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2-DISTANCE 4-COLORABILITY OF PLANAR SUBCUBIC GRAPHS WITH GIRTH AT LEAST 22

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Abstract

The trivial lower bound for the 2-distance chromatic number $\chi_2(G)$ of any graph G with maximum degree Δ is $\Delta+1$. It is known that $\chi_2 = \Delta+1$ if the girth g of G is at least 7 and Δ is large enough. There are graphs with arbitrarily large Δ and $g \leq 6$ having $\chi_2(G) \geq \Delta+2$. We prove the 2-distance 4-colorability of planar subcubic graphs with $g \geq 22$.

Keywords: planar graph, subcubic graph, 2-distance coloring.

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1. Introduction

By a graph we mean a non-oriented graph without loops and multiple edges. By $V(G)$, $E(G)$, $\Delta(G)$, and $g(G)$ denote the sets of vertices and edges, maximum degree, and girth of a graph G, respectively. (We will drop the argument when the graph is clear from context.)

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Definition. A coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$ of G is 2-*distance* if any two vertices at distance at most two from each other get different colors. The minimum number of colors in 2-distance colorings of G is its 2-*distance chromatic number*, denoted by $\chi_2(G)$.

In 1977, Wegner [21] (see also Jensen and Toft's monograph [17]) posed the following

Conjecture 1. *Each planar graph has:*

$$\chi_2(G) \leq \begin{cases} 7 & if \quad \Delta = 3, \\ \Delta + 5 & if \quad 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1 & \quad otherwise. \end{cases}$$

The following upper bounds have been established: $\lfloor \frac{9\Delta}{5} \rfloor + 2$ for $\Delta \geq 749$ by Agnarsson and Halldorsson [1, 2] and $\lceil \frac{9\Delta}{5} \rceil + 1$ for $\Delta \geq 47$ by Borodin, Broersma, Glebov, and van den Heuvel [3, 4]. The best known upper bounds for large Δ are due to Molloy and Salavatipour [18, 19]: $\lceil \frac{5\Delta}{3} \rceil + 78$ for all Δ and $\lceil \frac{5\Delta}{3} \rceil + 25$ for $\Delta \geq 241$.

In [5, 9] we give sufficient conditions (in terms of g and Δ) for the 2-distance chromatic number of a planar graph to equal the trivial lower bound $\Delta + 1$. In particular, we determine the least g such that $\chi_2 = \Delta + 1$ if Δ is large enough (depending on g) to be equal to seven.

Theorem 2. *If G is a planar graph, then $\chi_2 = \Delta + 1$ in each of the cases* (i–viii)*:*

(i) $\Delta = 3,\ g \geq 24,$

(ii) $\Delta = 4,\ g \geq 15,$

(iii) $\Delta = 5,\ g \geq 13,$

(iv) $\Delta = 6,\ g \geq 12,$

(v) $\Delta \geq 7,\ g \geq 11,$

(vi) $\Delta \geq 9,\ g = 10,$

(vii) $\Delta \geq 15,\ g \geq 8,$

(viii) $\Delta \geq 30,\ g = 7.$

There exist planar graphs with $g \leq 6$ such that $\chi_2 = \Delta + 2$ for arbitrarily large Δ.

Borodin, Ivanova, and Neustroeva [10, 11] proved that $\chi_2 = \Delta + 1$ whenever $\Delta \geq 31$ for planar graphs of girth six with the additional assumption that each edge is incident with a vertex of degree two.

Dvořák, Kràl, Nejedlỳ, and Škrekovski [12] proved that every planar graph with $\Delta \geq 8821$ and $g \geq 6$ has $\chi_2 \leq \Delta + 2$, and Borodin and Ivanova [6, 7] weakened the restriction on Δ to 18.

Ivanova [15] improved Theorem 2 for $\Delta \geq 5$ as follows.

