Discussiones Mathematicae
Graph Theory 20 (2000) 93–103

CLASSES OF HYPERGRAPHS WITH SUM NUMBER ONE

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Abstract

A hypergraph $\mathcal{H}$ is a *sum hypergraph* iff there are a finite $S \subseteq \mathbb{N}^+$ and $\underline{d}, \overline{d} \in \mathbb{N}^+$ with $1 < \underline{d} \leq \overline{d}$ such that $\mathcal{H}$ is isomorphic to the hypergraph $\mathcal{H}_{\underline{d},\overline{d}}(S) = (V, \mathcal{E})$ where $V = S$ and $\mathcal{E} = \{e \subseteq S : \underline{d} \leq |e| \leq \overline{d} \wedge \sum_{v \in e} v \in S\}$. For an arbitrary hypergraph $\mathcal{H}$ the sum number $\sigma = \sigma(\mathcal{H})$ is defined to be the minimum number of isolated vertices $w_1, \ldots, w_\sigma \notin V$ such that $\mathcal{H} \cup \{w_1, \ldots, w_\sigma\}$ is a sum hypergraph.

For graphs it is known that cycles C_n and wheels W_n have sum numbers greater than one. Generalizing these graphs we prove for the hypergraphs $\mathcal{C}_n$ and $\mathcal{W}_n$ that under a certain condition for the edge cardinalities $\sigma(\mathcal{C}_n) = \sigma(\mathcal{W}_n) = 1$ is fulfilled.

Keywords: hypergraphs, sum number, vertex labelling.

1991 Mathematics Subject Classification: 05C65, 05C78.

1 Introduction and Definitions

The notion of sum graphs was introduced by Harary [4]. This graph theoretic concept can be generalized to hypergraphs as follows.

All hypergraphs considered here are supposed to be nonempty and finite, without loops and multiple edges. In standard terminology we follow Berge [1]. By $\mathcal{H} = (V, \mathcal{E})$ we denote a hypergraph with vertex set V and edge set $\mathcal{E} \subseteq \mathcal{P}(V) \backslash \{\emptyset\}$. Further we use the notations $\underline{d} = \underline{d}(\mathcal{H}) = \min \{|e| : e \in \mathcal{E}\}$ and $\overline{d} = \overline{d}(\mathcal{H}) = \max \{|e| : e \in \mathcal{E}\}$; if $\underline{d} = \overline{d} = d$, we say $\mathcal{H}$ is a *d-uniform hypergraph*. A hypergraph is *linear* if no two edges intersect in more than one vertex.

Let $S \subseteq \mathbb{N}^+$ be finite and $\underline{d}, \overline{d} \in \mathbb{N}^+$ such that $1 < \underline{d} \leq \overline{d}$. Then $\mathcal{H}_{\underline{d},\overline{d}}(S) = (V, \mathcal{E})$ is called a $(\underline{d}, \overline{d})$-*sum hypergraph of* S iff $V = S$ and $\mathcal{E} = \{e \subseteq S : \underline{d} \leq |e| \leq \overline{d} \wedge \sum_{v \in e} v \in S\}$. Furthermore, a hypergraph $\mathcal{H}$ is a *sum hypergraph* iff there exist $S \subseteq \mathbb{N}^+$ and $\underline{d}, \overline{d} \in \mathbb{N}^+$ such that $\mathcal{H}$ is isomorphic to $\mathcal{H}_{\underline{d},\overline{d}}(S)$. For $\underline{d} = \overline{d} = 2$ we obtain the known concept for graphs. For an arbitrary hypergraph $\mathcal{H}$ the *sum number* $\sigma = \sigma(\mathcal{H})$ is defined to be the minimum number of isolated vertices $w_1, \ldots, w_\sigma \notin V$ such that $\mathcal{H} \cup \{w_1, \ldots, w_\sigma\}$ is a sum hypergraph.

