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A NOTE ON JOINS OF ADDITIVE HEREDITARY GRAPH PROPERTIES

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Abstract

Let L^a denote a set of additive hereditary graph properties. It is a known fact that a partially ordered set $(L^a, \subseteq)$ is a complete distributive lattice. We present results when a join of two additive hereditary graph properties in $(L^a, \subseteq)$ has a finite or infinite family of minimal forbidden subgraphs.

Keywords: hereditary property, lattice of additive hereditary graph properties, minimal forbidden subgraph family, join in the lattice.

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1. Introduction and Preliminaries

Let us denote by $\mathcal{I}$ the class of all finite simple graphs possessing at least one vertex. *A property* $\mathcal{P}$ (*of graphs*) is any nonempty isomorphic closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called *hereditary* if it is closed to subgraphs and $\mathcal{P}$ is called *additive* if it is closed with respect to disjoint union of graphs.

For example, some well-known additive hereditary graph properties are given in the list below.

$\mathcal{O} \ = \{G \in \mathcal{I} : E(G) = \emptyset\}$,
$\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of G has at most $k+1$ vertices$\}$,
$\mathcal{D}_1 = \{G \in \mathcal{I} : G$ does not contain cycles$\}$,
$\mathcal{I}_k \ = \{G \in \mathcal{I} : G$ does not contain $K_{k+2}\}$.

If $\mathcal{P}$ is a hereditary property, then the set of *minimal forbidden subgraphs* of $\mathcal{P}$ is defined as follows:

$\mathbf{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P}$ but for each proper subgraph H of G, $H \in \mathcal{P}\}$. For instance, $\mathbf{F}(\mathcal{O}_k) = \{G \in \mathcal{I} : G$ is a tree on $k+2$ vertices$\}$.

To investigate the structure of an additive hereditary property $\mathcal{P}$ it is enough to find the family $\mathbf{F}(\mathcal{P})$, because $\mathcal{P}$ is uniquely determined by this family [5].

Let L^a stand for a set of all additive hereditary graph properties. It is known that L^a partially ordered by the set inclusion is a lattice. To denote it we will use the notation $(L^a, \subseteq)$. A property $\mathcal{P}$ is called $\wedge$-*reducible* in $(L^a, \subseteq)$ ($\vee$-*reducible* in $(L^a, \subseteq)$) if there exist properties $\mathcal{P}_1, \mathcal{P}_2 \in L^a$ both different from $\mathcal{P}$ such that $\mathcal{P} = \mathcal{P}_1 \wedge \mathcal{P}_2$ ($\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2$), otherwise $\mathcal{P}$ is called $\wedge$-*irreducible* in $(L^a, \subseteq)$ ($\vee$-*irreducible* in $(L^a, \subseteq)$).

A graph G has a property $\mathcal{P}_1 \circ \ldots \circ \mathcal{P}_k$, $\mathcal{P}_1, \ldots, \mathcal{P}_k \in L^a$ if its vertex set $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that V_i is an empty set or the subgraph $G[V_i]$ of G induced by V_i is an element of $\mathcal{P}_i$, $i = 1, 2, \ldots k$. Such a partition is said to be $(P_1, P_2, \ldots, P_k)$-*partition*. A property $\mathcal{P}$ is called *reducible over* L^a if there exist properties $\mathcal{P}_1, \mathcal{P}_2 \in L^a$, both different from $\mathcal{I}$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$, otherwise $\mathcal{P}$ is called *irreducible over* L^a.

The paper has been motivated by the following observation: if an additive hereditary graph property has finitely many minimal forbidden subgraphs, then it will have a polynomial-time membership test.

The recognition of additive hereditary graph properties possessing a finite family of minimal forbidden subgraphs seems to be a very difficult problem. This property is not monotone with respect to the set inclusion. To see it we consider properties $\mathcal{I}_1, \mathcal{D}_1, \mathcal{O}$. It is clear that $\mathcal{O} \subseteq \mathcal{D}_1 \subseteq \mathcal{I}_1$ and the families of minimal forbidden subgraphs for properties $\mathcal{O}$ and $\mathcal{I}_1$ are finite unlike the family for the property $\mathcal{D}_1$.

