

A NOTE ON UNIQUELY H -COLOURABLE GRAPHS

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Abstract

For a graph H , we compare two notions of uniquely H -colourable graphs, where one is defined via automorphisms, the second by vertex partitions. We prove that the two notions of uniquely H -colourable are not identical for all H , and we give a condition for when they are identical. The condition is related to the first homomorphism theorem from algebra.

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1. Introduction

A *homomorphism* from G to H is an edge-preserving vertex-mapping. If there is a homomorphism from G to H , then we say that G is *H -colourable*. For background and notation on graph homomorphisms, the reader is directed to [6]. Uniquely H -colourable graphs, where H is a fixed finite graph, have been studied by many authors; see, for example, [1, 2, 4, 9, 10]. The usual definition of uniquely H -colourable uses automorphisms of H , and as such, makes no explicit mention of vertex partitions. To be more explicit, recall that a graph H is a *core* if every endomorphism of H is an automorphism. For a core H , G is *uniquely H -colourable* if G is H -colourable, every

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homomorphism from G to H is surjective, and for all homomorphisms f, h from G to H , there is an automorphism g of H so that $f = gh$. We denote the class of uniquely H -colourable graphs by $C_u(H)$.

Given a homomorphism f from G to H , define $\ker(f) = \{(x, y) \in V(G) \times V(G) : f(x) = f(y)\}$. Then $\ker(f)$ is an equivalence relation whose equivalence classes, called *colour blocks*, are independent sets partitioning $V(G)$. A graph G is *weakly uniquely H -colourable* if the last condition in the definition of uniquely H -colourable is replaced by: for all homomorphisms f, h from G to H , $\ker(f) = \ker(h)$. The class of weakly uniquely H -colourable graphs is written $C_{wu}(H)$. It is straightforward to see that $C_u(H) \subseteq C_{wu}(H)$. The classical notion of uniquely n -colourable graph introduced in [6] corresponds to weakly uniquely K_n -colourable graphs. Further, $C_u(K_n) = C_{wu}(K_n)$.

However, perhaps surprisingly, there are cores H such that $C_u(H) \subsetneq C_{wu}(H)$. We demonstrate that there are infinitely many graphs H with this property; see Theorem 2. If $C_u(H) = C_{wu}(H)$, then we say that the core H is *good*; otherwise, we say that H is *bad*. Our goal in this short note is to present a condition for a core to be good that applies to a large number of cases; see Theorem 3 (2). We give an equivalent algebraic formulation of this condition in Theorem 3 (1). For related work on H -colourings and properties of cores, the reader is directed to [7, 8].

2. Good, Great, and Bad Cores

All graphs we consider are finite, undirected, and simple. Define $\text{Hom}(G, H)$ be the set of homomorphisms from G into H . The monoid $\text{Hom}(H, H)$ of endomorphisms of H under composition is denoted $\text{End}(H)$. We write $\text{Aut}(H)$ for the group of automorphisms of H under composition. If X is a set, then we write $\text{Sym}(X)$ for the set of bijections from X to itself.

Before we can give examples of bad cores, we need the following straightforward lemma.

Lemma 1. *Let H be a core graph. Then $G \in C_{wu}(H)$ if and only if G is H -colourable, every element of $\text{Hom}(G, H)$ is surjective, and for all $f, h \in \text{Hom}(G, H)$, there is a $g \in \text{Sym}(V(H))$ so that $f = gh$.*

Proof. Assume $|V(H)| = n$, for some $n \geq 1$. For the forward direction, fix $f, h \in \text{Hom}(G, H)$. Label the colour blocks of $\ker(f)$ as B_1, \dots, B_n . For each

$1 \leq i \leq n$, choose $b_i \in B_i$. Define $g : V(H) \rightarrow V(H)$ by $g(h(b_i)) = f(b_i)$. Then g is well-defined as $\ker(f) = \ker(h)$ and as h is surjective. As f is surjective, it follows that g is surjective, and hence, $g \in \text{Sym}(V(H))$. As $f = gh$, the forward direction follows.