Generalizing known results for classes of graphs, the sum number for the corresponding classes of hypergraphs was determined in the following three cases:

- Trees $T : \sigma(T) = 1$ (Ellingham [3]);
  hypertrees $\mathcal{T}$ with $\overline{d} < 2\underline{d} - 1 : \sigma(\mathcal{T}) = 1$ (Sonntag and Teichert [9]).
- Complete graphs K_n with $n \geq 4 : \sigma(K_n) = 2n - 3$ (Bergstrand et al. [2]); d-uniform complete hypergraphs $\mathcal{K}_n^d$ with $n \geq d+2 : \sigma(\mathcal{K}_n^d) = d(n-d)+1$ (Sonntag and Teichert [10]).
- Complete bipartite graphs K_{n_1,n_2} with $2 \leq n_1 \leq n_2 : \sigma(K_{n_1,n_2}) = \left\lceil \frac{1}{2}(3n_1 + n_2 - 3) \right\rceil$ (Hartsfield and Smyth [6]);
  d-partite complete hypergraphs $\mathcal{K}_{n_1,\ldots,n_d}^d$ with $2 \leq n_{d-1} \leq n_d$:
  $\sigma(\mathcal{K}_{n_1,\ldots,n_d}^d) = 1 + \sum_{i=1}^{d} (n_i - 1) + \min \left\{ 0, \left\lceil \frac{1}{2} \left(\sum_{i=1}^{d-1} (n_i - 1) - n_d \right) \right\rceil \right\}$
  (Teichert [11]).

In this paper, we determine the sum number for two other classes of hypergraphs. As a generalization of cycles C_n and wheels $W_n, n \geq 3$, we obtain the linear hypergraphs $\mathcal{C}_n$ and $\mathcal{W}_n$, respectively. These hypergraphs have the same number of edges as the corresponding graphs, but now each edge e_j consists of an arbitrary number $d_j \geq 2$ of vertices. Note that any two edges of $\mathcal{C}_n$ and $\mathcal{W}_n$, respectively, have only a vertex in common if this is explicitly claimed in the following definitions. In detail, a *hypercycle* $\mathcal{C}_n = (V_n, \mathcal{E}_n)$ is defined by

$$V_n = \bigcup_{i=1}^{n} \{v_1^i, \ldots, v_{d_i-1}^i\},$$

$$\mathcal{E}_n = \{e_1, \ldots, e_n\} \quad \text{with} \quad e_i = \{v_1^i, \ldots, v_{d_i}^i = v_1^{i+1}\} \tag{1}$$

$$\text{where } i+1 \text{ is taken mod } n.$$

A *hyperwheel* $\mathcal{W}_n = (V'_n, \mathcal{E}'_n)$ is defined by

$$V'_n = V_n \cup \{c\} \cup \bigcup_{i=1}^{n} \{v_2^{n+i}, \ldots, v_{d_{n+i}-1}^{n+i}\},$$

$$\mathcal{E}'_n = \mathcal{E}_n \cup \{e_{n+1}, \ldots, e_{2n}\} \tag{2}$$

$$\text{with} \quad e_{n+i} = \{v_1^{n+i} = c, v_2^{n+i}, \ldots, v_{d_{n+i}-1}^{n+i}, v_{d_{n+i}}^{n+i} = v_1^i\}.$$

The edges of $\mathcal{E}_n$ are called the *rim* of the hyperwheel, the vertex c is the *centre* and the edges $e_{n+1}, \ldots, e_{2n}$ are the *spokes* of $\mathcal{W}_n$. Obviously, for $d_i = 2, i = 1, \ldots, n$ and $i = 1, \ldots, 2n$ it follows $\mathcal{C}_n = C_n$ and $\mathcal{W}_n = W_n$, respectively.

