

IMPROVED UPPER BOUNDS FOR NEARLY ANTIPODAL CHROMATIC NUMBER OF PATHS*

YU-FA SHEN^a, GUO-PING ZHENG^a, WEN-JIE HE^b

^a*Department of Mathematics and Physics*
Hebei Normal University of Science and Technology
Qinhuangdao 066004, P.R. China

^b*Applied Mathematics Institute*
Hebei University of Technology
Tianjin 300130, P.R. China

e-mail: syf030514@163.com (Yu-Fa Shen).

Abstract

For paths P_n , G. Chartrand, L. Nebeský and P. Zhang showed that $ac'(P_n) \leq \binom{n-2}{2} + 2$ for every positive integer n , where $ac'(P_n)$ denotes the nearly antipodal chromatic number of P_n . In this paper we show that $ac'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 7$ if n is even positive integer and $n \geq 10$, and $ac'(P_n) \leq \binom{n-2}{2} - \frac{n-1}{2} - \lfloor \frac{13}{n} \rfloor + 8$ if n is odd positive integer and $n \geq 13$. For all even positive integers $n \geq 10$ and all odd positive integers $n \geq 13$, these results improve the upper bounds for nearly antipodal chromatic number of P_n .

Keywords: radio colorings, nearly antipodal chromatic number, paths.

2000 Mathematics Subject Classification: 05C12, 05C15, 05C78.

*This research was supported by the Doctoral Foundation of Hebei Normal University of Science and Technology, P.R. China, and the Science Foundation of the Education Department of Hebei Province, P.R. China (Grant No. 2005108).

1. Introduction

Radio k -colorings are generalizations of ordinary colorings of graphs, which were inspired by (FM Radio) Channel Assignments Problem (see [5, 7]) and introduced by G. Chartrand, D. Erwan, F. Harary and P. Zhang [1]. For a connected graph G of order n and diameter d and a integer k with $1 \leq k \leq d$, a radio k -coloring of G is a function $c: V(G) \rightarrow \mathbf{N}$, such that $d(u, v) + |c(u) - c(v)| \geq k + 1$ for every pair u and v of distinct vertices of G , where $d(u, v)$ denotes the distance between u and v (the length of a shortest $u - v$ path) in G . Clearly, radio 1-colorings and ordinary colorings are synonymous. The *value* $rc_k(c)$ of a radio k -coloring c of G is the maximum color assigned to a vertex of G ; while the *radio k -chromatic number* $rc_k(G)$ of G is $\min\{rc_k(c)\}$ taken over all k -coloring c of G . In particular, radio d -colorings are referred to as *radio labelings* and the *radio d -chromatic number* is called the *radio number*. Radio $(d - 1)$ -colorings are referred to as *radio antipodal coloring* or, more simply, as an *antipodal coloring*, and the *radio $(d - 1)$ -chromatic number* is called the *antipodal chromatic number*, denoted by $ac(G)$. Radio k -coloring and radio labeling of graphs were studied in [1, 2]. Radio antipodal coloring of paths were studied in [3, 4, 6].

Furthermore, G. Chartrand, L. Nebeský and P. Zhang gave the concepts of *nearly antipodal colorings* in [4]. For a connected graph G of diameter d , a nearly antipodal coloring of G is a function $c: V(G) \rightarrow \mathbf{N}$, such that $d(u, v) + |c(u) - c(v)| \geq d - 1$ for every two distinct vertices u and v of G . The *value* $ac'(c)$ of a nearly antipodal coloring c of G is the maximum color assigned to a vertex of G . The *nearly antipodal chromatic number* $ac'(G)$ of G is $\min\{ac'(c)\}$ taken over all nearly antipodal colorings of G (In fact, for $d \geq 3$, a nearly antipodal coloring is a radio $(d - 2)$ -coloring).

Clearly, if G is a connected graph of diameter 1 or 2, then $ac'(G) = 1$; while if $\text{diam}(G) = 3$, then $ac'(G)$ is the chromatic number of G . Thus nearly antipodal colorings are most interesting for connected graphs of diameter 4 or more. For this reason, the nearly antipodal chromatic number of paths P_n were investigated in [4] by G. Chartrand, L. Nebeský and P. Zhang. And they showed that $ac'(P_5) = 5$, $ac'(P_6) = 7$, $ac'(P_7) = 11$ and $ac'(P_8) = 16$. Moreover, they presented an upper bound for the nearly antipodal chromatic number of paths P_n for every positive integer n as follows.

Theorem 1.1 ([4]). *If n is a path of order $n \geq 1$, $ac'(P_n) \leq \binom{n-2}{2} + 2$.*

2. Our Results and the Idea of the Proof

In this paper we will provide an improved version for Theorem 1.1. We will show that

Theorem 2.1.

1. If P_n is even and $n \geq 10$, then $\text{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 7$;
2. If n is odd and $n \geq 13$, then $\text{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n-1}{2} - \lfloor \frac{13}{n} \rfloor + 8$.

Clearly, it holds that $-\frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 7 \leq 1$ for all even integers $n \geq 10$, and $-\frac{n-1}{2} - \lfloor \frac{13}{n} \rfloor + 8 \leq 1$ for all odd integers $n \geq 13$. Thus, for all even integers $n \geq 10$ and all odd integers $n \geq 13$, Theorem 2.1 improves the upper bounds of $\text{ac}'(P_n)$.