Theorem 3. *If G is a planar graph, then $\chi_2(G) = \Delta + 1$ in each of the cases* (i–iv)*:*

(i) $\Delta \geq 16, \quad g = 7,$

(ii) $\Delta \geq 10, \quad 8 \leq g \leq 9,$

(iii) $\Delta \geq 6, \quad 10 \leq g \leq 11,$

(iv) $\Delta = 5, \quad g \geq 12.$

A lot of research is devoted to coloring graphs with $\Delta = 3$ (called *subcubic*). For such planar graphs Dvořák, Škrekovski, and Tancer [13] proved that $\chi_2 = 4$ if $g \geq 24$ (i.e., they independently obtained (i) in Theorem 2) and $\chi_2 \leq 5$ if $g \geq 14$. The second of these results was also obtained by Montassier and Raspaud [20], which was improved by Ivanova and Solov'eva [16] and Havet [14] to $g \geq 13$. Borodin and Ivanova [8] proved that $\chi_2 = 4$ if $g \geq 23$, and the purpose of the present paper is

Theorem 4. *Every planar subcubic graph with girth at least* 22 *is* 2*-distance* 4*-colorable.*

We would like to attract attention to the following problem.

Problem 5. Find the smallest k such that every planar subcubic graph with girth at least k is 2-distance 4-colorable.

In the proof of Theorem 4 we use a new trick (see Claim 8) that makes it possible to produce new reducible configurations from already known ones.

2. Proof of Theorem 4

Let G' be a counterexample to Theorem 4, i.e., with $\Delta(G') = \Delta = 3$, $g(G') \geq 22$, and $\chi_2(G') > 4$. Now let G be a graph with the fewest edges such that $\Delta(G) \leq \Delta$, $g(G) = g \geq g(G')$, and $\chi_2(G) > 4$. The set of graphs with these properties is non-empty, since at least G' has all of them. Our proof of Theorem 4 consists in showing that G does not exist, which contradicts the assumption that G' exists.

Without loss of generality, we can assume that G is 2-connected and thus has no pendant edges. Euler's formula $|V| - |E| + |F| = 2$ can be rewritten as $94(20|E| - 22|V|) + (2|E| - 22|F|) = -44$ where F is the set of faces of G.

Hence,

$$\sum_{v \in V} \Big(10d(v) - 22\Big) + \sum_{f \in F} (r(f) - 22) = -44, \tag{1}$$

where $d(v)$ is the degree of vertex v, and $r(f)$ is the size of face f. The *charge* $\mu(v)$ of every vertex v of G is defined to be $10d(v) - 22$, while the *charge* $\mu(f)$ of every face f of G, to be $r(f) - 22$. Since the charge of every face is nonnegative, (1) implies that

$$\sum_{v \in V} \Big(10d(v) - 22\Big) < 0. \tag{2}$$

Note that the charge of a 2-vertex is -2, while the charge of a 3-vertex is 8. We first describe some structural properties of G; then, based on these, we redistribute the charges, preserving their sum, so that all new charges $\mu^*(v)$ are non-negative (which will give a contradiction with (2)).

2.1. Coloring and structural properties of G

Definition. By a *k-path* we mean a path consisting of precisely k vertices of degree 2. A *$\geq t$-path* is any k-path with $k \geq t$. By (k, l, m) denote a vertex of degree 3 incident with a $\geq k$-path, an $\geq l$-path, and an $\geq m$-path.

Definition. A pair of vertices (k, l, m) and (m, n, p) joined by an m-path will be denoted by $(klm - mnp)$. By $(klm - mnp - prs)$ we denote a triple of vertices (k, l, m), (m, n, p), (p, r, s), where the (m, n, p)-vertex is joined by an m-path and a p-path to the (k, l, m)-vertex and the (p, r, s)-vertex, respectively.

We present the proofs of some already published simplest properties of our G since our proofs of the main new structural properties of G (see Lemmas 12, 13) are built on the proofs of these old facts rather than on these facts themselves.