The sum numbers of cycles (Harary [5]) and wheels (Hartsfield and Smyth [7], Miller et al. [8]) are known:

$$\sigma(C_n) = \begin{cases} 2, & \text{if} \quad n \neq 4, \\ 3, & \text{if} \quad n = 4. \end{cases} \qquad \sigma(W_n) = \begin{cases} \frac{n}{2} + 2, & \text{if} \quad n \text{ even} \quad \text{and} \quad n \geq 4, \\ n, & \text{if} \quad n \text{ odd} \quad \text{and} \quad n \geq 5. \end{cases}$$

Now observe that both, trees and hypertrees, have sum number one and that the sum number for complete graphs and complete bipartite graphs can be obtained from the more general formula for the corresponding hypergraphs by setting $d = 2$. In contrast to these observations, we show in the following that under a certain condition for the edge cardinalities $\sigma(\mathcal{C}_n) = \sigma(\mathcal{W}_n) = 1$ is fulfilled.

2 An Algorithm for Labelling Hypercycles and Hyperwheels

The algorithm given below is a modification of an algorithm for labelling the vertices of hypertrees (Sonntag, Teichert [9]). It starts with labelling the vertices of $e_1, \ldots, e_n$, therefore it can be used for labelling both, $\mathcal{C}_n$ and $\mathcal{W}_n$.

Let $\tilde{v} \notin V'_n$ be an isolated vertex, $\tilde{\mathcal{C}}_n = \mathcal{C}_n \cup \{\tilde{v}\}$ and $\tilde{\mathcal{W}}_n = \mathcal{W}_n \cup \{\tilde{v}\}$. We consider the vertex set $V(\tilde{\mathcal{H}}_n)$ of the hypergraph $\tilde{\mathcal{H}}_n \in \{\tilde{\mathcal{C}}_n, \tilde{\mathcal{W}}_n\}$ and construct a labelling $r : V(\tilde{\mathcal{H}}_n) \to \mathbb{N}^+$. This vertex labelling r induces the mapping r^*:

$$\mathcal{P}(V(\tilde{\mathcal{H}}_n)) \ni M \mapsto r^*(M) = \sum_{v \in M} r(v) \in \mathbb{N}^+.$$

In the following assume

$$\underline{d}(\tilde{\mathcal{H}}_n) \geq 3. \tag{3}$$

Before formulating the algorithm in detail we give a summary of the main steps: In the algorithm the vertices are labelled edge by edge. Whenever a vertex gets its label we add it to the set L of labelled vertices.

- We start with initialization (i.e., labelling the vertices of e_1 similarly as described in the next step).
- If $e_i, 1 \leq i \leq n-1$, is completly labelled but e_{i+1} not then $r(v_1^{i+1}) = r(v_{d_i}^i)$ is given, v_2^{i+1} is labelled by the sum of the labels of all vertices of e_i. The other nonlabelled vertices of e_{i+1} get labels greater than the sum of all vertices of L.
- The value $r^*(e_n)$ is assigned to $\tilde{v}$ (if $\tilde{\mathcal{H}}_n = \tilde{\mathcal{C}}_n$) or c (if $\tilde{\mathcal{H}}_n = \tilde{\mathcal{W}}_n$). In the first case we are done; in the other case we label the vertices $v_2^i, \ldots, v_{d_i-1}^i$ of the spokes $e_{n+1}, \ldots, e_{2n}$ similarly as described above and assign $r^*(e_{2n})$ to $\tilde{v}$.

  We use the notations introduced in (1) and (2) for $\mathcal{C}_n$ and $\mathcal{W}_n$, respectively.

ALGORITHM: Let $\tilde{\mathcal{H}}_n \in \{\tilde{\mathcal{C}}_n, \tilde{\mathcal{W}}_n\}$ be a hypergraph containing the isolated vertex $\tilde{v}$.