We will prove Theorem 2.1 in Section 3, and the proof will virtually provide a nearly antipodal coloring c for paths P_n with $\text{ac}'(c)$ that is equal to the bound presented in Theorem 2.1. The idea of performing the coloring c is based on pseudo greedy algorithm: Let $V(P_n) = \{p_1, p_2, \dots, p_n\}$. At first, we use the color $c_1 = 1$ to color some vertex $p_{n_1} \in \{p_1, p_2, \dots, p_n\}$, where p_{n_1} is the (a) *central vertex* of P_n . Suppose that for $1 \leq i \leq n-1$ the vertices in $\{p_{n_1}, p_{n_2}, \dots, p_{n_i}\} \subset \{p_1, p_2, \dots, p_n\}$ have been colored with $c(p_{n_j}) = c_j$ for all $1 \leq j \leq i$, then we choose a color $c_{i+1} \in \mathbf{N}$ as small as possible to color one vertex $p_{n_{i+1}} \in V(P_n) \setminus \{p_{n_1}, p_{n_2}, \dots, p_{n_i}\}$, such that $d(p_{n_{i+1}}, p_{n_j}) + |c(p_{n_{i+1}}) - c(p_{n_j})| \geq d - 1$ for all $1 \leq j \leq i$. And if there are two vertices can be chosen for $p_{n_{i+1}}$, then we take $p_{n_{i+1}}$ close to central vertices of P_n as near as possible. Finally, we obtain that $\text{ac}'(c) = c(p_{n_n})$ and hence $\text{ac}'(P_n) \leq \text{ac}'(c)$. In Section 4 we will give some examples which present the nearly antipodal coloring c for some paths P_n with $\text{ac}'(c)$ showed in Theorem 2.1 by our methods.

3. Proof of Theorem 2.1

Proof. 1. n is even and $n \geq 10$. Firstly, we let $n \geq 12$, note that $-\lfloor \frac{10}{n} \rfloor = 0$, it suffices to show that $\text{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} + 7$. Write $n = 2k = 10 + 2(4p + q)$, where $p \in \{0, 1, 2, \dots\}$ and $q \in \{1, 2, 3, 4\}$. Then we have that $k = 5 + (4p + q)$ and $d - 1 = \text{diam}(P_n) - 1 = 2k - 2$.

We denote the vertices of P_n by $x'_1, x'_2, x'_3; v'_1, v'_2, \dots, v'_{2p-1}, v'_{2p}; w_1, w_2, \dots, w_q; v_{2p}, v_{2p-1}, \dots, v_2, v_1; x_2, x_1; y_1, y_2; u_1, u_2, \dots, u_{2p-1}, u_{2p};$

$z_q, \dots, z_2, z_1; u'_{2p}, u'_{2p-1}, \dots, u'_2, u'_1; y'_3, y'_2, y'_1$ (see Figure 1). And we write

$$V_1 = \{x_1, x_2; y_1, y_2; x'_1, x'_2, x'_3; y'_1, y'_2, y'_3\},$$

$$V_2 = \{v_1, u_2, v_3, u_4, \dots, v_{2p-1}, u_{2p}; v'_1, v'_2, \dots, v'_{2p-1}, v'_{2p}; u'_1, u'_2, \dots, u'_{2p-1}, u'_{2p}\},$$

$$V_3 = \{w_1, w_2, \dots, w_q; z_1, z_2, \dots, z_q; v_{2p}, u_{2p-1}, \dots, v_4, u_3, v_2, u_1\}.$$

In the following we will present a coloring c for P_n by three steps, such that

$$(1) \quad d(u, v) + |c(u) - c(v)| \geq d - 1 = 2k - 2$$

holds for all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, and $ac'(c) = \binom{n-2}{2} - \frac{n}{2} + 7$ (note that $V_2 = \emptyset$ if $p = 0$, and it is easy to see that the following proof is also suitable for $V_2 = \emptyset$).

Step 1. Color the vertices in V_1 (see Figure 1).

Let

$$\begin{aligned} c(x_1) &= 1 \text{ (} x_1 \text{ is an central vertex of } P_n\text{);} \\ c(y'_1) &= c(x_1) + (k - 2) = k - 1, & c(x'_1) &= c(x_1) + (k - 1) = k; \\ c(y_1) &= c(x'_1) + (k - 2) = 2k - 2; \\ c(x'_2) &= c(y_1) + k - 1 = 3k - 3, & c(y'_2) &= c(x'_2) + 1 = 3k - 2; \\ c(x_2) &= c(x'_2) + (k + 1) = 4k - 2; \\ c(y'_3) &= c(x_2) + (k - 1) = 5k - 3, & c(x'_3) &= c(y'_3) + 3 = 5k; \\ c(y_2) &= c(x'_3) + (k - 1) = 6k - 1. \end{aligned}$$

Then by the definition of c and the value of $d(u, v)$ for $u, v \in V_1$, it is easy to verify that the following claim holds.

Claim 3.1. For all distinct vertices $u, v \in V_1$, the inequality (1) holds. At the same time, $\max_{v \in V_1} c(v) = c(y_2) = 6k - 1$ and $\max_{v \in V_1 \setminus \{y_2\}} c(v) = c(x'_3) = 5k$.

Step 2. Color the vertices in V_2 (see Figure 1).

