in the proof of Claim 3.5. Hence, the assumption that u_1 is adjacent to u^+_{m-4} is false.

Thus, u_1 is not adjacent to u^+_{m-4}, i.e., $r_0 = m-4$. This means that u_1 is adjacent to each of $u^+_1, \dots, u^+_{m-5}$ and u^+_{m-3}. By Claim 3.6, $u^+_1 = u^-_2$. Therefore, u_{m-3} cannot be adjacent to u^-_2 because otherwise $C = u_1u^+_{j_0}\overleftarrow{P}u^+_1u_{m-3}\overleftarrow{P}u^{++}_{j_0}v_n\overleftarrow{P}u^+_{m-3}u_1$ would be a Hamilton cycle of G, a contradiction. Thus, u_{m-3} is adjacent to each of vertices $u^-_3, \dots, u^-_{m-3}$ and u^+_{m-3}. Therefore, if $j_0 > 2$, then $C = u_1\overrightarrow{P}u^-_{j_0}u_{m-3}\overleftarrow{P}u^{++}_{j_0}v_n\overleftarrow{P}u^+_{m-3}u_{j_0}u^+_{j_0}u_1$ is a Hamilton cycle of G, a contradiction; and if $j_0 = 2$, then $C = u_1u^+_3\overrightarrow{P}u_{m-3}$ $u^-_3u_3u^+_{m-3}\overrightarrow{P}v_nu^{--}_3\overleftarrow{P}u_1$ is a Hamilton cycle of G, a contradiction again.

This final contradiction shows the assumption that G does not satisfy the last conclusion of Lemma 17 is false.

The proof of Lemma 17 is complete. ■

Lemma 18. *Let $G = S(I \cup K, E)$ be a maximal nonhamiltonian Burkard-Hammer graph with $m = |I| \geq 7, n = |K|$ and $\delta(G) = |I| - 4$. Furthermore, let G possess a vertex $v \in B_3$ such that some vertex $u \in \overline{N_I(v)} = I \setminus N_I(v)$ has* $\deg(u) \geq |I| - 3$*. Then G is isomorphic to the expansion $H^{4,t}[G_2, v_2^*]$ where $t = |K| - |I| + 5$ and $G_2 = S(I_2 \cup K_2, E_2)$ is a complete split graph with $|K_2| - 1 = |I_2| = |I| - 5$.*

Proof. By Lemma 11, without loss of generality, we may assume here that $K = N(I)$. Let P, u_1 and v_n be as in Lemma 12. Set $u_m = v_n^- \in I$ and let the vertices $u_1, u_2, \ldots, u_{m-3}$ of I and the subpaths $P_1, P_2, \ldots, P_{m-3}$ of P be defined as before Lemma 13. By the assumption of our lemma, without loss of generality, we may assume that u_1 and v_n are such that

$$\deg(u_1) \geq m - 3.$$

Together with Lemma 13, this implies the following Claim.

Claim 3.7. $N(u_1) = \{u_1^+, u_2^+, \ldots, u_{m-3}^+\}$.

By Lemma 14 and Claim 3.7, for any $j \in \{1, 2, \ldots, m-3\}$ we have $\deg(u_j) \leq m - 3$. But $\deg(u_j) \geq \delta(G) = m - 4$. It follows that $\deg(u_j) = m - 4$ or $m - 3$ for any $j \in \{1, 2, \ldots, m - 3\}$. By Lemma 15,

$$N(u_j) \subseteq \{u_2^-, u_3^-, \ldots, u_j^-, u_j^+, u_{j+1}^+, \ldots, u_{m-3}^+\}$$

for $j = 2, 3, \ldots, m - 3$.

Claim 3.8. There exists a number $j_0 \in \{1, 2, \ldots, m - 4\}$ such that $u_{j_0}^+ \neq u_{j_0+1}^-$ but $u_{j_0+1}^+ = u_{j_0+2}^-, \ldots, u_{m-4}^+ = u_{m-3}^-$.

Proof. Suppose that $u_j^+ = u_{j+1}^-$ for each $j \in \{1, 2, \ldots, m - 4\}$. Then for $I' = \{u_1, u_2, \ldots, u_{m-3}\}$, by Claim 3.7 and $N(u_j) \subseteq \{u_2^-, u_3^-, \ldots, u_j^-, u_j^+, u_{j+1}^+, \ldots, u_{m-3}^+\}$ for each $j = 2, 3, \ldots, m - 3$ just proved above, we have $N(I') = \{u_1^+, u_2^+, \ldots, u_{m-3}^+\}$. So $|N(I')| = |I'|$, contradicting Lemma 6. This means that Claim 3.8 must hold. ■

We have $u_{j_0}^{++}, u_{j_0+1}^{--} \notin \overline{N_I(v_n)} = \{u_1, \ldots, u_{m-3}\}$. Therefore, both $u_{j_0}^{++}$ and $u_{j_0+1}^{--}$ are adjacent to v_n. The reader should remember this because later we frequently use it, without mentioning it, to construct a Hamilton cycle in a graph G.