for $i = 1$ **to** n **do**
  if $i = 1$ **then** $[r(v_1^i) := 1;\ L := \{v_1^i\};\ \alpha := 10]$;
  $r(v_2^i) := \alpha;\ L := L \cup \{v_2^i\}$;
  if $i < n$ **then** $k := d_i$ **else** $k := d_i - 1$;
  for $j = 3$ **to** k **do**
    $r(v_j^i) := 10 \cdot r^*(L);\ L := L \cup \{v_j^i\}$;
  end for;
  $\alpha := r^*(e_i)$;
end for;
if $\tilde{\mathcal{H}}_n = \tilde{\mathcal{C}}_n$ **then goto** LAB;
for $i = 1$ **to** n **do**
  if $i = 1$ **then** $[r(c) := \alpha;\ L := L \cup \{c\};\ r(v_2^{n+i}) := n \cdot r^*(L)$;
    $L := L \cup \{v_2^{n+i}\}]$;
      else $[r(v_2^{n+i}) := \alpha;\ L := L \cup \{v_2^{n+i}\}]$;
  for $j = 3$ **to** $d_{n+i} - 1$ **do**
    $r(v_j^{n+i}) := 10 \cdot r^*(L);\ L := L \cup \{v_j^{n+i}\}$;
  end for;

$\alpha := r^*(e_{n+i})$;
end for;
LAB: $r(\tilde{v}) := \alpha$;
stop.

From the algorithm it follows immediately that $|e_n| \geq 3$ is needed and, furthermore, in case of $\tilde{\mathcal{H}}_n = \tilde{\mathcal{W}}_n$ the spokes $e_{n+1}, \ldots, e_{2n}$ must contain at least three vertices. We call all those vertices $\hat{v}$ of $\tilde{\mathcal{H}}_n$ with $r(\hat{v}) := \alpha$ in the algorithm *α-vertices* and the others *β-vertices*. Let A and B denote the set of α-vertices and β-vertices, respectively. Corresponding to the order of labelling we use the notation

$$\overline{v}_1 = v_1^1, \overline{v}_2 = v_2^1, \overline{v}_3, \overline{v}_4, \ldots, \overline{v}_q = \tilde{v}$$

$$\text{with} \quad q = \begin{cases} 1 + \sum\limits_{i=1}^{n} (d_i - 1), & \text{if} \quad \tilde{\mathcal{H}}_n = \tilde{\mathcal{C}}_n, \\ (n+2) + \sum\limits_{i=1}^{2n} (d_i - 2), & \text{if} \quad \tilde{\mathcal{H}}_n = \tilde{\mathcal{W}}_n. \end{cases} \tag{4}$$

Concluding this section we give some useful properties of the labelling r needed in the following.

Lemma 1. *Assume that the vertices of $\tilde{\mathcal{H}}_n$ are labelled by the algorithm and consider the sequence $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_q$. Then*

$$\forall\, \overline{v}_i \in B \backslash \{\overline{v}_1\} : r(\overline{v}_i) > r^*(\{\overline{v}_1, \ldots, \overline{v}_{i-1}\}), \tag{5}$$

$$\forall\, \overline{v}_i \in \{c\} \cup V_n \backslash \{\overline{v}_1, \overline{v}_2\} : r(\overline{v}_i) > r(\overline{v}_{i-1}) + r(\overline{v}_{i-2}). \tag{6}$$

Proof. Condition (5) and condition (6) for β-vertices $\overline{v}_i$ follow from the algorithm immediately. From (3) it follows that (6) is also true for α-vertices $\overline{v}_i$. ∎

Lemma 2. *Let $M \subseteq V_n'$, $t \in \{1, \ldots, n\}$ and a vertex subset $M' \subseteq \bigcup\limits_{i=1}^{t-1} e_i$ be arbitrarily chosen. Then*

$$r^*(M) = r(v_1^t) + r(v_2^t) + r^*(M') \Rightarrow \{v_1^t, v_2^t\} \subseteq M.$$

Proof. Suppose $v_2^t \notin M$. Then $t \geq 2$ and we have to substitute $r(v_2^t) = r^*(e_{t-1})$ in the equation by using labels of other vertices. By Lemma 1, (5) this yields $B_{t-1} := \{v_3^{t-1}, \ldots, v_{d_{t-1}}^{t-1}\} \subseteq M$, hence

$$r^*(M \backslash B_{t-1}) = r(v_1^t) + r(v_1^{t-1}) + r(v_2^{t-1}) + r^*(M').$$