For $i = 1, 2, \dots, p$, let

$$\begin{aligned}
c(v'_{2i-1}) &= c(y_2) + (2i-1)k + 3(2i-2) + 2[1+2+\dots+(2i-2)] \\
&\quad + (2i-2)(k-1), \\
c(u'_{2i-1}) &= c(y_2) + (2i-1)k + 3(2i-1) + 2[1+2+\dots+(2i-1)] \\
&\quad + (2i-2)(k-1); \\
c(v_{2i-1}) &= c(y_2) + (2i-1)k + 3(2i-1) + 2[1+2+\dots+(2i-1)] \\
&\quad + (2i-1)(k-1); \\
c(u'_{2i}) &= c(y_2) + (2i)k + 3(2i-1) + 2[1+2+\dots+(2i-1)] \\
&\quad + (2i-1)(k-1), \\
c(v'_{2i}) &= c(y_2) + (2i)k + 3(2i) + 2[1+2+\dots+(2i)] + (2i-1)(k-1); \\
c(u_{2i}) &= c(y_2) + (2i)k + 3(2i) + 2[1+2+\dots+(2i)] + (2i)(k-1).
\end{aligned}$$

Then we have the following claim.

Claim 3.2. For all distinct vertices $u, v \in V_1 \cup V_2$, the inequality (1) holds. At the same time, it holds that $\max_{v \in V_1 \cup V_2} c(v) = c(u_{2p}) = 6k - 1 + 2p(2k + 2p + 3)$ and $\max_{v \in (V_1 \cup V_2) \setminus \{u_{2p}\}} c(v) = c(v'_{2p}) = 5k + 2p(2k + 2p + 3)$.

In fact, note $d - 1 = 2k - 2$. Since that $d(y_2, v'_1) = k - 2$, $d(y_2, u'_1) = k - 5$, $d(v'_1, u'_1) = 2k - 7$, $c(v'_1) = c(y_2) + k$ and $c(u'_1) = c(y_2) + k + 5$, then for all distinct vertices $u, v \in \{y_2, v'_1, u'_1\}$, the inequality (1) holds. As $\max_{v \in V_1 \setminus \{y_2\}} c(v) = c(x'_3)$ by Claim 3.1, $c(v'_1) = c(y_2) + k = c(x'_3) + 2k - 1$ and $c(u'_1) > c(v'_1)$, we have that $c(v'_1) - c(x'_3) \geq d - 1$ and $c(u'_1) - c(x'_3) \geq d - 1$. Therefore for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1\}$, the inequality (1) holds.

Since that $d(u'_1, v_1) = k - 1$, $d(v_1, v'_1) = k - 6$, and $c(v_1) = c(u'_1) + (k - 1) = c(v'_1) + 5 + (k - 1)$, then for all distinct vertices $u, v \in \{v_1, v'_1, u'_1\}$, the inequality (1) holds. As $\max_{v \in V_1} c(v) = c(y_2)$ by Claim 3.1, and $c(v_1) = c(y_2) + k + 5 + (k - 1)$, we have that $c(v_1) - c(y_2) \geq d - 1$. Therefore for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\}$, the inequality (1) holds.

Note the fact that $d(v_1, u'_2) = k - 2$, $d(v_1, v'_2) = k - 5 - 2$, $d(u'_2, v'_2) = 2k - 7 - 2$, $c(u'_2) = c(v_1) + k$, $c(v'_2) = c(v_1) + k + 5 + 2$; and $d(v'_2, u_2) = k - 1$, $d(u_2, u'_2) = k - 6 - 2$, $c(u_2) = c(v'_2) + (k - 1) = c(u'_2) + 5 + 2 + (k - 1)$. Similar to the above discussion we can obtain that for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \cup \{u'_2, v'_2, u_2\}$, the inequality (1) holds.

Continue the above discussion we can conclude that for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \cup \{u'_2, v'_2, u_2\} \cup \dots \cup \{v'_{2p-1}, u'_{2p-1}, v_{2p-1}\} \cup \{u'_{2p}, v'_{2p}, u_{2p}\} = V_1 \cup V_2$, the inequality (1) holds.

By the definition of c , it is easy to verify that $\max_{v \in V_1 \cup V_2} c(v) = c(u_{2p}) = 6k - 1 + 2p(2k + 2p + 3)$ and $\max_{v \in (V_1 \cup V_2) \setminus \{u_{2p}\}} c(v) = c(v'_{2p}) = 5k + 2p(2k + 2p + 3)$.

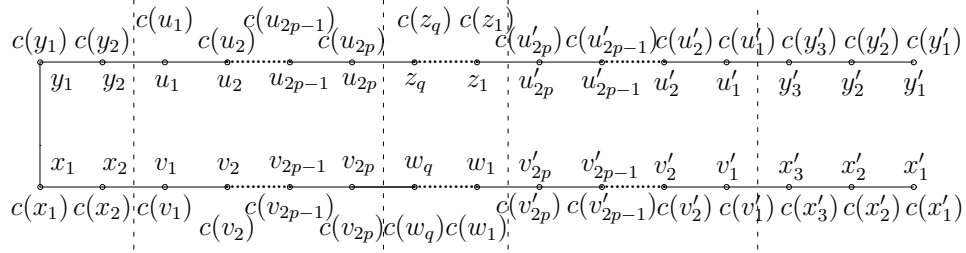


Figure 1: A nearly antipodal coloring for P_n ($n = 2k \geq 10$).

Step 3. Color the vertices in V_3 (see Figure 1).

Step 3.1. Color the vertices in $\{w_1, w_2, \dots, w_q; z_1, z_2, \dots, z_q\}$.

According the value of q , there are four cases.