In what follows, by $c(v)$ we denote the color of a vertex v in a partial 2-distance coloring c of G, and $A(v)$ is the set of colors that are admissible for an uncolored vertex v, i.e., they do not appear on the already colored 2-distance neighbors of v.

The next two claims deal with the following problem. Given a small graph H with a small list L of admissible colors on its vertices, is it possible to 2-distance color H according to L? In fact, H in Claims 1 and 2 is a path of length three or four, respectively.

Claim 6 [14]**.** *A path $x_1x_2x_3x_4$ of vertices having 2, 2, 3, and 2 admissible colors, respectively, is 2-distance colorable with admissible colors.*

Proof. If x_1 and x_4 have an admissible color in common, then after putting $c(x_1) = c(x_4)$ we can color x_2 and x_3 in this order. Now let x_1 and x_4 have no common color. Putting $c(x_3) \notin A(x_2)$ implies that $|A(x_1)| \geq 2$ or $|A(x_4)| \geq 2$. We first color that of vertices x_1 and x_4 now having just one admissible color. Then we color x_2, and finally color the yet uncolored vertex from $\{x_1, x_4\}$. $\square$

Claim 7 [8]**.** *A path $x_1 \dots x_5$ of vertices having 2, 2, 3, 3, and 2 admissible colors, respectively, is 2-distance colorable with admissible colors.*

Proof. If $A(x_2) \cap A(x_5) \neq \emptyset$, then after putting $c(x_2) = c(x_5)$ we can color x_1, x_3, and x_4 in this order. Now let $A(x_2) \cap A(x_5) = \emptyset$. Putting $c(x_3) \notin A(x_1)$ implies that $|A(x_2)| \geq 2$ or $|A(x_5)| \geq 2$. We first color that of vertices x_2 and x_5 now having just one admissible color, then we color x_4, then the yet uncolored vertex from $\{x_2, x_5\}$, and finally color x_1. □

The forthcoming Claim 8 is the main tool in the proof of Theorem 4. It is used in Lemmas 12 and 13 for proving that certain relatively small trees are reducible. Let us explain the idea behind Claim 8. Suppose, for simplicity, that a tree has a 3-vertex u adjacent to a vertex v and incident with two pendant 3-paths. After some manipulations with these 3-paths, vertex u starts to behave from the viewpoint of v exactly the same as if u were a 2-vertex rather than a 3-vertex. (In fact, u is colored, but precisely two restrictions on the choice of color for v arrive along the edge uv.) In other words, the k-path that contains v becomes a "virtual" $(k+1)$-path.

Claim 8. *Let a vertex u be incident with an m-path $uu_1u_2\dots u_m$, an n-path $uu'_1u'_2\dots u'_n$, where $m+n \geq 6$, $m \geq n \geq 1$, and with a path uvw, where $v \notin \{u_1, u'_1\}$. If the $m+n-1$ vertices $u_{m-1}, \dots, u_1$, u, $u'_1, \dots, u'_{n-1}$ have 2, 3, $4, \dots, 4$, 3, 2 admissible colors, respectively, then we can color them so that v gets restrictions only from u and u'_1, while w gets a restriction only from u.*

Proof. As proved in [9, 13] (see Lemma 10(a) below), we have $n \leq 5$. If $m+n \geq 7$, then we simply color $u_{m-1}, \dots, u_{6-n}$ in this order. So, we can assume that $m+n = 6$ and still $m \geq n \geq 1$ due to the symmetry. Thus $|A(u)| = 4$ if $n = 3$, $|A(u)| = 3$ if $n = 2$, and $|A(u)| = 2$ if $n = 1$. Put $c(u) = c(u_3)$. (This can be done for $1 \leq n \leq 2$, since any pair of colors from $\{1, 2, 3, 4\}$ has a common color with any triple from this set.) Observe that u_1 and u_2 can be colored in the last place in this order. So, we first color the yet uncolored vertices from $\{u'_2, u'_1, u_4\}$ as follows: if $n = 3$, then we color u'_2 and u'_1 in this order; if $n = 2$, then only u'_1 is colored; if $n = 1$, then we color only u'_1. □

Remark 9. One of the referees suggested to emphasize even more that in the proof of Claim 8 it is important that the vertices u_1 and u_2 get their colors at the very end of the coloring process; in particular, after v and w have been colored.