Is it possible to say anything about the dependence between families of minimal forbidden subgraphs for properties that one of them is included in the other one? We recall such a result below.

Theorem 1 [3]**.** *Let $\mathcal{P}_1, \mathcal{P}_2 \in L^a$. Then $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if for every $H \in \mathbf{F}(\mathcal{P}_2)$ there exists a graph $H' \in \mathbf{F}(\mathcal{P}_1)$ such that $H' \subseteq H$.*

The simple result characterizes the $\wedge$-irreducible property $\mathcal{P}$ as the property satisfying $\mathbf{F}(\mathcal{P}) = \{G\}$ for a unique connected graph G (see [2]). Moreover, a similar characterization was given for $\wedge$-irreducible and $\vee$-irreducible properties ([2]). In 2001 Berger [1] showed that for any additive reducible over L^a

property, the class of minimal forbidden subgraphs is infinite. It was stated in [2] that $\vee$-reducible properties are not reducible over L^a. In the light of the results presented, it is of interest to look carefully at $\vee$-reducible properties in order to make a decision about finiteness of their minimal forbidden subgraphs families.

2. Results

In what follows $\bar{\mathcal{P}} = \mathcal{I} \setminus \mathcal{P}$.

Let $\mathcal{P}_1, \mathcal{P}_2 \in L^a$, $\mathcal{P}_i \nsubseteq \mathcal{P}j, i, j = 1, 2\ i \neq j$. We define the following sets:

$A_{\mathcal{P}_1 \vee \mathcal{P}_2} = \{G \in \mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2) : G \in (\mathbf{F}(\mathcal{P}_1) \cap \bar{\mathcal{P}}_2) \cup (\mathbf{F}(\mathcal{P}_2) \cap \bar{\mathcal{P}}_1)\}$,

$B_{\mathcal{P}_1 \vee \mathcal{P}_2} = \{G \in \mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2) \setminus A_{\mathcal{P}_1 \vee \mathcal{P}_2}$: for every edge $e \in E(G)$, $G - e \in \mathcal{P}_1 \cup \mathcal{P}_2\}$,

$C_{\mathcal{P}_1 \vee \mathcal{P}_2} = \{G \in \mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2)$: there exists an edge $e \in E(G)$ such that $G - e \in \bar{\mathcal{P}}_1 \cap \bar{\mathcal{P}}_2\}$.

It is obvious that $\mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2) = A_{\mathcal{P}_1 \vee \mathcal{P}_2} \cup B_{\mathcal{P}_1 \vee \mathcal{P}_2} \cup C_{\mathcal{P}_1 \vee \mathcal{P}_2}$ and sets $A_{\mathcal{P}_1 \vee \mathcal{P}_2}$, $B_{\mathcal{P}_1 \vee \mathcal{P}_2}$, $C_{\mathcal{P}_1 \vee \mathcal{P}_2}$ are pairwise disjoint.

In next theorems we denote by $G_1 v_1 \stackrel{k}{\longleftrightarrow} v_2 G_2$ a graph obtained from disjoint graphs G_1, G_2 by joining the marked vertex v_1 of G_1 and the marked vertex v_2 of G_2 using a path of length k (with k edges).

Lemma 2. *Let $\mathcal{P}_1, \mathcal{P}_2 \in L^a$, $\mathcal{P}_i \nsubseteq \mathcal{P}j, i, j = 1, 2,\ i \neq j$. Then every graph $G \in C_{\mathcal{P}_1 \vee \mathcal{P}_2}$ is of the form $G_1 v_1 \stackrel{k}{\longleftrightarrow} v_2 G_2$ where $G_1 \in \mathbf{F}(\mathcal{P}_1) \cap \mathcal{P}_2$ and $G_2 \in \mathbf{F}(\mathcal{P}_2) \cap \mathcal{P}_1$.*