Case 1. $q = 1$. Let

$$\begin{aligned} c(w_1) &= c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3), \\ c(z_1) &= c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5). \end{aligned}$$

Case 2. $q = 2$. Let

$$\begin{aligned} c(w_1) &= c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3), \\ c(z_1) &= c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5), \\ c(w_2) &= c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5), \\ c(z_2) &= c(w_2) + 3 + 2(2p + 2) = 8k + 10 + 2p(2k + 2p + 7). \end{aligned}$$

Case 3. $q = 3$. Let

$$\begin{aligned} c(w_1) &= c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3), \\ c(z_1) &= c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5), \\ c(w_3) &= c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5), \\ c(z_2) &= c(w_3) + k = 9k + 3 + 2p(2k + 2p + 5), \\ c(w_2) &= c(z_2) + 3 + 2(2p + 2) = 9k + 10 + 2p(2k + 2p + 7), \\ c(z_3) &= c(w_2) + k = 10k + 10 + 2p(2k + 2p + 7). \end{aligned}$$

Case 4. $q = 4$. Let

$$\begin{aligned}
 c(w_1) &= c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3), \\
 c(z_1) &= c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5), \\
 c(w_4) &= c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5), \\
 c(z_2) &= c(w_4) + k = 9k + 3 + 2p(2k + 2p + 5), \\
 c(w_2) &= c(z_2) + 3 + 2(2p + 2) = 9k + 10 + 2p(2k + 2p + 7), \\
 c(z_3) &= c(w_2) + (k - 1) = 10k + 9 + 2p(2k + 2p + 7), \\
 c(w_3) &= c(z_3) + 3 + 2(2p + 3) = 10k + 18 + 2p(2k + 2p + 9), \\
 c(z_4) &= c(w_3) + (k + 1) = 11k + 19 + 2p(2k + 2p + 9).
 \end{aligned}$$

Step 3.2. Color the vertices in $\{v_{2p}, u_{2p-1}, \dots, v_4, u_3, v_2, u_1\}$.

For any case above ($q = 1, 2, 3, 4$), we let

$$\begin{aligned}
 c(v_{2p}) &= c(z_q) + [(k + q) - 1], \\
 c(u_{2p-1}) &= c(v_{2p}) + [(k + q - 1) + 2], \\
 c(v_{2p-2}) &= c(u_{2p-1}) + [(k + q - 1) + 2 \cdot 2], \\
 c(u_{2p-3}) &= c(v_{2p-2}) + [(k + q - 1) + 2 \cdot 3], \\
 &\dots\dots\dots, \\
 c(v_2) &= c(u_3) + [(k + q - 1) + 2(2p - 2)], \\
 c(u_1) &= c(v_2) + [(k + q - 1) + 2(2p - 1)] \\
 &= c(z_q) + 2p(k + q - 1) + 2 \cdot \frac{2p(2p-1)}{2} \\
 &= c(z_q) + 2p(k + q + 2p - 2).
 \end{aligned}$$

Then by a similar method to prove Claim 3.2, we can obtain the following claim.

Claim 3.3. For all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, the inequality (1) holds. And $\max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 2)$.

By Claim 3.3, we have shown that for all even integers $n \geq 12$, c is a nearly antipodal coloring for P_n . Therefore $\text{ac}'(P_n) \leq \text{ac}'(c) = \max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 2)$. To finish the proof of Theorem 2.1 for all even integers $n \geq 12$, it suffices to prove the following claim.

Claim 3.4. For any $p \in \{0, 1, 2, \dots\}$ and any $q \in \{1, 2, 3, 4\}$, it holds that $c(u_1) = c(z_q) + 2p(k + q + 2p - 2) = \binom{n-2}{2} - \frac{n}{2} + 7$, where $n = 2k = 2(5 + 4p + q)$.

In fact, if $q = 1$, then $k = 4p + 6$, $2p = \frac{k-6}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_1) + 2p(k + q + 2p - 2) = 7k + 4 + 2p(2k + 2p + 5) \\ &\quad + 2p(k + 2p - 1) \\ &= 2k^2 - 6k + 10 = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

If $q = 2$, then $k = 4p + 7$, $2p = \frac{k-7}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_2) + 2p(k + q + 2p - 2) = 8k + 10 + 2p(2k + 2p + 7) \\ &\quad + 2p(k + 2p) \\ &= 8k + 10 + 2p(3k + 4p + 7) = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

If $q = 3$, then $k = 4p + 8$, $2p = \frac{k-8}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_3) + 2p(k + q + 2p - 2) = 10k + 10 + 2p(2k + 2p + 7) \\ &\quad + 2p(k + 2p + 1) \\ &= 10k + 10 + 2p(3k + 4p + 8) = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

If $q = 4$, then $k = 4p + 9$, $2p = \frac{k-9}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_4) + 2p(k + q + 2p - 2) = 11k + 19 + 2p(2k + 2p + 9) \\ &\quad + 2p(k + 2p + 2) \\ &= 11k + 19 + 2p(3k + 4p + 11) = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

Thus Claim 3.4 holds and hence $\text{ac}'(P_n) \leq \text{ac}'(c) = \binom{n-2}{2} - \frac{n}{2} + 7$ for all even integers $n \geq 12$.

Secondly, for $n = 10$, in the above proof we take $p = 0$ and $q = 0$. Namely, $V_2 = V_3 = \emptyset$, $V(P_{10}) = V_1 = \{x'_1, x'_2, x'_3; x_2, x_1; y_1, y_2; y'_3, y'_2, y'_1\}$ (also see Figure 1 and let $p = q = 0$). Then coloring $c|_{v \in V_1}(v)$ is a nearly antipodal coloring for P_{10} . Thus by Claim 3.1, $\text{ac}'(P_{10}) \leq \text{ac}'(c|_{v \in V_1}) = \max_{v \in V_1} c(v) = c(y_2) = (6k - 1)|_{k=5} = 29 = \binom{10-2}{2} + 1$. Since $-\lfloor \frac{10}{n} \rfloor = -1$ for $n = 10$, it follows that $\text{ac}'(P_{10}) \leq \text{ac}'(c|_{v \in V_1}) = \binom{10-2}{2} + 1 = \binom{10-2}{2} - \frac{10}{2} - \lfloor \frac{10}{10} \rfloor + 7$.