Now observe that $v_1^t = v_{d_{t-1}}^{t-1} \in B_{t-1}$; furthermore, the condition (3) implies that v_1^t is a β-vertex. By the first observation $r(v_1^t)$ has to be substituted by using labels of other vertices and the second observation implies (by Lemma 1, (5)) that this is not possible. Hence $v_2^t \in M$ is necessary. Now the equation in Lemma 2 can be written as

$$r^*(M\backslash\{v_2^t\}) = r(v_1^t) + r^*(M')$$

and it follows immediately by Lemma 1, (5) that also the β-vertex v_1^t must be contained in M. ∎

3 The Sum Number of Hypercycles and Hyperwheels

In the following we use the labelling r and determine $\sigma(\mathcal{C}_n)$ and $\sigma(\mathcal{W}_n)$. Observe that in every sum hypergraph the vertex with highest label is an isolated vertex.

Theorem 3. *Let $\mathcal{C}_n$ be a hypercycle with $\underline{d}(\mathcal{C}_n) \geq 3$. Then*

$$\sigma(\mathcal{C}_n) = 1. \tag{7}$$

Proof. We denote the vertices and edges of $\mathcal{C}_n = (V_n, \mathcal{E}_n)$ according to (1), take an isolated vertex $\tilde{v} \notin V_n$ and assume that the vertices of $V_n \cup \{\tilde{v}\}$ are labelled by the algorithm given in the previous section. For convenience we denote by

$$\begin{aligned} B_t &= e_t\backslash\{v_1^t, v_2^t\}, \quad \text{if} \quad 1 \leq t \leq n-1, \\ B_n &= e_n\backslash\{v_1^n, v_2^n, v_{d_n}^n = v_1^1\} \end{aligned} \tag{8}$$

subsets of β-vertices. Thus — considering $B_1, \ldots, B_n$ as vertex sequences — (4) can be written into

$$v_1^1, v_2^1, B_1, v_2^2, B_2, v_2^3, B_3, \ldots, v_2^n, B_n, v_2^{n+1} := \tilde{v},$$

where v_2^t is the t-th α-vertex for $t = 1, \ldots, n+1$. Using $r(v_{d_n}^n) = r(v_1^1) = 1$ this yields

$$\begin{aligned} &r(v_2^{t+1}) = r(v_1^t) + r(v_2^t) + r^*(B_t) + x \\ &\text{with} \quad x = \begin{cases} 0, & \text{if} \quad 1 \leq t \leq n-1, \\ 1, & \text{if} \quad t = n. \end{cases} \end{aligned} \tag{9}$$

Let $\mathcal{E}_{sum}$ denote the set of edges generated by the labelling r and the sum hypergraph property. Then the algorithm yields $\mathcal{E}_n \subseteq \mathcal{E}_{sum}$; we prove (7) by showing that even equality is fulfilled, i.e.,

$$\begin{gathered} \forall\ \overline{v}_i \in V_n \cup \{\tilde{v}\} \quad \forall\ M \subseteq \{\overline{v}_1, \ldots, \overline{v}_{i-1}\} : \\ r(\overline{v}_i) = r^*(M) \quad \wedge \quad \underline{d}(\mathcal{C}_n) \geq 3 \quad \Rightarrow \quad M \in \mathcal{E}_n. \end{gathered} \tag{10}$$

By Lemma 1, (5) the condition $r(\overline{v}_i) = r^*(M)$ can only be true if $\overline{v}_i$ is an α-vertex v_2^{t+1} with $1 \leq t \leq n$. Hence by (9)

$$r^*(M) = r(v_2^{t+1}) = r(v_1^t) + r(v_2^t) + r^*(B_t) + x.$$

Using (8) and Lemma 1, (5) we obtain $B_t \subseteq M$. Thus

$$r^*(M \backslash B_t) = r(v_1^t) + r(v_2^t) + x. \tag{11}$$

By Lemma 2 it follows that $\{v_1^t, v_2^t\} \subseteq M$ and (11) can be written into

$$r^*(M \backslash B_t \backslash \{v_1^t, v_2^t\}) = x. \tag{12}$$

In case of $1 \leq t \leq n-1$ (i.e., $x = 0$) this yields $M = B_t \cup \{v_1^t, v_2^t\} = e_t$. For $t = n$ (i.e., $x = 1$) the only possibility is $M = B_n \cup \{v_1^n, v_2^n, v_{d_n}^n\} = e_n$. ■