Proof. Let $G \in C_{\mathcal{P}_1 \vee \mathcal{P}_2}$ and $e \in E(G)$ be an edge guaranteed by the definition of $C_{\mathcal{P}_1 \vee \mathcal{P}_2}$. According to additivity of $\mathcal{P}_1 \vee \mathcal{P}_2$ we know that G is connected. Moreover, $G - e = G_1^* \cup G_2^*$ such that $G_1^* \in \mathcal{P}_1 \cap \bar{\mathcal{P}}_2$ and $G_2^* \in \mathcal{P}_2 \cap \bar{\mathcal{P}}_1$. Thus e is a bridge. It is clear that there exist $G_1 \subseteq G_1^*$, $G_2 \subseteq G_2^*$ such that $G_i \in \mathcal{P}_i \cap \mathbf{F}(\mathcal{P}_j)$, $i, j = 1, 2,\ i \neq j$. Assume $e^* \in E(G_i^*) \setminus E(G_i)$. By the fact $G - e^* \in \mathcal{P}_1 \vee \mathcal{P}_2$ it follows that every component of $G_i^* - e^*$ does not contain $G_1 \cup G_2$. Hence, e^* is a bridge and e^* lies on every path joining G_1 and G_2. The above implies that the only form of G is $G_1 v_1 \stackrel{k}{\longleftrightarrow} v_2 G_2$ for some $k \in N$ and $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. ■

Lemma 3. *Let $\mathcal{P}_1, \mathcal{P}_2 \in L^a$, $\mathcal{P}_i \nsubseteq \mathcal{P}j, i, j = 1, 2,\ i \neq j$. If $\mathbf{F}(\mathcal{P}_1) \cup \mathbf{F}(\mathcal{P}_2)$ is a finite set, then $B_{\mathcal{P}_1 \vee \mathcal{P}_2}$ is a finite set.*

Proof. Let $G \in B_{\mathcal{P}_1 \vee \mathcal{P}_2}$ and the assumptions are satisfied. We define sets $E_i(G) = \{e \in E(G) : G - e \in \mathcal{P}_i\}$, $i = 1, 2$. It is evident that $E(G) = E_1(G) \cup E_2(G)$ (of course it is not necessarily the disjoint sum). Moreover, the subgraph induced by E_i in G has to be the subgraph of each graph from $\mathbf{F}(\mathcal{P}_i)$ contained in G, respectively. This is the argument, which implies that the cardinality of $E(G)$ can be bounded above by the sum $|E(G_1')| + |E(G_2')|$, where G_1', G_2' are some forbidden subgraphs of $\mathcal{P}_1$, $\mathcal{P}_2$, respectively. Finiteness of families $\mathbf{F}(\mathcal{P}_i)$, $i = 1, 2$ implies that there exists a constant, which bounds above the number $|E(G)|$. By additivity of $\mathcal{P}_1 \vee \mathcal{P}_2$ there follows connectivity of elements in $\mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2)$ and the lemma follows. ∎

A symbol $\delta(G)$ stands for a minimum vertex degree in G.

Theorem 4. *Let $\mathcal{P}_1, \mathcal{P}_2 \in L^a$, $\mathcal{P}_i \not\subseteq \mathcal{P}j, i, j = 1, 2$, $i \neq j$ and for every graph $G \in \mathbf{F}(\mathcal{P}_1) \cup \mathbf{F}(\mathcal{P}_2)$ holds $\delta(G) > 1$. Then $\mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2)$ is infinite.*

Proof. A definition of sets $A_{\mathcal{P}_1 \vee \mathcal{P}_2}$,$B_{\mathcal{P}_1 \vee \mathcal{P}_2}$, $C_{\mathcal{P}_1 \vee \mathcal{P}_2}$ and assumptions guarantee for $F \in \mathbf{F}(\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2)$ that $\delta(F) \geq 2$. Suppose to the contrary that the set $\mathbf{F}(\mathcal{P})$ is finite. Because of $\mathcal{P}_i \not\subseteq \mathcal{P}j, i, j = 1, 2$ there exist two graphs $F_i \in \mathbf{F}(\mathcal{P}_i) \cap \mathcal{P}_j$, $i \neq j$, $i, j = 1, 2$. We consider a graph $F_1 v_1 \overset{s}{\longleftrightarrow} v_2 F_2$ with arbitrary vertices $v_1 \in V(F_1)$, $v_2 \in V(F_2)$ and s being greater than the length of the longest path taken over all graphs in $\mathbf{F}(\mathcal{P})$.