Thus we complete the proof of assertion 1 in Theorem 2.1.

2. n is odd and $n \geq 13$. Firstly, we let $n \geq 15$, note that $-\lfloor \frac{13}{n} \rfloor = 0$, it suffices to show that $\text{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} + 8$. Write $n = 2k + 1 = 13 + 2(4p + q)$, where $p \in \{0, 1, 2, \dots\}$ and $q \in \{1, 2, 3, 4\}$. Then we have that $k = 6 + (4p + q)$ and $d - 1 = \text{diam}(P_n) - 1 = 2k - 1$.

We denote the vertices of P_n by $x'_1, x'_2, x'_3, x'_4; v'_1, v'_2, \dots, v'_{2p-1}, v'_{2p}; w_1, w_2, \dots, w_q; v_{2p}, v_{2p-1}, \dots, v_2, v_1; x_2, x_1; x_0; y_1, y_2; u_1, u_2, \dots, u_{2p-1}, u_{2p}; z_q, \dots, z_2, z_1; u'_{2p}, u'_{2p-1}, \dots, u'_2, u'_1; y'_4, y'_3, y'_2, y'_1$ (see Figure 2). And we write

$$V_1 = \{x_0; x_1, x_2; y_1, y_2; x'_1, x'_2, x'_3, x'_4; y'_1, y'_2, y'_3, y'_4\},$$

$$V_2 = \{v_1, u_2, v_3, u_4, \dots, v_{2p-1}, u_{2p}; v'_1, v'_2, \dots, v'_{2p-1}, v'_{2p}; u'_1, u'_2, \dots, u'_{2p-1}, u'_{2p}\},$$

$$V_3 = \{w_1, w_2, \dots, w_q; z_1, z_2, \dots, z_q; v_{2p}, u_{2p-1}, \dots, v_4, u_3, v_2, u_1\}.$$

Similar to the method of proof assertion 1, we will present a coloring c for P_n by three steps, such that

$$(2) \quad d(u, v) + |c(u) - c(v)| \geq d - 1 = 2k - 1$$

holds for all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, and $\text{ac}'(c) = \binom{n-2}{2} - \frac{n}{2} + 8$ (note that $V_2 = \emptyset$ if $p = 0$, and it is easy to see that the following proof is also suitable for $V_2 = \emptyset$).

Step 1. Color the vertices in V_1 (see Figure 2).

Let

$$\begin{aligned} c(x_0) &= 1 \quad (x_0 \text{ is the central vertex of } P_n); \\ c(x'_1) &= c(x_0) + (k - 1) = k, & c(y'_1) &= c(x_0) + (k - 1) = k; \\ c(x_1) &= c(x'_1) + k = 2k; \\ c(y'_2) &= c(x_1) + (k - 1) = 3k - 1, & c(x'_2) &= c(x_1) + (k + 1) = 3k + 1; \\ c(y_1) &= c(y'_2) + (k + 1) = 4k; \\ c(x'_3) &= c(y_1) + k = 5k, & c(y'_3) &= c(y'_3) + 3 = 5k + 3; \\ c(x_2) &= c(x'_3) + (k + 3) = 6k + 3; \\ c(y'_4) &= c(x_2) + k = 7k + 3, & c(x'_4) &= c(y'_4) + 5 = 7k + 8; \\ c(y_2) &= c(x'_4) + k = 8k + 8. \end{aligned}$$

Then by the definition of c and the value of $d(u, v)$ for $u, v \in V_1$, it is easy to verify that the following claim holds.

Claim 3.5. For all distinct vertices $u, v \in V_1$, the inequality (2) holds. At the same time, $\max_{v \in V_1} c(v) = c(y_2) = 8k + 8$ and $\max_{v \in V_1 \setminus \{y_2\}} c(v) = c(x'_4) = 7k + 8$.

Step 2. Color the vertices in V_2 (see Figure 2).

For $i = 1, 2, \dots, p$, let

$$\begin{aligned}
 c(v'_{2i-1}) &= c(y_2) + (2i-1)(k+1) + 5(2i-2) + 2[1+2+\dots+(2i-2)] \\
 &\quad + (2i-2)k, \\
 c(u'_{2i-1}) &= c(y_2) + (2i-1)(k+1) + 5(2i-1) + 2[1+2+\dots+(2i-1)] \\
 &\quad + (2i-2)k; \\
 c(v_{2i-1}) &= c(y_2) + (2i-1)(k+1) + 5(2i-1) + 2[1+2+\dots+(2i-1)] \\
 &\quad + (2i-1)k; \\
 c(u'_{2i}) &= c(y_2) + (2i)(k+1) + 5(2i-1) + 2[1+2+\dots+(2i-1)] \\
 &\quad + (2i-1)k, \\
 c(v'_{2i}) &= c(y_2) + (2i)(k+1) + 5(2i) + 2[1+2+\dots+(2i)] + (2i-1)k; \\
 c(u_{2i}) &= c(y_2) + (2i)(k+1) + 5(2i) + 2[1+2+\dots+(2i)] + (2i)k.
 \end{aligned}$$

Then we have the following claim.