Lemma 10. *G has no*

(a) ≥ 6-*path* ([9, 13]),
(b) $(1, 4, 5)$-*vertex* ([9, 13]),
(c) $(2, 3, 4)$-*vertex* ([13]).

Proof. (a) Suppose we have a ≥ 6-path $u_0u_1\dots$, where u_0 is a 3-vertex. Take a 2-distance 4-coloring of $G - u_2$, which exists by the minimality of G, uncolor vertices u_3, u_4, and u_5, and then extend this coloring to u_2 and the uncolored vertices. Note that $|A(u_2)| = |A(u_5)| = 2$ and $|A(u_3)| = |A(u_4)| = 3$, so we can use Claim 6.

(b) Let a $(1,4,5)$-vertex u be incident with paths $uu_1u_2u_3u_4u_5$, $uu'_1u'_2u'_3u'_4$, and adjacent to a vertex $u''_1 \notin \{u_1, u'_1\}$. Take a 2-distance coloring of $G-u$ and uncolor u_1, u_2, u_3, u_4, u'_1, u'_2, and u'_3. Note that $|A(u_4)| = |A(u)| = |A(u'_3)| = 2$, $|A(u_3)| = |A(u_1)| = |A(u'_1)| = |A(u'_2)| = 3$, and $|A(u_2)| = 4$. Put $c(u_3) \notin A(u_4)$. Now u_2, u_4 can be colored in the last place in this order, while vertices u_1, u, u'_1, u'_2, u'_3 have 2, 2, 3, 3, 2 admissible colors, respectively, and so can be colored by Claim 7.

(c) Let a $(2,3,4)$-vertex u be incident with paths $uu_1u_2u_3u_4$, $uu'_1u'_2u'_3$, and $uu''_1u''_2$. Take a 2-distance coloring of $G-u$ and uncolor u_1, u_2, u_3, u'_1, u'_2, u''_1. Note that $|A(u_3)| = |A(u'_2)| = |A(u''_1)| = 2$, $|A(u)| = |A(u_2)| = |A(u'_1)| = 3$, and $|A(u_1)| = 4$. If $A(u''_1) \cap A(u'_2) \neq \emptyset$, then put $c(u'_2) = c(u''_1)$. Now u'_1, u, u_1, u_2, u_3 have 2, 2, 3, 3, 2 admissible colors, respectively, and so can be colored by Claim 7. If $A(u''_1) \cap A(u'_2) = \emptyset$, then put $c(u) = c(u_3)$ (this can be done, since any pair and triple of colors from $\{1,2,3,4\}$ have a common color). Now u_1 and u_2 can be colored in the last place in this order. Note that $|A(u'_1)| \geq 2$ and either $c(u) \notin A(u''_1)$ or $c(u) \notin A(u'_1)$. In the first case we color vertices u'_2, u'_1, and u''_1 in this order, in the second case the order is u''_1, u'_1, u'_2. ■

Lemma 11 [8]**.** *G has no $(3,3,3)$-vertex.*

Proof. Let a $(3,3,3)$-vertex u be incident with paths $uu_1u_2u_3$, $uu'_1u'_2u'_3$, and $uu''_1u''_2u''_3$. Take a 2-distance coloring of $G-u$ and uncolor u_1, u_2, u'_1, u'_2, u''_1, and u''_2. Suppose that $c(u'_3) = 1$, and let the set of admissible colors for $z \in U = \{u_2, u_1, u''_1, u''_2\}$ be $A(z)$. Put $c(u) = 1$. Now u'_1 and u'_2 can be colored in the last place in this order. There is only one case when we cannot extend this coloring to the remaining vertices: $A(u_2) = \{1, x\}$, $A(u_1) = \{1, x, y\}$, $A(u''_1) = \{1, y, z\}$, and $A(u''_2) = \{1, z\}$.