It is clear that such a graph has not the property $\mathcal{P}$. This implies the existence of $F \in \mathbf{F}(\mathcal{P})$, $F \subseteq F_1 v_1 \overset{s}{\longleftrightarrow} v_2 F_2$. According to $\delta(F) \geq 2$ and $F \not\subseteq F_1$, $F \not\subseteq F_2$ the only possible form of F is $G_1 v_1 \overset{s}{\longleftrightarrow} v_2 G_2$ with $G_1 \subseteq F_1$, $G_2 \subseteq F_2$, contrary to the assumption about the longest path for graphs in $\mathbf{F}(\mathcal{P})$. ∎

Let us consider the property $\mathcal{P} = \mathcal{I}_1 \vee \mathcal{O}_2$ and its arbitrary minimal forbidden subgraph F. It is evident that F has to contain at least one tree with four vertices and K_3 as subgraphs. According to additivity of $\mathcal{P}$, F is connected. It implies $K_3 v \overset{1}{\longleftrightarrow} v K_1 \subseteq F$. On the other hand, we can immediately check that $K_3 v \overset{1}{\longleftrightarrow} v K_1 \in \mathbf{F}(\mathcal{P})$. It follows that we found the unique minimal forbidden subgraph of $\mathcal{P} = \mathcal{I}_1 \vee \mathcal{O}_2$. A quite different situation arises for the property $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2$ defined by $\mathbf{F}(\mathcal{P}_1) = \{C_3 v \overset{1}{\longleftrightarrow} v K_1\}$ and $\mathbf{F}(\mathcal{P}_2) = \{C_4 v \overset{1}{\longleftrightarrow} v K_1\}$, respectively. As a simple observation we can write that $\{C_4 v \overset{s}{\longleftrightarrow} v K_3 : s \in N\} \subseteq \mathbf{F}(\mathcal{P})$ what gives infinitely many minimal forbidden subgraphs of $\mathcal{P}$.

What can we say about a family of minimal forbidden subgraphs for $\vee$-reducible property $\mathcal{P}$ satisfying $\delta(F) = 1$ for any $F \in \mathbf{F}(\mathcal{P})$? We will give a partial answer to this question in the next theorem.

Theorem 5. *Let $\mathcal{P}_1, \mathcal{P}_2 \in L^a$, $\mathcal{P}_i \nsubseteq \mathcal{P}j, i, j = 1, 2,\ i \neq j$. If there exists a positive integer k such that $\mathcal{P}_1 \subseteq \mathcal{O}_k$, then $\mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2)$ is finite if and only if $\mathbf{F}(\mathcal{P}_2)$ is finite.*

Proof. By the assumption $\mathcal{P}_1 \subseteq \mathcal{O}_k$, for fixed $k \in N$, every component of a graph in $\mathbf{F}(\mathcal{P}_1)$ has a bounded number of vertices. Its connectivity (by additivity of $(\mathcal{P}_1 \vee \mathcal{P}_2)$) implies immediately that $\mathbf{F}(\mathcal{P}_1)$ is finite. Assume that $\mathbf{F}(\mathcal{P}_2)$ is infinite. We observe that $A_{\mathcal{P}_1 \vee \mathcal{P}_2}$ is infinite because every connected graph with $k+2$ vertices cannot be in $\mathcal{P}_1$. Consequently, the family $\mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2)$ is infinite. In the case where $\mathbf{F}(\mathcal{P}_2)$ is finite we have finiteness of $A_{\mathcal{P}_1 \vee \mathcal{P}_2}$, $B_{\mathcal{P}_1 \vee \mathcal{P}_2}$ by the definition and Lemma 3, respectively. Moreover, Lemma 2 guarantees a form of $G \in C_{\mathcal{P}_1 \vee \mathcal{P}_2}$ as $G_1 v_1 \stackrel{n}{\longleftrightarrow} v_2 G_2$, where $G_1 \in \mathbf{F}(\mathcal{P}_1) \cap \mathcal{P}_2$ and $G_2 \in \mathbf{F}(\mathcal{P}_2) \cap \mathcal{P}_1$, $v_1 \in V(G_1), v_2 \in V(G_2)$. It is a simple observation that the parameter n has to be smaller than or equal to $k+1$. To deduce it, take the edge e of a path joining G_1 with G_2, which is close to G_1, delete it, and observe that if $n \geq k+2$, then a graph $G - e$ is not an element of $\mathcal{P}_1 \vee \mathcal{P}_2$. Hence, the construction $G_1 v_1 \stackrel{n}{\longleftrightarrow} v_2 G_2$ works only finite times, even if we count the changes of marked vertices in G_1 and G_2. ■