Claim 3.6. For all distinct vertices $u, v \in V_1 \cup V_2$, the inequality (2) holds. At the same time, it holds that $\max_{v \in V_1 \cup V_2} c(v) = c(u_{2p}) = 8k + 8 + 2p(2k + 2p + 7)$ and $\max_{v \in (V_1 \cup V_2) \setminus \{u_{2p}\}} c(v) = c(v'_{2p}) = 7k + 8 + 2p(2k + 2p + 7)$.

In fact, note $d-1 = 2k-1$. Since that $d(y_2, v'_1) = k-2$, $d(y_2, u'_1) = k-6$, $d(v'_1, u'_1) = 2k-8$, $c(v'_1) = c(y_2) + (k+1)$ and $c(u'_1) = c(y_2) + (k+1) + 7$, then for all distinct vertices $u, v \in \{y_2, v'_1, u'_1\}$, the inequality (2) holds. As $\max_{v \in V_1 \setminus \{y_2\}} c(v) = c(x'_4)$ by Claim 3.5, $c(v'_1) = c(y_2) + (k+1) = c(x'_4) + 2k + 1$ and $c(u'_1) > c(v'_1)$, we have that $c(v'_1) - c(x'_4) \geq d-1$ and $c(u'_1) - c(x'_4) \geq d-1$. Therefore for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1\}$, the inequality (2) holds.

Since that $d(u'_1, v_1) = k-1$, $d(v_1, v'_1) = k-7$, and $c(v_1) = c(u'_1) + k = c(v'_1) + 7 + k$, then for all distinct vertices $u, v \in \{v_1, v'_1, u'_1\}$, the inequality (2) holds. As $\max_{v \in V_1} c(v) = c(y_2)$ by Claim 3.5, and $c(v_1) = c(y_2) + (k+1) + 7 + k$, we have that $c(v_1) - c(y_2) \geq d-1$. Therefore for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\}$, the inequality (2) holds.

Note the fact that $d(v_1, u'_2) = k-2$, $d(v_1, v'_2) = k-6-2$, $d(u'_2, v'_2) = 2k-8-2$, $c(u'_2) = c(v_1) + (k+1)$, $c(v'_2) = c(v_1) + (k+1) + 7 + 2$; and

$d(v'_2, u_2) = k - 1$, $d(u_2, u'_2) = k - 7 - 2$, $c(u_2) = c(v'_2) + k = c(u'_2) + 7 + 2 + k$. Similar to the above discussion we can obtain that for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \cup \{u'_2, v'_2, u_2\}$, the inequality (2) holds.

Continue the above discussion we can conclude that for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \cup \{u'_2, v'_2, u_2\} \cup \dots \cup \{v'_{2p-1}, u'_{2p-1}, v_{2p-1}\} \cup \{u'_{2p}, v'_{2p}, u_{2p}\} = V_1 \cup V_2$, the inequality (2) holds.

By the definition of c , it is easy to see that $\max_{v \in V_1 \cup V_2} c(v) = c(u_{2p}) = 8k + 8 + 2p(2k + 2p + 7)$, and $\max_{v \in (V_1 \cup V_2) \setminus \{u_{2p}\}} c(v) = c(v'_{2p}) = 7k + 8 + 2p(2k + 2p + 7)$.

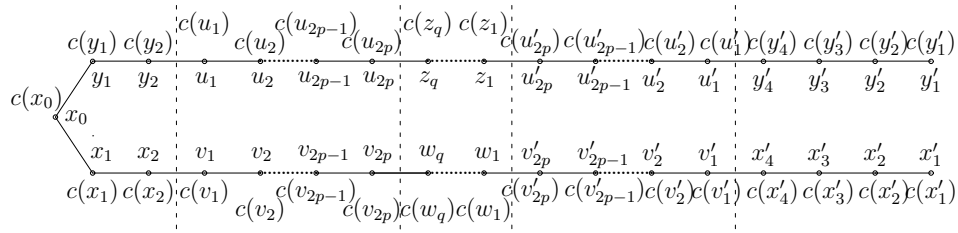


Figure 2. A nearly antipodal coloring for P_n ($n = 2k + 1 \geq 13$).

Step 3. Color the vertices in V_3 (see Figure 2).

Step 3.1. Color the vertices in $\{w_1, w_2, \dots, w_q; z_1, z_2, \dots, z_q\}$.

According the value of q , there are four cases.

Case 1. $q = 1$. Let

$$\begin{aligned} c(w_1) &= c(u_{2p}) + (k + 1) = 9k + 9 + 2p(2k + 2p + 7), \\ c(z_1) &= c(w_1) + 5 + 2(2p + 1) = 9k + 16 + 2p(2k + 2p + 9). \end{aligned}$$

Case 2. $q = 2$. Let

$$\begin{aligned} c(w_1) &= c(u_{2p}) + (k + 1) = 9k + 9 + 2p(2k + 2p + 7), \\ c(z_1) &= c(w_1) + 5 + 2(2p + 1) = 9k + 16 + 2p(2k + 2p + 9), \\ c(w_2) &= c(z_1) + k = 10k + 16 + 2p(2k + 2p + 9), \\ c(z_2) &= c(w_2) + 5 + 2(2p + 2) = 10k + 25 + 2p(2k + 2p + 11). \end{aligned}$$