Remark. The labelling r from the algorithm cannot be used to show $\sigma(\mathcal{C}_n) = 1$ if there is an edge $e_{t-1}, t-1 \in \{2, \ldots, n-1\}$ with $|e_{t-1}| = 2$. (This is true because $v_1^t = v_2^{t-1}$ is an α-vertex in this case and therefore $r(v_2^{t+1}) = r^*(M) = r^*(e_t \backslash \{v_1^t\}) + r^*(e_{t-2})$, but $e_{t-2} \cup e_t \backslash \{v_1^t\} \notin \mathcal{E}_n$).

Now we turn to hyperwheels $\mathcal{W}_n$ and start with two properties needed for the proof of the main result.

Lemma 4. *Suppose that the hyperwheel* $\mathcal{W}_n = (V_n', \mathcal{E}_n')$ *with* $\underline{d}(\mathcal{W}_n) \geq 3$ *is denoted according to* (2). *Further let* $M \subseteq V_n'$ *with* $|M| \geq 2$. *If* $t \in \{1, \ldots, n\}$, $r^*(M) = r(v_2^{n+t+1})$ *and* $v_2^{2n+1} := \tilde{v}$, *then*

(i)

$$\mu := |M \cap \{v_2^{n+1}, \ldots, v_2^{n+t}\}| = 1. \tag{13}$$

(ii) *If* $j \in \{1, \ldots, t\}$ *with* $v_2^{n+j} \in M$, *then*

$$M \cap \bigcup_{i=1}^{j-1} (e_{n+i} \backslash \{v_1^i, c\}) = \emptyset. \tag{14}$$

Proof. (i) We use the notation (4) and consider the maximum index k in that sequence such that $\overline{v}_k$ is an α-vertex belonging to M:

$$k = \max\{j \in \{1, \ldots, q\} : \overline{v}_j \in A \wedge \overline{v}_j \in M\}.$$

Then the algorithm yields that each β-vertex $\overline{v}_m$ with $m > k$ belongs to M.

If $M \cap \{v_2^{n+2}, \ldots, v_2^{n+t}\} = \emptyset$, it follows that v_2^{n+1} is such a β-vertex $\overline{v}_m$. Hence $v_2^{n+1} \in M$, i.e., $\mu = 1$ in this case.

If $M \cap \{v_2^{n+2}, \ldots, v_2^{n+t}\} \neq \emptyset$, suppose that $j \in \{2, \ldots, t\}$ is the maximum value such that $v_2^{n+j} \in M$. Then we obtain by Lemma 1, (5)

$$M_j = \left(\bigcup_{i=j}^{t} (e_{n+i} \setminus \{v_1^{n+i} = c, v_2^{n+i}, v_{d_{n+i}}^{n+i} = v_1^i\}) \right) \subseteq M. \tag{15}$$

Note that $M_j = \emptyset$ is possible. Now assume $\mu \geq 2$, i.e., there are $k, j \in \{1, \ldots, t\}, k < j$ such that $\{v_2^{n+k}, v_2^{n+j}\} \subseteq M$. With (15) and $r^*(M) = r(v_2^{n+t+1})$ follows

$$r^*(M \setminus M_j \setminus \{v_2^{n+j}\}) = (t - j + 1) \cdot r(c) + r^* \left(\bigcup_{i=j}^{t} \{v_1^i\} \right). \tag{16}$$

Using $j \geq 2, t \leq n$ and $r(v_2^{n+1}) = n \cdot r^*(V_n \cup \{c\})$ (cf., the algorithm) we obtain

$$\begin{aligned} r^*(M \setminus M_j \setminus \{v_2^{n+j}\}) &\leq (n-1) \cdot r(c) + r^* \left(\bigcup_{i=2}^{n} \{v_1^i\} \right) \\ &< r(v_2^{n+1}) \leq r(v_2^{n+k}). \end{aligned} \tag{17}$$

This is a contradiction to the assumption $v_2^{n+k} \in M$ and therefore (13) is true.