Claim 3.9. At least one of vertices u_2 or u_{m-4} is adjacent to u_{m-3}^+.

Proof. Suppose, on the contrary, that neither u_2 nor u_{m-4} is adjacent to u_{m-3}^+. Then since $\deg(u_j) \geq m-4$ and $N(u_j) \subseteq \{u_2^-, u_3^-, \ldots, u_j^-, u_j^+, u_{j+1}^+, \ldots, u_{m-3}^+\}$ for $j = 2, 3, \ldots, m-3$, we have

$$\begin{aligned} N(u_2) &= \{u_2^-, u_2^+, u_3^+, \ldots, u_{m-4}^+\}, \\ N(u_{m-4}) &= \{u_2^-, u_3^-, \ldots, u_{m-4}^-, u_{m-4}^+\}. \end{aligned}$$

By applying Lemma 16 with $a = 2$ and $b = m-4$ we get $u_1^+ = u_2^-, u_2^+ = u_3^-$. If there exists $x \in \{3, \ldots, m-5\}$ such that $u_x^+ \neq u_{x+1}^-$, then u_x^{++} is adjacent to v_n. So since $\deg(u_j) \geq m-4 \geq 3$ for every $j \in \{1, 2, \ldots, m-3\}$, $C = u_1 u_x^+ \overleftarrow{P} u_2 u_{m-4}^+ \overrightarrow{P} v_n u_x^{++} \overrightarrow{P} u_{m-4} u_2^- u_1$ is a Hamilton cycle of G, contradicting the nonhamiltonicity of G. Thus, we also have $u_3^+ = u_4^-, u_4^+ = u_5^-, \ldots, u_{m-5}^+ = u_{m-4}^-$. It follows that $j_0 = m-4$ and therefore u_{m-4}^{++} is adjacent to v_n.

If u_{m-3} is adjacent to u_2^-, then since $u_1^+ = u_2^-$, $C = u_{m-3} u_2^- \overrightarrow{P} u_{m-4}^+ u_1 u_{m-3}^+ \overrightarrow{P} v_n u_{m-4}^{++} \overrightarrow{P} u_{m-3}$ is a Hamilton cycle of G, a contradiction. So u_{m-3} is not adjacent to u_2^-. It follows that $N(u_{m-3}) = \{u_3^-, u_4^-, \ldots, u_{m-3}^-, u_{m-3}^+\}$ because $\deg(u_{m-3}) \geq m-4$ and $N(u_{m-3}) \subseteq \{u_2^-, u_3^-, \ldots, u_{m-3}^-, u_{m-3}^+\}$. Since $m \geq 7$ and $u_1^+ = u_2^-, \ldots, u_{m-5}^+ = u_{m-4}^-$, we always have $u_1^+ = u_2^-$ and $u_2^+ = u_3^-$. Therefore, $C = u_{m-3} u_3^- \overrightarrow{P} u_{m-4}^+ u_2 u_1^+ u_1 u_{m-3}^+ \overrightarrow{P}\ v_n u_{m-4}^{++} \overrightarrow{P} u_{m-3}$ is a Hamilton cycle of G, a contradiction again. The proof of Claim 3.9 is complete. ■

We continue the proof of Lemma 18. If $u_{m-3}^{++} \in K$, then $C = u_1 u_{m-3}^+ \overleftarrow{P} u_{j_0+1}^- u_{m-3}^{++} \overrightarrow{P} v_n u_{j_0+1}^{--} \overleftarrow{P} u_1$ is a Hamilton cycle of G, contradicting the nonhamiltonicity of G. So $u_{m-3}^{++} \in I$. It follows that $|N_I(u_{m-3}^+)| \geq 4$ because $u_1, u_{m-3}, u_{m-3}^{++}$ and at least one of vertices u_2 or u_{m-4} by Claim 3.9 are in $N_I(u_{m-3}^+)$. By Theorem 9, $N_I(u_{m-3}^+) = I$, i.e., u_{m-3}^+ is adjacent to all vertices of G.