Theorem 5 by the assumption $\mathcal{P} \subseteq \mathcal{O}_k$, for a fixed $k \in N$ has assured the existence of at least one tree in $\mathbf{F}(\mathcal{P})$ (see Theorem 1 and the form of $\mathbf{F}(\mathcal{O}_k)$). The assumption $\delta(G) > 1$ for all $G \in \mathbf{F}(\mathcal{P}_1) \cup \mathbf{F}(\mathcal{P}_2)$ in Theorem 4 excludes such a possibility for both properties $\mathcal{P}_1, \mathcal{P}_2$. Hence, Theorems 4 and 5 actually deal with the disjoint sets of properties.

In some cases, we are able to determine whether the family of minimal forbidden subgraphs of a $\vee$-reducible property $\mathcal{P}_1 \vee \mathcal{P}_2$ is finite or infinite if we have some knowledge about $\mathbf{F}(\mathcal{P}_1) \cup \mathbf{F}(\mathcal{P}_2)$. Precisely, when all graphs in $\mathbf{F}(\mathcal{P}_1) \cup \mathbf{F}(\mathcal{P}_2)$ are two-connected or without bridges (in general without vertices of degree one), even if the set $\mathbf{F}(\mathcal{P}_1) \cup \mathbf{F}(\mathcal{P}_2)$ is finite, we have the infinity of $\mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2)$. On the other hand, it is possible to give examples of joins $\mathcal{P}_1 \vee \mathcal{P}_2$ so that $\mathbf{F}(\mathcal{P}_1) \cup \mathbf{F}(\mathcal{P}_2)$ is finite and $\mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2)$ is finite, too (Theorem 5). Moreover, there exists an example of properties $\mathcal{P}_1, \mathcal{P}_2 \in L^a$ such that each of them possesses an infinite family of minimal forbidden subgraphs but $\mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2)$ is finite. Namely, let $\mathcal{P}_1, \mathcal{P}_2$ be properties satisfying the requirement that each component of a graph in $\mathcal{P}_1$ is a cycle of odd length or a path and each component of a graph in $\mathcal{P}_2$ is a cycle of even length or a path. Then $\mathbf{F}(\mathcal{P}_1) = \{C_n : n \text{ is even}\} \cup \{K_{1,3}\}$, $\mathbf{F}(\mathcal{P}_2) = \{C_n : n \text{ is odd}\} \cup \{K_{1,3}\}$ are infinite contrary to $\mathbf{F}(\mathcal{P}_1 \vee \mathcal{P}_2) = \{K_{1,3}\}$.

It seems sufficient to know the form of all elements in $\mathbf{F}(\mathcal{P})$ to describe all possible cases in which a $\vee$-reducible property $\mathcal{P}$ has a finite number of minimal forbidden subgraphs. Lemmas 2, 3 have given the permissible shape of mentioned sets elements but we are still not able to solve our problem for a $\vee$-reducible property $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2$ satisfying the following conditions:

- there exists a graph $F \in \mathbf{F}(\mathcal{P}_1) \cup \mathbf{F}(\mathcal{P}_2)$ with the property $\delta(F) = 1$,
- $\mathcal{P}_1 \nsubseteq \mathcal{O}_k$ and $\mathcal{P}_2 \nsubseteq \mathcal{O}_l$ for any natural k, l.

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