For the converse, let $f, h \in \text{Hom}(G, H)$. By hypothesis, there is a $g \in \text{Sym}(V(H))$ so that $f = gh$. Then for $x, y \in V(G)$, $f(x) = f(y)$ if and only if $g^{-1}f(x) = g^{-1}f(y)$. But the latter is equivalent to $h(x) = h(y)$. Hence, $(x, y) \in \ker(f)$ if and only if $(x, y) \in \ker(h)$, so $\ker(f) = \ker(h)$. ■

For an integer $n \geq 2$, let $G = K_{2n-2}$ with 1, 2 fixed distinct vertices of G . Define a graph H_n as follows. Let G_1 and G_2 be two disjoint copies of G , so that the vertices of G_2 are $\{x' : x \in V(G)\}$. The vertices of H_n are the vertices of G_1 and G_2 , along with three new vertices a, b , and c . The edges of H_n include the edges of G_1 and G_2 ; the vertex a is joined to every vertex in $V(G_1) \cup V(G_2)$ and no other vertex; the vertex b is joined to vertices in $\{c\} \cup V(G_1)$ and no other vertex; the vertex c is joined to vertices in $\{b\} \cup V(G_2)$ and no other vertex; the only edge “between” $V(G_1)$ and $V(G_2)$ is $12'$. See Figure 1 for H_2 .

If x is a vertex of G , then $G - x$ is the graph that results when x is deleted. A nontrivial graph is *critical* if $\chi(G - x) < \chi(G)$ for all $x \in V(G)$.

Theorem 2. *Each graph H_n is a bad core.*

Proof. The graph $H = H_n$ is critical with $\chi(H) = 2n$. Hence, H is a core. Define J by deleting the edge $12'$, so that 1 and 2 have the same neighbors in J . Then J is critical with $\chi(J) = 2n$.

We next show that $J \in C_{wu}(H) \setminus C_u(H)$, which will witness that H is bad. To see this, note first that $J \in C_{wu}(H)$: clearly J is H -colourable; since J and H are critical with chromatic number $2n$, every $f \in \text{Hom}(J, H)$ is surjective; and as $|V(J)| = |V(H)|$, $\ker(f)$ has only singleton colour blocks, so any two elements of $\text{Hom}(J, H)$ have the same kernel. Let $f = id_J \in \text{Hom}(J, H)$ and define h so that it interchanges 1 and 2 and fixes all other vertices. Then $h \in \text{Hom}(J, H)$ as 1 and 2 have the same neighbors in J . If there is a $g \in \text{Aut}(H)$ so that $f = gh$, then g interchanges 1 and 2 leaving all other vertices of H fixed. But 1 and 2 have different neighbours in H , so that $g \notin \text{End}(H)$, which is a contradiction. ■

By Theorem 2, there are infinitely many bad cores. By a direct check, each core of order at most 6 is good. Hence, the minimum order of a bad core is 7, with an explicit example given in Figure 1.

have all singleton blocks. In particular,

$$(1.1) \quad (H - e) / \ker(f) \cong H - e.$$

The mapping $g : V(H) \rightarrow V(H)$ defined by $g(f(x)) = h(x)$ is well-defined and bijective as $\ker(f) = \ker(h)$ and f is surjective. Hence, $gf = h$ for some $g \in \text{Sym}(V(H))$. As f and h were arbitrary, $H - e \in C_{wu}(H)$ by Lemma 1. Since $C_{wu}(H)$ satisfies the first homomorphism theorem, $\eta_f : V((H - e) / \ker(f)) \rightarrow V(H)$ is an isomorphism, so that $H - e \cong H$ by (1), which is a contradiction.