Indeed, if there is a $v \in U$ such that $1 \notin A(z)$, then v can be colored in the last place, so that it suffices to color $U - v$, which is easy. Therefore, we are done unless u_2 remains with just one admissible color, say x, after coloring u with 1. Similarly, u''_2 should be forcedly colored, say with y. Now u_1 and u''_1 cannot be colored only if they are left with just one (and the same) color, say z, as desired.

Thus, we see that the color 1 is not suitable for u with this list on U. Now we have two options.

(A) $x = z = 2$, $y = 3$. Put $c(u) = 4$, then color u'_2 and u'_1, and finally color u_1, u''_1, u_2, u''_2 in this order.

(B) $x = 2$, $z = 4$, $y = 3$. Put $c(u) = 3$. We can color u_2 and u''_2 in the last place, since each has just one restriction (from u_1 and u''_1, respectively). We first color u'_2 and u'_1. Vertices u_1 and u''_1 can be colored whatever color on u'_1, since after coloring u they have different pairs of admissible colors: $\{1,2\}$ and $\{1,4\}$, respectively. ■

In the proof of Lemma 12, we use the following notation for the vertex pairs defined at the beginning of Section 2.1. Let an (i,j,k)-vertex u and a (k,n,s)-vertex w be joined by a path $uv'_1 \dots v'_k w$, where $0 \le k \le 2$, and incident with paths $uu_1 \dots u_i$, $uu'_1 \dots u'_j$, $ww_1 \dots w_n$, and $ww'_1 \dots w'_s$, where $i \ge j$, $n \ge s$. By internal vertices of this configuration we mean all those listed above, except for u_i, u'_j, w_n, and w'_s (see Figure 1).

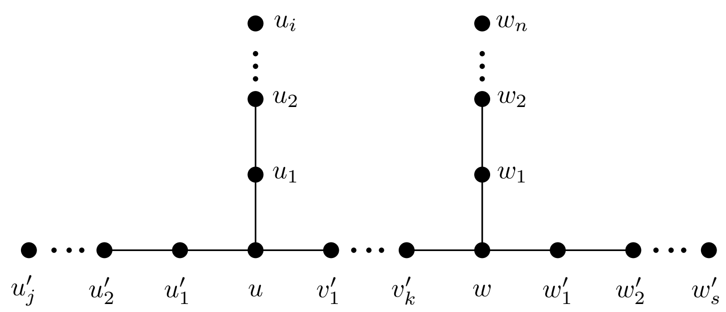

Figure 1. Notation in Lemma 12.

Lemma 12. *G has none of the following pairs of vertices:*

(a) $(332-224)$,
(b) $(422-224)$,
(c) $(331-134)$,
(d) $(421-134)$,
(e) $(512-224)$,
(f) $(420-045)$,
(g) $(330-045)$.

Proof. Let a (i,j,k)-vertex u and a (k,n,s)-vertex w form one of the pairs (a)–(g). Take a 2-distance coloring of $G-w$ and uncolor the internal vertices. Put $c(u)=c(u_3)$ and apply Claim 8 to u and the 2-vertices of its incident i- and j-paths. Now the numbers of admissible colors on vertices w, $v'_1,\dots,v'_k$, $w_1,\dots,w_{n-1}$, $w'_1,\dots,w'_{s-1}$ correspond to those on a $(2,3,4)$-vertex and 2-vertices of its incident paths in Cases (a)–(e), while Cases (f), (g) correspond to a $(1,4,5)$-vertex and its incident paths. Thus, our (k,n,s)-vertex w in a sense "transforms" into a $(k+1,n,s)$-vertex. Hence, we can color w, $v'_1,\dots,v'_k$, $w_1,\dots,w_{n-1}$, $w'_1,\dots,w'_{s-1}$ due to Lemma 10(b, c).

