

## A NOTE ON PACKING OF TWO COPIES OF A HYPERGRAPH

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### Abstract

A *2-packing* of a hypergraph  $\mathcal{H}$  is a permutation  $\sigma$  on  $V(\mathcal{H})$  such that if an edge  $e$  belongs to  $\mathcal{E}(\mathcal{H})$ , then  $\sigma(e)$  does not belong to  $\mathcal{E}(\mathcal{H})$ .

We prove that a hypergraph which does not contain neither empty edge  $\emptyset$  nor complete edge  $V(\mathcal{H})$  and has at most  $\frac{1}{2}n$  edges is 2-packable.

A 1-uniform hypergraph of order  $n$  with more than  $\frac{1}{2}n$  edges shows that this result cannot be improved by increasing the size of  $\mathcal{H}$ .

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## 1. Introduction

Let  $\mathcal{H} = (V, \mathcal{E})$  be a hypergraph where  $V$  is the *vertex set* and  $\mathcal{E} \subset 2^V$  is the *edge set*. We shall assume that  $V$  and  $\mathcal{E}$  are non-empty but allow in general empty edges for technical reasons. So, a complete simple hypergraph of order  $n$  has  $2^n$  edges. We consider only finite hypergraphs.

An edge  $e \in \mathcal{E}$  is called a *singleton* if  $|e| = 1$ . A vertex is *isolated* if no edge contains it. The number  $d(v)$  of edges containing a vertex  $v$  is called the *degree of*  $v \in V$ . A hypergraph is *t-uniform* if  $|e| = t$  for all  $e \in \mathcal{E}$ .

Let  $\mathcal{H}$  be a hypergraph of order  $n$ . A *packing* of two copies of  $\mathcal{H}$  (*2-packing* of  $\mathcal{H}$ ) is a permutation  $\sigma$  on  $V(\mathcal{H})$  such that if an edge  $e = \{x_1, \dots, x_k\}$  belongs to  $\mathcal{E}(\mathcal{H})$ , then the edge  $\sigma(e) = \{\sigma(x_1), \dots, \sigma(x_k)\}$  does not belong to  $\mathcal{E}(\mathcal{H})$ . Such a permutation (a *packing permutation*) is called also an *embedding* of  $\mathcal{H}$  into its complement.

Let us consider a hypergraph  $\mathcal{H}$  and a permutation  $\sigma$  on  $V$ . We have  $\sigma(V) = V$  and  $\sigma(\emptyset) = \emptyset$ . So, if  $V \in \mathcal{E}$  or  $\emptyset \in \mathcal{E}$  then  $\mathcal{H}$  cannot be packable. The hypergraph  $\mathcal{H}$  such that neither  $\emptyset \in \mathcal{E}(\mathcal{H})$  nor  $V \in \mathcal{E}(\mathcal{H})$  is called *admissible*. We consider only admissible hypergraphs.

Let  $\mathcal{H}$  be an admissible hypergraph. Let us consider a hypergraph  $\tilde{\mathcal{H}} = (V, \tilde{\mathcal{E}})$  with the same vertex set  $V$  and the edge set,  $\tilde{\mathcal{E}}$  obtained from  $\mathcal{E}$  in the following way: if  $e \in \mathcal{E}$  has at most  $\frac{n}{2}$  vertices then  $e$  belongs to  $\tilde{\mathcal{E}}$  and if  $e$  has more than  $\frac{n}{2}$  vertices, then  $e$  is replaced by  $V \setminus e$  with convention that each double edge eventually created in this way is replaced by a single one.

**Remark.** Let  $\mathcal{H}$  be an admissible hypergraph of order  $n$  and size at most  $\frac{1}{2}n$ . Let us observe that a hypergraph  $\tilde{\mathcal{H}}$  is 2-packable iff the hypergraph  $\mathcal{H}$  is 2-packable.

Therefore, we will consider 2-packing of  $\tilde{\mathcal{H}}$  in the proof of Theorem 2, although we shall write  $\mathcal{H}$ .

A 2-uniform hypergraph is called a graph. The packing problems for graphs have been studied for about thirty years (see for instance chapters in the books by B. Bollobás or H.P. Yap ([2, 7]) or survey papers by H.P. Yap or M. Woźniak ([8, 5, 6] and [4])). One of the first results in this area was the following theorem (see [3]).

**Theorem 1.** *A graph  $G$  of order  $n$  and size at most  $n - 2$  is 2-packable.*

## 2. Main Result

The aim of this note is to prove the following theorem.

**Theorem 2.** *An admissible hypergraph of order  $n$  and size at most  $\frac{1}{2}n$  is 2-packable.*

First, let us observe that this bound is best possible. Namely, if  $\mathcal{H}$  is a hypergraph of order  $n$  and has more than  $\frac{1}{2}n$  edges and each edge is a singleton, then evidently  $\mathcal{H}$  is not packable.

**Proof of Theorem 2.** It is easy to see that the theorem is true for  $n = 2$  and  $n = 3$ . So, let  $n \geq 4$ .

By Remark in the previous section, we may consider only hypergraphs which have only edges of cardinality at most  $\frac{n}{2}$ . Let  $\mathcal{H}$  be an admissible hypergraph. Denote by  $m_k$  the number of edges of cardinality  $k$  and let  $m$  be the size of  $\mathcal{H}$ . Thus

$$\frac{n}{2} \geq m = m_1 + m_2 + \dots + m_{\lfloor \frac{n}{2} \rfloor}.$$

The proof will be divided into two parts.