For the reverse direction of (1), fix H a great core. Fix $G \in C_{wu}(H)$ and $h \in \text{Hom}(G, H)$. If the natural map $\eta_h : V(G / \ker(h)) \rightarrow V(H)$ is not an isomorphism, then it is not an embedding. Hence, there are $x, y \in V(H)$ such that $xy \in E(H)$ but the colour blocks $h^{-1}(x)$ and $h^{-1}(y)$ are not joined in $G / \ker(h)$. Let $e = xy \in E(H)$. Note that $h : V(G) \rightarrow V(H - e)$ is a homomorphism. As H is great, there is some $f \in \text{Hom}(H - e, H)$ that is not surjective. But then $fh : V(G) \rightarrow V(H)$ is a homomorphism that is not surjective, which contradicts that $G \in C_{wu}(H)$.

For item (2), as H is great, $C_{wu}(H)$ satisfies the first homomorphism theorem by item (1). Fix $G \in C_{wu}(H)$ and fix $f, h \in \text{Hom}(G, H)$. Then by Lemma 1 there is a $g \in \text{Sym}(V(H))$ so that $f = gh$. We show that $g \in \text{Aut}(H)$. As H is a core, it is enough to show that $g \in \text{End}(H)$. To see this, fix $xy \in E(H)$. By hypothesis, $h^{-1}(x)h^{-1}(y) \in E(G / \ker(h))$, so that there is some $a \in h^{-1}(x)$ and $b \in h^{-1}(y)$ with $ab \in E(G)$. Now $f(a) = g(h(a)) = g(x)$; similarly, $f(b) = g(y)$. As f is a homomorphism, we have that $f(a)f(b) \in E(H)$, and hence, $g(x)g(y) \in E(H)$. ■

Cliques and odd cycles are great cores, and hence, are good by Theorem 3 (2). If G and H are graphs, recall that their *join*, written $G + H$, is the graph formed by adding edges between each vertex of G and H . If G is a great core, then an analysis of cases demonstrates that $G + K_n$ is a great core for each $n \geq 1$. In particular, the odd wheels W_{2n+1} for $n \geq 2$ are great cores. For a large class of great cores, we consider a recent construction of [3]. If G is a graph, then define $C(G)$ to be G with edge replaced by a path with 3 edges. As proven in [3], if G is connected with at least three vertices, then G is a core if and only if $C(G)$ is. It is not hard to see that for all graphs G , $C(G)$ is great.

We note that not all good cores are great. Direct checking (which is tedious, and so omitted) demonstrates that the Petersen graph is a good but not great core. Hence, the converse of Theorem 3 (2) is false.

References

- [1] A. Bonato, *A family of universal pseudo-homogeneous G -colourable graphs*, Discrete Math. **247** (2002) 13–23.
- [2] A. Bonato, *Homomorphisms and amalgamation*, Discrete Math. **270** (2003) 32–41.
- [3] K. Collins and B. Shemmer, *Constructions of 3-colorable cores*, SIAM J. Discrete Math. **16** (2002) 74–80.
- [4] D. Duffus, B. Sands and R.E. Woodrow, *On the chromatic number of the product of graphs*, J. Graph Theory **9** (1985) 487–495.
- [5] F. Harary, S.T. Hedetniemi and R.W. Robinson, *Uniquely colorable graphs*, J. Combin. Theory **6** (1969) 264–270.
- [6] P. Hell and J. Nešetřil, *Graphs and Homomorphisms* (Oxford University Press, Oxford, 2004).
- [7] J. Kratochvíl and P. Mihók, *Hom-properties are uniquely factorizable into irreducible factors*, Discrete Math. **213** (2000) 189–194.
- [8] J. Kratochvíl, P. Mihók and G. Semanišin, *Graphs maximal with respect to hom-properties*, Discuss. Math. Graph Theory **17** (1997) 77–88.
- [9] X. Zhu, *Uniquely H -colorable graphs with large girth*, J. Graph Theory **23** (1996) 33–41.
- [10] X. Zhu, *Construction of uniquely H -colorable graphs*, J. Graph Theory **30** (1999) 1–6.

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