Finally, we color u_1 and u_2. ■

In the proof of Lemma 13, we use the following notation for the vertex triples defined at the beginning of Section 2.1. Let an (i,j,k)-vertex u and a (k,l,m)-vertex v be joined by a path $uv'_1 \dots v'_k v$, where $0 \le k \le 2$, and incident with paths $uu_1 \dots u_i$, $uu'_1 \dots u'_j$, $vv_1 \dots v_l$, and $vv'_{k+1} \dots v'_{k+m}$, where $i \ge j$, $l \ge m$. Furthermore, let v and an (m,n,s)-vertex w be joined by a path $vv'_{k+1} \dots v'_{k+m} w$, and let w be incident with paths $ww_1 \dots w_n$, $ww'_1 \dots w'_s$, where $n \ge s$. By internal vertices of this configuration we mean all those listed above, except for u_i, u'_j, v_l, w_n, w'_s (see Figure 2).

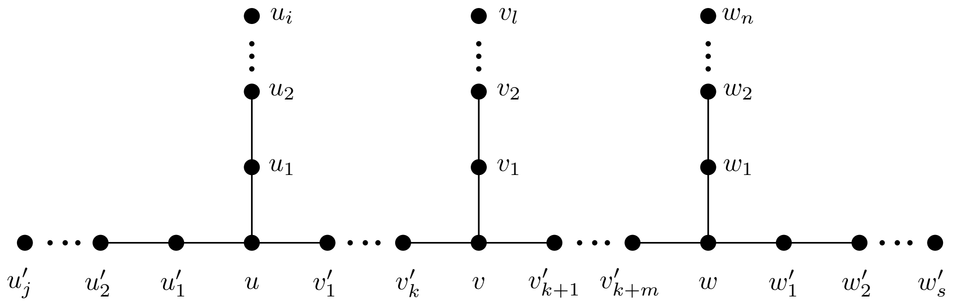

Figure 2. Notation in Lemma 13.

Lemma 13. *G has none of the following triples of vertices:*

(a) $(550-041-133)$,

(b) $(431-141-133)$,

(c) $(422-241-133)$,

(d) $(550-050-055)$.

Proof. (a)–(c) Let vertices u, v, and w form one of the triples (a)–(c). Take a 2-distance coloring of $G-w$ and uncolor the internal vertices. Put $c(w)=c(w_3)$ and apply Claim 8 to vertex w and the 2-vertices of its 3-paths. Now the numbers of admissible colors on vertices u, $u_1, \dots, u_{i-1}$, $u'_1, \dots, u'_{j-1}$, v, $v_1, \dots, v_{l-1}$, and $v'_{k+1}, \dots, v'_{k+m}$ correspond to those on the following pairs of vertices: $(550-042)$ in (a), $(431-142)$ in (b), and $(422-242)$ in (c). The first of these pairs can be colored by Lemma 12(f), the second pair by Lemma 12(d), and the third pair by Lemma 12(b). Finally, we color w_1 and w_2 in this order.

(d) Let u, v, and w form a triple $(550-050-055)$. Put $c(u'_3) \notin A(u'_4)$, $c(u_3) \notin A(u_4)$, $c(v_3) \notin A(v_4)$, $c(w_3) \notin A(w_4)$, and $c(w'_3) \notin A(w'_4)$. Note that vertices u'_2, u'_4, u_2, u_4, v_2, v_4, w_2, w_4, w'_2, and w'_4 can be colored in the last place. Since $|A(u_1)| = |A(v_1)| = |A(w_1)| = 3$, we can put $c(u_1)=c(v_1)=c(w_1)$. Now $|A(u'_1)| = |A(w'_1)| = 2$ and $|A(u)| = |A(v)| = |A(w)| = 3$, so vertices u'_1, w'_1, u, v, and w can be colored by Claim 7. ■

2.2. Completing the proof of Theorem 4

We use the following rules of discharging:

- **R1.** Each 2-vertex gets charge 1 from both ends of its incident k-path.
- **R2.** Each $(5,5,0)$-vertex gets charge 2 from the adjacent 3-vertex.
- **R3.** Each $(5,4,0)$-vertex gets charge 1 from the adjacent 3-vertex.
- **R4.** Each $(4,4,1)$-vertex and $(5,3,1)$-vertex gets charge 1 from the other end vertex of the incident 1-path.
- **R5.** Each $(5,2,2)$-vertex gets charge $\frac{1}{2}$ from the other end vertex of each incident 2-path.