(ii) Property (14) follows immediately from (17) because v_2^{n+1} is the vertex with minimum label in $\bigcup_{i=1}^{j-1} (e_{n+i} \setminus \{v_1^i, c\})$. ∎

Theorem 5. *Let $\mathcal{W}_n$ be a hyperwheel with $\underline{d}(\mathcal{W}_n) \geq 3$ and $\overline{d}(\mathcal{W}_n) < 2\underline{d}(\mathcal{W}_n) - 1$. Then*

$$\sigma(\mathcal{W}_n) = 1. \tag{18}$$

Proof. 1. We denote the vertices and edges of $\mathcal{W}_n = (V'_n, \mathcal{E}'_n)$ according to (2), take an isolated vertex $\tilde{v} \notin V'_n$ and assume that the vertices of $V'_n \cup \{\tilde{v}\}$ are labelled by the algorithm given in the previous section. Further we denote with

$$(19) \qquad B_{n+t} = e_{n+t} \backslash \{v_1^{n+t} = c, v_2^{n+t}, v_{d_{n+t}}^{n+t} = v_1^t\}, \quad 1 \le t \le n$$

subsets of β-vertices. Using (8) and (19) — again considering $B_1, \ldots, B_{2n}$ as vertex sequences — we can write (4) as

$$v_1^1, v_2^1, B_1, v_2^2, B_2, \ldots, v_2^n, B_n, v_1^{n+1} = c, v_2^{n+1}, B_{n+1},$$
$$v_2^{n+2}, B_{n+2}, \ldots, v_2^{2n}, B_{2n}, v_2^{2n+1} := \tilde{v},$$

where v_2^s (for $s \neq n+1$) and v_1^s (for $s = n+1$) is the *s-th* α-vertex for $s = 1, \ldots, 2n+1$.

2. Let $\mathcal{E}'_{sum}$ denote the set of edges generated by the labelling r and the sum hypergraph property. Obviously, the algorithm yields $\mathcal{E}'_n \subseteq \mathcal{E}'_{sum}$; we prove (18) by showing that even equality is given, i.e.,

$$(20) \quad \begin{array}{c} \forall\ \overline{v}_i \in V'_n \cup \{\tilde{v}\} \quad \forall\ M \subseteq \{\overline{v}_1, \ldots, \overline{v}_{i-1}\} : \\ r(\overline{v}_i) = r^*(M) \ \wedge\ \underline{d}(\mathcal{W}_n) \ge 3 \ \wedge\ |M| \le \overline{d} < 2\underline{d} - 1 \ \Rightarrow\ M \in \mathcal{E}'_n. \end{array}$$

In the proof of Theorem 3 the validity of (20) is shown for all vertices $\overline{v}_i$ of the rim and the centre (because $r(c) = r^*(e_n)$). Hence it suffices to consider only vertices of $V'_n \backslash V_n \backslash \{c\}$ in the following. Because v_2^{n+1} is a β-vertex the condition $r(\overline{v}_i) = r^*(M)$ in (20) implies that $\overline{v}_i$ is an α-vertex v_2^{n+t+1} with $1 \le t \le n$; hence

$$r^*(M) = r(v_2^{n+t+1}) = r^*(e_{n+t}) = r(c) + r(v_2^{n+t}) + r^*(B_{n+t}) + r(v_1^t).$$

$B_{n+t} \subseteq M$ follows from Lemma 1 (5) and we obtain

$$(21) \qquad r^*(M \backslash B_{n+t}) = r(c) + r(v_2^{n+t}) + r(v_1^t).$$

It remains to show that (21) together with the condition $|M| \le \overline{d} < 2\underline{d} - 1$ from (20) implies $M = e_{n+t}$.