Case 3. $q = 3$. Let

$$\begin{aligned} c(w_1) &= c(u_{2p}) + (k+1) = 9k + 9 + 2p(2k + 2p + 7), \\ c(z_1) &= c(w_1) + 5 + 2(2p+1) = 9k + 16 + 2p(2k + 2p + 9), \\ c(w_3) &= c(z_1) + k = 10k + 16 + 2p(2k + 2p + 9), \\ c(z_2) &= c(w_3) + (k+1) = 11k + 17 + 2p(2k + 2p + 9), \\ c(w_2) &= c(z_2) + 5 + 2(2p+2) = 11k + 26 + 2p(2k + 2p + 11), \\ c(z_3) &= c(w_2) + (k+1) = 12k + 27 + 2p(2k + 2p + 11). \end{aligned}$$

Case 4. $q = 4$. Let

$$\begin{aligned} c(w_1) &= c(u_{2p}) + (k+1) = 9k + 9 + 2p(2k + 2p + 7), \\ c(z_1) &= c(w_1) + 5 + 2(2p+1) = 9k + 16 + 2p(2k + 2p + 9), \\ c(w_4) &= c(z_1) + k = 10k + 16 + 2p(2k + 2p + 9), \\ c(z_2) &= c(w_4) + (k+1) = 11k + 17 + 2p(2k + 2p + 9), \\ c(w_2) &= c(z_2) + 5 + 2(2p+2) = 11k + 26 + 2p(2k + 2p + 11), \\ c(z_3) &= c(w_2) + k = 12k + 26 + 2p(2k + 2p + 11), \\ c(w_3) &= c(z_3) + 5 + 2(2p+3) = 12k + 37 + 2p(2k + 2p + 13), \\ c(z_4) &= c(w_3) + (k+2) = 13k + 39 + 2p(2k + 2p + 13). \end{aligned}$$

Step 3.2. Color the vertices in $\{v_{2p}, u_{2p-1}, \dots, v_4, u_3, v_2, u_1\}$.

For each case above ($q = 1, 2, 3, 4$), we let

$$\begin{aligned} c(v_{2p}) &= c(z_q) + (k+q), \\ c(u_{2p-1}) &= c(v_{2p}) + [(k+q) + 2], \\ c(v_{2p-2}) &= c(u_{2p-1}) + [(k+q) + 2 \cdot 2], \\ c(u_{2p-3}) &= c(v_{2p-2}) + [(k+q) + 2 \cdot 3], \\ &\dots\dots\dots, \\ c(v_2) &= c(u_3) + [(k+q) + 2(2p-2)], \\ c(u_1) &= c(v_2) + [(k+q) + 2(2p-1)] \\ &= c(z_q) + 2p(k+q) + 2 \cdot \frac{2p(2p-1)}{2} \\ &= c(z_q) + 2p(k+q+2p-1). \end{aligned}$$

Then by a similar method to prove Claim 3.6, we can obtain the following claim.

Claim 3.7. For all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, the inequality (2) holds. And $\max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 1)$.

By Claim 3.7, we have shown that for all odd integers $n \geq 15$, c is a nearly antipodal coloring for P_n . Therefore $\text{ac}'(P_n) \leq \text{ac}'(c) = \max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 1)$. To finish the proof of Theorem 2.1 for all odd integers $n \geq 15$, it suffices to prove the following claim.

Claim 3.8. For any $p \in \{0, 1, 2, \dots\}$ and any $q \in \{1, 2, 3, 4\}$, it holds that $c(u_1) = c(z_q) + 2p(k + q + 2p - 1) = \binom{n-2}{2} - \frac{n-1}{2} + 8$, where $n = 2k + 1 = 13 + 2(4p + q)$.

In fact, if $q = 1$, then $k = 4p + 7$, $4p = k - 7$, $2p = \frac{k-7}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_1) + 2p(k + q + 2p - 1) = 9k + 16 + 2p(2k + 2p + 9) + 2p(k + 2p) \\ &= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8. \end{aligned}$$

If $q = 2$, then $k = 4p + 8$, $4p = k - 8$, $p = \frac{k-8}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_2) + 2p(k + q + 2p - 1) \\ &= 10k + 25 + 2p(2k + 2p + 11) + 2p(k + 2p + 1) \\ &= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8. \end{aligned}$$

If $q = 3$, then $k = 4p + 9$, $4p = k - 9$, $p = \frac{k-9}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_3) + 2p(k + q + 2p - 1) \\ &= 12k + 27 + 2p(2k + 2p + 11) + 2p(k + 2p + 2) \\ &= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8. \end{aligned}$$

If $q = 4$, then $k = 4p + 10$, $4p = k - 10$, $2p = \frac{k-10}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_4) + 2p(k + q + 2p - 1) \\ &= 13k + 39 + 2p(2k + 2p + 13) + 2p(k + 2p + 3) \\ &= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8. \end{aligned}$$

Thus Claim 3.8 holds and hence $ac'(P_n) \leq ac'(c) = \binom{n-2}{2} - \frac{n-1}{2} + 8$ for all odd integers $n \geq 15$.

Secondly, for $n = 13$, in the above proof we take $p = 0$ and $q = 0$. Namely, $V_2 = V_3 = \emptyset$, $V(P_{13}) = V_1 = \{x'_1, x'_2, x'_3, x'_4; x_2, x_1; x_0; y_1, y_2; y'_4, y'_3, y'_2, y'_1\}$ (also see Figure 2 and let $p = q = 0$). Then coloring $c|_{v \in V_1}(v)$ is a nearly antipodal coloring for P_{13} . Thus by Claim 3.5, $ac'(P_{13}) \leq ac'(c|_{v \in V_1}) = \max_{v \in V_1} c(v) = c(y_2) = (8k + 8)|_{k=6} = 56 = \binom{13-2}{2} + 1$. Since $-\lfloor \frac{13}{n} \rfloor = -1$ for $n = 13$, it follows that $ac'(P_{13}) \leq ac'(c|_{v \in V_1}) = \binom{13-2}{2} + 1 = \binom{13-2}{2} - \frac{13-1}{2} - \lfloor \frac{13}{13} \rfloor + 8$.