Note that a 3-vertex gives charge k along each incident k-path by R1, and Rules R2–R5 are well defined, since no two receivers of charge are adjacent in G due to Lemmas 12(b, c, g).

We now check that $\mu^*(v) \geq 0$ for each $v \in V(G)$, which contradicts (2) and completes our proof.

If $d(v) = 2$, then $\mu^*(v) = -2 + 2 = 0$ by R1.

Let $d(v) = 3$. Recall that $\mu(v) = 8$. Note that after applying R1 the charge of each $(5,5,0)$-vertex becomes -2, each of the vertices $(5,4,0)$, $(5,3,1)$, $(4,4,1)$, and $(5,2,2)$ has -1, while the charges of all other vertices are nonnegative due to Lemmas 10 and 11.

Clearly, all above mentioned vertices have $\mu^*(v) = 0$ after applying R2–R5. It remains to check that the charges of all other vertices are still nonnegative.

If v is incident with two 0-paths, then $\mu^*(v) \geq 8 - 2 \times 2 - 4 = 8 - 2 - 1 - 5 = 0$ by R1–R3 and Lemma 13(d).

Suppose v is incident with one 0-path and a 1-path. Now v can give charge at most 2 to an adjacent 3-vertex by R2–R3. If v participates in R2, then $\mu^*(v) \geq 8 - 2 - 6 = 0$ due to Lemma 13(a) and R1, R4. But if v gives charge at most 1 along its 0-path, then the other two incident paths take away from v at most 7, whence $\mu^*(v) \geq 8 - 1 - 7 = 0$.
Let v be incident with a 0-path, a 2-path, and not incident with a 1-path; then $\mu^*(v) \geq 8 - 7\frac{1}{2} > 0$ by Lemma 12(f) and R1, R2, R3, and R5.

Now suppose that v is incident with two 1-paths, but not incident with a 0-path; then v either twice gives charge 1 by R4 and is incident with a ≤ 4-path, or it is incident with a 5-path and then participates in R4 at most once due to Lemma 13(b). This implies that $\mu^*(v) \geq 0$.

Let v be incident with a 1-path, a 2-path and not incident with a 0-path. Now v either participates both in R4 and in R5 and is incident with a ≤ 3-path, or it is incident with a 4-path and participates in at most one of R4 and R5 due to Lemma 13(c), or else it is incident with a 5-path and does not participate in R4, R5 by Lemma 12(d, e). Hence, $\mu^*(v) \geq 0$.

Suppose that v is incident with two 2-paths and not incident with a 0- or 1-path. Now if v is incident with a 4-path, then it does not participate in R5 due to 12(b), whence $\mu^*(v) \geq 8 - 2 - 2 - 4 = 0$ by R1. Otherwise, $\mu^*(v) \geq 8 - 2 - 2 - 3 - 2 \times \frac{1}{2} = 0$ by R1 and R5.

It remains to assume that v is incident with a 3-path. Now the other two paths take away from v at most 5 in total. Indeed, v can neither simultaneously participate in R2 and be incident with a $\geq$ 4-path, nor participate in R3 or in R2 and be incident with a 5-path due to Lemma 12(g). Similarly, v cannot give 1 by R4 and be incident with a 4- or 5-path due to Lemma 12(c). Finally, v cannot participate in R5 and be incident with another $\geq$ 3-path due to Lemma 12(a).

So, $\mu^*(v) \geq 0$ for each $v \in V$, which contradicts (2) and completes the proof of Theorem 4.

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