3. We show that $v_2^{n+t} \in M$ is necessary. By Lemma 4, (13) M contains exactly one vertex $v_2^{n+j} \in \{v_2^{n+t}, \ldots, v_2^{n+1}\}$; now assume $j < t$. Then $t \ge 2$ and Lemma 1, (5) yields $B_{n+t} \cup B_{n+t-1} \subseteq M$. We obtain with (19)

(22) $\quad |M \backslash B_{n+t} \backslash B_{n+t-1} \backslash \{v_2^{n+j}\}| < (2\underline{d}-1) - 2(\underline{d}-3) - 1 = 4.$

On the other hand, by (21) and $r(v_2^{n+t}) = r^*(e_{n+t-1})$ it follows that

$$r^*(M \backslash B_{n+t} \backslash B_{n+t-1} \backslash \{v_2^{n+j}\}) > 2r(c).$$

Due to Lemma 4, (14) only (pairwise distinct) vertices of $V_n \cup \{c\}$ are available to provide this value. By Lemma 1, (6) we need at least three such vertices to substitute $r(c)$ once, i.e., (22) cannot be true. Hence $v_2^{n+t} \in M$ and (21) can be written into

(23) $$r^*(M \backslash B_{n+t} \backslash \{v_2^{n+t}\}) = r(c) + r(v_1^t).$$

4. Next we show that $c \in M$ must be fulfilled. Assuming the contrary it follows with $r(c) = r^*(e_n)$ that $B_n \subseteq M$. Hence by (8)

$$r^*(M \backslash B_{n+t} \backslash \{v_2^{n+t}\} \backslash B_n) = r(v_1^n) + r(v_2^n) + r(v_{d_n}^n) + r(v_1^t).$$

Because of $v_{d_n}^n = v_1^1$ Lemma 2 yields $\{v_1^n, v_2^n\} \subseteq M$, i.e.,

$$r^*(M \backslash B_{n+t} \backslash B_n \backslash \{v_2^{n+t}, v_1^n, v_2^n\}) = r(v_1^t) + 1.$$

Now observe that there is no vertex $\hat{v}$ with $r(\hat{v}) = r(v_1^t) + 1$ but this is a contradiction to

$$|M \backslash B_{n+t} \backslash B_n \backslash \{v_2^{n+t}, v_1^n, v_2^n\}| < (2\underline{d}-1) - 2(\underline{d}-3) - 3 = 2.$$

Thus $c \in M$ and (23) can be written as

$$r^*(M \backslash B_{n+t} \backslash \{v_2^{n+t}, c\}) = r(v_1^t).$$

Because v_1^t is a β-vertex we obtain by Lemma 1, (5) that $r(v_1^t)$ cannot be replaced by a sum of other labels, i.e., $v_1^t \in M$. Summarizing the results we have $M = B_{n+t} \cup \{v_2^{n+t}, c, v_1^t\} = e_{n+t}$ and the proof is completed. ■

Remark. The labelling r from the algorithm can only be used to show $\sigma(\mathcal{W}_n) = 1$ if the condition $\overline{d}(\mathcal{W}_n) < 2\underline{d}(\mathcal{W}_n) - 1$ is true. (Otherwise observe that for some $t \in \{2, \ldots, n-1\}$ holds $r(v_2^{n+t+1}) = r^*(e_{n+t}) = r^*(e_{n+t} \backslash \{c\}) + r^*(e_n)$; but $e_n \cup e_{n+t} \backslash \{c\} \notin \mathcal{E}'_n$ and $|e_n \cup e_{n+t} \backslash \{c\}| \geq 2\underline{d} - 1$).

Acknowledgement

The author wishes to thank the referee for his valuable suggestions.

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Received 4 May 1999
Revised 28 September 1999