Thus the assertion 2 in Theorem 2.1 holds. ■

4. Examples

In this section we give some examples which present the nearly antipodal coloring c for some P_n with $ac'(c)$ presented in Theorem 2.1 by our methods.

Example 4.1. A nearly antipodal coloring c for P_{10} with $ac'(c) = \binom{10-2}{2} - \frac{10}{2} - \lfloor \frac{10}{10} \rfloor + 7 = \binom{10-2}{2} + 1 = 29$ (see Figure 3).

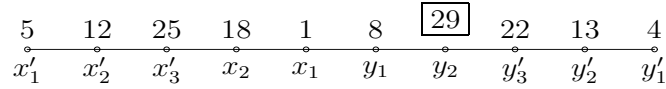


Figure 3. A nearly antipodal coloring for P_{10} .

Example 4.2. A nearly antipodal coloring c for P_{13} with $ac'(c) = \binom{13-2}{2} - \frac{13-1}{2} - \lfloor \frac{13}{13} \rfloor + 8 = \binom{13-2}{2} + 1 = 56$ (see Figure 4).

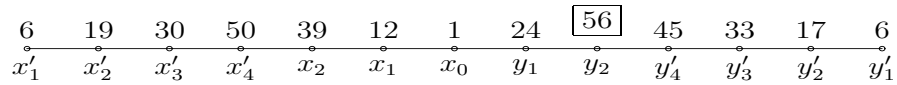


Figure 4. A nearly antipodal coloring c for P_{13} .

Example 4.3 A nearly antipodal coloring c for P_{32} with $ac'(c) = \binom{32-2}{2} - \frac{32}{2} + 7 = \binom{32-2}{2} - 9 = 426$ (see Figure 5).

Here $n = 2k = 10 + 2(4p + q) = 32$, then $k = 16$, $p = 2$ and $q = 3$.

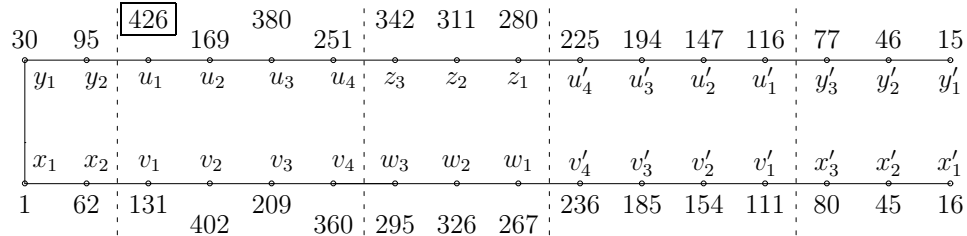


Figure 5. A nearly antipodal coloring for P_{32} .

Example 4.4. A nearly antipodal coloring c for P_{33} with $ac'(c) = \binom{33-2}{2} - \frac{33-1}{2} + 8 = \binom{33-2}{2} - 8 = 457$ (see Figure 6).

Here $n = 2k + 1 = 13 + 2(4p + q) = 33$, then $k = 16$, $p = 2$ and $q = 2$.

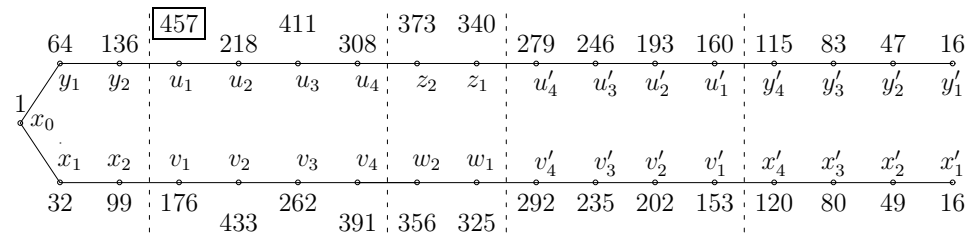


Figure 6. A nearly antipodal coloring c for P_{33} .

Acknowledgements

We would like to express our gratitude to the referees for their careful reading and valuable comments and suggestions about this paper.

References

- [1] G. Chartrand, D. Erwin, F. Harary and P. Zhang, *Radio labelings of graphs*, Bull. Inst. Combin. Appl. **33** (2001) 77–85.
- [2] G. Chartrand, D. Erwin and P. Zhang, *A graph labeling problem suggested by FM channel restrictions*, Bull. Inst. Combin. Appl. **43** (2005) 43–57.
- [3] G. Chartrand, D. Erwin and P. Zhang, *Radio antipodal colorings of graphs*, Math. Bohem. **127** (2002) 57–69.

- [4] G. Chartrand, L. Nebeský and P. Zhang, *Radio k -colorings of paths*, Discuss. Math. Graph Theory **24** (2004) 5–21.
- [5] D. Fotakis, G. Pantziou, G. Pentaris and P. Spirakis, *Frequency assignment in mobile and radio networks*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **45** (1999) 73–90.
- [6] R. Khennoufa and O. Togni, *A note on radio antipodal colorings of paths*, Math. Bohem. **130** (2005) 277–282.
- [7] J. Van den Heuvel, R.A. Leese and M.A. Shepherd, *Graph labeling and radio channel assignment*, J. Graph Theory **29** (1998) 263–283.

Received 21 February 2006

Revised 31 October 2006