If u_{m-3} is adjacent to u_2^- then by applying Lemma 16 with $a = 2$ and $b = m-3$ we get $u_1^+ = u_2^-, u_2^+ = u_3^-$. Therefore, $j_0 \geq 3$ and $C =$

$u_1 u_{j_0}^+ \overleftarrow{P} u_2^- u_{m-3} \overleftarrow{P} u_{j_0}^{++} v_n \overleftarrow{P}\ u_{m-3}^+ u_1$ is a Hamilton cycle of G, a contradiction. Thus, u_{m-3} is not adjacent to u_2^-. Hence $N(u_{m-3}) = \{u_3^-, u_4^-, \ldots, u_{m-3}^-, u_{m-3}^+\}$ because $\deg(u_{m-3}) \geq m-4$ and $N(u_{m-3}) \subseteq \{u_2^-, u_3^-, u_4^-, \ldots, u_{m-3}^-, u_{m-3}^+\}$. By applying now Lemma 16 with $a = 3, \ldots,\ m-4$ and $b = m-3$ we get $u_2^+ = u_3^-, u_3^+ = u_4^-, \ldots, u_{m-4}^+ = u_{m-3}^-$.

Thus, $j_0 = 1$. If u_2^- is adjacent to some u_j with $j \in \{3, 4, \ldots, m-3\}$ then $C = u_j u_2^- \overrightarrow{P} u_{j-1}^+ u_1 \overrightarrow{P} u_2^{--} v_n \overleftarrow{P} u_j$ is a Hamilton cycle of G, a contradiction. So u_2^- is not adjacent to any vertices $u_3, u_4, \ldots, u_{m-3}$. It follows that for $j = 3, 4, \ldots, m-3$,

$$N(u_j) = \{u_2^+, u_3^+, \ldots, u_{m-3}^+\}$$

because $\deg(u_j) \geq m-4$ and $N(u_j) \subseteq \{u_2^-, u_3^-, \ldots, u_j^-, u_j^+, \ldots, u_{m-3}^+\}$.

We have proved before that $u_{m-3}^{++} \in I$. If $u_{m-3}^{++} = u_m = v_n^-$ and u_m has no neighbours in $P_1 = u_1 \overrightarrow{P} u_2^-$, then $B_G(I' \cup K', E')$ with $I' = \{u_1, u_2, \ldots, u_{m-3}, u_m\}$ and $K' = N(I') \setminus \{u_2^+, u_3^+, \ldots, u_{m-3}^+\}$ has three H-components, namely $B_G(\{u_1\} \cup \{u_1^+\}, \{u_1 u_1^+\})$, $B_G(\{u_2\} \cup \{u_2^-\}, \{u_2 u_2^-\})$ and $B_G(\{u_m\} \cup \{v_n\}, \{u_m v_n\})$ and $m-5$ T-components, each of which consists of a single vertex from $\{u_3, \ldots, u_{m-3}\}$. Therefore, $k(I', K') + \max\left\{1, \frac{h(I',K')}{2}\right\} = m - 5 + \frac{3}{2}$. But $|N(I')| - |K'| = |\{u_2^+, u_3^+, \ldots, u_{m-3}^+\}| = m-4$. This contradicts the fact that G is a Burkard-Hammer graph. Thus, if $u_{m-3}^{++} = u_m$ then u_m has to have a neighbour v in P_1. If $v \neq u_1^+$ then v^- is adjacent to v_n and therefore $C = u_m v \overrightarrow{P} u_{m-3}^+ u_1 \overrightarrow{P} v^- v_n u_m$ is a Hamilton cycle of G. If $v = u_1^+$ then v^+ is adjacent to v_n and therefore $C = u_m v u_1 u_{m-3}^+ \overleftarrow{P} v^+ v_n u_m$ is a Hamilton cycle of G. We have got a contradiction in any situations.

Thus, $u_{m-3}^{++} \neq u_m$. Set $u_{m-2} = u_{m-3}^{++}$, $\overrightarrow{R_1} = u_1^+ \overrightarrow{P} u_2^-$ and $\overrightarrow{R_2} = u_{m-2} \overrightarrow{P}$ $v_n u_{m-2}$. Then $\overrightarrow{R_1}$ has at least two vertices and $\overrightarrow{R_2}$ is a cycle of length at least 4.

Claim 3.10. If there exist a vertex y of the path $\overrightarrow{R_1}$ and a vertex z of the cycle $\overrightarrow{R_2}$ such that either both yz and y^+z^+ are edges of G or both yz^+ and y^+z are edges of G, where y^+ and z^+ are the successor of y and the successor of z with respect to $\overrightarrow{R_1}$ and $\overrightarrow{R_2}$, respectively, then G has a Hamilton cycle.