*Case 1.*  $m_1 = 0$

First, by using a ‘probabilistic’ argument we shall show that the packing permutation exists if  $\mathcal{H}$  has no singleton.

Let  $e$  and  $f$  be two edges of  $\mathcal{H}$  of the same cardinality and let  $\sigma$  be a random permutation on  $V$ . We say that an edge  $e$  covers an edge  $f$  (with respect to  $\sigma$ ), if  $\sigma(e) = f$ . We write:  $(e \curvearrowright f)$ .

Let  $e$  and  $f$  be two edges of cardinality  $k$ . The probability of the event  $A$  that  $e$  covers  $f$  (denoted by  $A(e \curvearrowright f)$ ) is equal to

$$Pr(A(e \curvearrowright f)) = \frac{k!(n-k)!}{n!} = \binom{n}{k}^{-1}.$$

Let us observe, that the number of events that an edge of cardinality  $k$  cover some edge in  $\mathcal{H}$  of cardinality  $k$  is equal to  $m_k^2$ .

So, we have

$$\begin{aligned}
Pr \left( \bigcup_{e,f \in \mathcal{H}} A(e \curvearrowright f) \right) &\leq \sum_{e,f \in \mathcal{H}} Pr(A(e \curvearrowright f)) \\
&= m_2^2 \binom{n}{2}^{-1} + m_3^2 \binom{n}{3}^{-1} + \dots + m_{\lfloor \frac{n}{2} \rfloor}^2 \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1}.
\end{aligned}$$

Since  $k \leq \frac{n}{2}$ , the sequence  $\left( \binom{n}{2}^{-1}, \binom{n}{3}^{-1}, \dots \right)$  is decreasing and we have

$$\begin{aligned}
&m_2^2 \binom{n}{2}^{-1} + m_3^2 \binom{n}{3}^{-1} + \dots + m_{\lfloor \frac{n}{2} \rfloor}^2 \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1} \\
&\leq \binom{n}{2}^{-1} \left( m_2^2 + m_3^2 + \dots + m_{\lfloor \frac{n}{2} \rfloor}^2 \right) \\
&\leq \binom{n}{2}^{-1} \left( \frac{n}{2} \right)^2 = \frac{n}{2(n-1)}.
\end{aligned}$$

It is easy to see that  $\frac{n}{2(n-1)} < 1$  for  $n > 2$ . In consequence, there exists a packing of an admissible hypergraph  $\mathcal{H}$  of order  $n$  and size at most  $\frac{1}{2}n$  into its complement, if  $\mathcal{H}$  does not have any singletons.

*Case 2.  $m_1 \geq 1$*

In this case we use the induction with respect to  $n$ . Let  $\mathcal{H}$  be an admissible hypergraph of order  $n$  and suppose that the theorem holds for  $n' < n$ .

Let  $\{x\}$  be a singleton and let  $y$  be a vertex of  $\mathcal{H}$  such that  $\{y\} \notin \mathcal{E}$  and  $\{x, y\} \notin \mathcal{E}$ . Such a  $y$  exists. For, otherwise each vertex other than  $x$  would be either a singleton or the end of an edge joining it with  $x$ , and we would get a contradiction with the size of  $\mathcal{H}$ .

Now, we construct a hypergraph  $\mathcal{H}' = (V', \mathcal{E}')$  such that  $V' = V - \{x, y\}$  and the set of edges is obtained from  $\mathcal{E}$  as follows: we delete the edge  $\{x\}$  and we replace all edges containig  $x$  or  $y$  (or  $x$  and  $y$ ) by new edges without these vertices. So  $\mathcal{H}'$  has one edge and two vertices less than  $\mathcal{H}$ . If  $m_1 \neq 0$  in  $\mathcal{H}'$  then a packing permutation  $\sigma'$  exists by the induction hypothesis. If  $m_1 = 0$  in  $\mathcal{H}'$  then a packing permutation  $\sigma'$  exists by Case 1.

By the choice of  $x$  and  $y$  and the property of  $\sigma'$ , it is easy to see that the permutation  $\sigma = \sigma' \circ (xy)$  where  $(xy)$  denotes a transposition, is a packing permutation of  $\mathcal{H}$  ■

### 3. Some Open Problems

One of the objectives of this paper is to draw attention of the reader to some open problems.

**I.** It would be interesting to consider an analogous problem for uniform hypergraphs. For  $t = 1$  the above result is still best possible. For  $t = 2$  the answer is given by Theorem 1. In the case  $t = 3$  we suppose (together with E. Győri) that the *right* bound of the size of a packable hypergraph is  $k = \frac{1}{6}(n-2)(n+3)$ . It is easy to see that this bound cannot be improved.

**II.** Does Theorem 2 remain true if instead of packing of two copies of the same hypergraph of order  $n$  we pack two distinct hypergraphs of order  $n$ , both of size at most  $\frac{1}{2}n$ ?

**III.** A. Benhocine and A.P. Wojda in [1] proved that a graph of order  $n$  and size at most  $n-1$  is 2-packable if and only if  $G$  is embeddable into a self complementary graph of the same order (for  $n \equiv 0, 1 \pmod{4}$ ). It would be interesting to get an analogous result for hypergraphs.

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