Proof. Suppose that both yz and y^+z^+ are edges of G. If $z \neq v_n$, then $C = y \overleftarrow{P} u_1 u_{m-3}^+ \overleftarrow{P} y^+ z^+ \overrightarrow{P} v_n u_{m-2} \overrightarrow{P} zy$ is a Hamilton cycle of G. If $z = v_n$, then $z^+ = u_{m-2}$. Therefore, $C = y\overleftarrow{P}u_1u_{m-3}^+\overleftarrow{P}y^+u_{m-2}\overrightarrow{P}v_ny$ is a Hamilton cycle of G.

If both yz^+ and y^+z are edges of G, then Claim 3.10 can be proved similarly. The proof of Claim 3.10 is complete. ∎

Let u_{m-1} be the remaining vertex of I. Then either $u_{m-1} \in \overrightarrow{R_1}$ or $u_{m-1} \in \overrightarrow{R_2}$.

If $u_{m-1} \in \overrightarrow{R_2}$ then all vertices of $\overrightarrow{R_1}$ are in K. Therefore, by using Claim 3.10, it is not difficult to see that $u_{m-2}^+ = u_{m-1}^-, u_{m-1}^+ = u_m^-$ and u_{m-2}, u_{m-1}, u_m have no neighbours in $\overrightarrow{R_1}$. Take $I' = I, K' = N(I') \setminus \{u_2^+, u_3^+, \ldots, u_{m-3}^+\}$. Then it is not difficult to see as before that $k(I', K') = m - 5$ and $h(I', K') = 3$. Therefore, $k(I', K') + \max\left\{1, \frac{h(I',K')}{2}\right\} = m - 5 + \frac{3}{2}$, $|N(I')| - |K'| = |\{u_2^+, u_3^+, \ldots, u_{m-3}^+\}| = m - 4$. It follows that $k(I', K') + \max\left\{1, \frac{h(I',K')}{2}\right\} > |N(I')| - |K'|$, contradicting the fact that G is a Burkard-Hammer graph. Thus, u_{m-1} cannot be a vertex of $\overrightarrow{R_2}$ and therefore $u_{m-1} \in \overrightarrow{R_1}$.

Suppose that $u_{m-2}^+ = u_m^-$. Since v_n is adjacent to every vertex of $\overrightarrow{R_1}$, again by using Claim 3.10, we see that u_{m-2} and u_m are not adjacent to any vertices of $\overrightarrow{R_1}$. Take $I' = \{u_1, u_2, \ldots, u_{m-3}, u_{m-2}, u_m\}$ and $K' = N(I') \setminus \{u_2^+, u_3^+, \ldots, u_{m-3}^+\}$. Then as before it is not difficult to check that G does not satisfy the Burkard-Hammer condition with respect to these I' and K', a contradiction. Thus, $u_{m-2}^+ \neq u_m^-$ and therefore again by Claim 3.10 we must have $u_1^+ = u_{m-1}^-, u_{m-1}^+ = u_2^-$ and u_{m-1} is not adjacent to any vertices in $u_{m-2}^+\overrightarrow{P}u_m^-$. Further, if u_{m-2} is adjacent to a vertex $v \in u_{m-2}^{++}\overrightarrow{P}u_m^-$ then since $v^- \in K$,

$$C = u_1u_{m-3}^+\overleftarrow{P}u_{m-1}v_n\overleftarrow{P}vu_{m-2}\overrightarrow{P}v^-u_1^+u_1$$

is a Hamilton cycle of G. Similarly, if u_m is adjacent to a vertex $v \in u_{m-2}^+\overrightarrow{P}u_m^{--}$ then since $v^+ \in K$,

$$C = u_1u_{m-3}^+\overleftarrow{P}u_{m-1}v_nu_{m-2}\overrightarrow{P}vu_m\overleftarrow{P}v^+u_1^+u_1$$

is a Hamilton cycle of G. We have got a contradiction in both situations. Thus, in $\overrightarrow{R_2}$ the vertex u_{m-2} is adjacent to only u_{m-2}^+ and v_n and the vertex u_m is adjacent to only u_m^- and v_n. It follows that if $u_{m-2}^+\overrightarrow{P}u_m^-$ has more than two vertices, then since $K = N(I)$ (by our assumption), $N(u_1) =$

$\{u_1^+, \ldots, u_{m-3}^+\}$ (by Claim 3.7) and $N(u_j) \subseteq \{u_2^-, \ldots, u_j^-, u_j^+, \ldots, u_{m-3}^+\}$ for $j = 2, 3, \ldots, m-3$ (by Claim 3.7 and Lemma 15) are true, the vertex u_{m-2}^{++} must be adjacent to u_{m-1}. By Claim 3.10, G has a Hamilton cycle, contradicting the nonhamiltonicity of G. Thus, $u_{m-2}^{++} = u_m^-$. It follows that $n = |K| = |N(I)| = m + 1$ and

$$\begin{aligned} I &= \{u_1, u_2, \ldots, u_m\}, \\ K &= \{u_1^+, u_2^+, \ldots, u_{m-3}^+, u_{m-2}^+, u_{m-1}^+, u_m^-, v_{m+1}\}. \end{aligned}$$

Let $H = S(I \cup K, E(H))$ be a split graph with

$$\begin{aligned} N_H(u_1) &= \{u_1^+, u_2^+, \ldots, u_{m-3}^+\}, \\ N_H(u_2) &= \{u_2^+, u_3^+, \ldots, u_{m-3}^+, u_{m-1}^+\}, \\ N_H(u_3) &= N_H(u_4) = \cdots = N_H(u_{m-3}) = \{u_2^+, u_3^+, \ldots, u_{m-3}^+\}, \\ N_H(u_{m-2}) &= \{u_2^+, u_3^+, \ldots, u_{m-3}^+, u_{m-2}^+, v_{m+1}\}, \\ N_H(u_{m-1}) &= \{u_1^+, u_2^+, \ldots, u_{m-3}^+, u_{m-1}^+, v_{m+1}\}, \\ N_H(u_m) &= \{u_2^+, u_3^+, \ldots, u_{m-3}^+, u_m^-, v_{m+1}\}. \end{aligned}$$

Set $I_2 = \{u_3, u_4, \ldots, u_{m-3}\}$, $K_2 = \{u_2^+, u_3^+, \ldots, u_{m-3}^+\}$ and $G_2 = H[I_2 \cup K_2]$. Then G_2 is a complete split graph $S(I_2 \cup K_2, E_2)$ with $|K_2| - 1 = |I_2| = |I| - 5$. Further, let $H^{4,6} = S(I^* \cup K^*, E(H^{4,6}))$ with $I^* = \{u_1^*, u_2^*, u_3^*, u_4^*, u_5^*\}$ and $K^* = \{v_1^*, v_2^*, v_3^*, v_4^*, v_5^*, v_6^*\}$ be a split graph defined in Table 1 and $H' = H^{4,6}[G_2, v_2^*]$. Then H' is a split graph $S(I' \cup K', E')$ with $I' = \{u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_3, u_4, \ldots, u_{m-3}\}$ and $K' = \{v_1^*, v_3^*, v_4^*, v_5^*, v_6^*, u_2^+, u_3^+, \ldots, u_{m-3}^+\}$. Consider the following mapping $\varphi : V(H) \to V(H')$ with

$$\begin{aligned} \varphi(u_1) &= u_1^*, \varphi(u_2) = u_2^*, \varphi(u_j) = u_j \text{ for } j = 3, 4, \ldots, m-3, \\ \varphi(u_{m-2}) &= u_3^*, \varphi(u_{m-1}) = u_4^*, \varphi(u_m) = u_5^*, \\ \varphi(u_1^+) &= v_1^*, \varphi(u_j^+) = u_j^+ \text{ for } j = 2, 3, \ldots, m-3, \\ \varphi(u_{m-2}^+) &= v_3^*, \varphi(u_{m-1}^+) = v_4^*, \varphi(u_m^-) = v_5^*, \varphi(v_{m+1}) = v_6^*. \end{aligned}$$

It is not difficult to see that φ is an isomorphism between the graphs H and H'. By Theorem 10, H' is a maximal nonhamiltonian Burkard-Hammer graph. So, by $H \cong H'$, $H = S(I \cup K, E(H))$ also is a maximal nonhamiltonian Burkard-Hammer graph.

By considerations before we see that $N_G(u_i) \subseteq N_H(u_i)$ for every $i = 1, 2, \ldots, m$, i.e., $G = S(I \cup K, E)$ is a spanning subgraph of $H = S(I \cup K, E(H))$. But G is a maximal nonhamiltonian Burkard-Hammer graph by our assumption. So G must coincide with H and therefore G is isomorphic to $H' = H^{4,6}[G_2, v_2^*]$.

The proof of Lemma 18 is complete. ∎

From Theorem 10 and Lemmas 17 and 18 we can obtain immediately Theorem 1 formulated in the introduction, which gives us the classification of maximal nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $|I| \neq 6, 7$ and $\delta(G) = |I| - 4$.

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