Discussiones Mathematicae
Graph Theory 27 (2007) 209–227

ON DISTANCE LOCAL CONNECTIVITY AND VERTEX DISTANCE COLOURING

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Abstract

In this paper, we give some sufficient conditions for distance local connectivity of a graph, and a degree condition for local connectivity of a k-connected graph with large diameter. We study some relationships between t-distance chromatic number and distance local connectivity of a graph and give an upper bound on the t-distance chromatic number of a k-connected graph with diameter d.

Keywords: degree condition, distance local connectivity, distance chromatic number.

2000 Mathematics Subject Classification: 05C15, 05C75.

1. Introduction

By a graph we mean a simple undirected graph. We use [2] for terminology and notation not defined here. Let $\mathrm{dist}_G(x, y)$ denote the distance between vertices x and y in G. An *x, y-path* is a path between vertices x and y in G. Let $d = \max \mathrm{dist}_G(xy) : x, y \in V(G)$ denote the diameter of G. An x, y-path P is called *diameter-path*, if $\mathrm{dist}_G(x, y) = d$ and $|E(P)| = d$. Let $d_G(x)$ denote the degree of a vertex x in G, $\delta(G)$ the minimum degree of G and $\Delta(G)$ the maximum degree of G. For a nonempty set $U \subseteq V(G)$, the induced subgraph on U is denoted by $\langle U\rangle$. For a nonempty

*Research Supported by Grant No. 1M0021620808 of the Czech Ministry of Education.

set $A \subset V(G)$, $G - A$ denotes the subgraph of G that we obtain by deleting all vertices of A and all edges adjacent to at least one vertex of A. Let $\sigma_k(G) = \min\{\sum_{i=1}^{k} d_G(x_i) | \{x_1, \ldots, x_k\} \subset V(G),\ \text{independent}\}$. The *square* of a graph G, denoted by G^2, is the graph in which $V(G^2) = V(G)$ and $E(G^2) = E(G) \cup \{\{u, v\} | \mathrm{dist}_G(u, v) = 2\}$.

Let $N_G(x) = \{y \in V(G), xy \in E(G)\}$, let $N_G[x] = N_G(x) \cup \{x\}$. The set $N_G(x)$ is called the *neighbourhood* of the first type of x in G. We say that x is a *locally connected vertex* of G, if $\langle N_G(x)\rangle$ is connected. We say that G is a *locally connected graph*, if every vertex of G is locally connected. Chartrand and Pippert [3] proved the following Ore-type condition for local connectivity of graphs:

Theorem A [3]. *Let G be a connected graph of order n. If*

$$d_G(u) + d_G(v) > \frac{4}{3}(n - 1)$$

for every pair of vertices u, $v \in V(G)$, then G is locally connected.

Let $N_2(x)$ be a subgraph induced by the set of edges uv, such that

$$\min\{\mathrm{dist}_G(x, u),\ \mathrm{dist}_G(x, v)\} = 1.$$

The subgraph $N_2(x)$ is called the *neighbourhood of the second type* of x in G. We say that x is an N_2*-locally connected vertex* of G, if $N_2(x)$ is connected. We say that G is N_2*-locally connected*, if every vertex of G is N_2-locally connected.

Now define the distance neighbourhood of the first type of a vertex of G as in [5]. Let m be a positive integer and let x be an arbitrary vertex of a graph G. The N_1^m*-neighbourhood* of x in G, denoted by $N_1^m(x)$, is the set of all vertices $y \in V(G)$, $y \neq x$, such that $\mathrm{dist}_G(x, y) \leq m$. Let $N_1^m[x] = N_1^m(x) \cup \{x\}$. A vertex x is called N_1^m*-locally connected* if $\langle N_1^m(x)\rangle$ is connected. A graph G is said to be N_1^m*-locally connected* if every vertex of G is N_1^m-locally connected.

The distance local connectivity of the second type is analogously defined as the neighbourhood of the second type. Let m be a positive integer and let x be an arbitrary vertex of a graph G. The N_2^m*-neighbourhood* of x, denoted by $N_2^m(x)$, is the subgraph induced by all edges $\{u, v\}$ of G, $u \neq x$, $v \neq x$, with $\min\{\mathrm{dist}_G(x, u), \mathrm{dist}_G(x, v)\} \leq m$. We say that x is N_2^m*-locally connected* in G if $N_2^m(x)$ is connected. A graph G is said to be N_2^m*-locally connected* if every vertex of G is N_2^m-locally connected in G.

Let t be a positive integer. The *t-distance chromatic number* of a graph G, denoted $\chi^{(t)}(G)$, is the minimum number of colours required to colour all vertices of G in such a way that any two vertices x, y with $\mathrm{dist}_G(x, y) \leq t$ have distinct colours. Let $\chi(x)$ denote the colour of a vertex x in G. Recall that the vertex distance colouring was introduced by Kramer and Kramer in [7] and [8]. In the 90's, several results on vertex distance colourings were presented, cf. Baldi in [1], Skupień in [11], Chen *et al.* in [4].

The following result was proved by Jendrol' and Skupień in [6].

Theorem B [6]**.** *Given a planar graph G, let $D = \max\{8, \Delta(G)\}$. Then the t-distance chromatic number of G is*

$$\chi^{(t)}(G) \leq 6 + \frac{3D+3}{D-2}((D-1)^{t-1} - 1).$$

Madaras and Marcinová strengthened this condition in [9].

Theorem C [9]**.** *Let G be a planar graph, let $D = \max\{8, \Delta(G)\}$. Then*

$$\chi^{(t)}(G) \leq 6 + \frac{2D+12}{D-2}((D-1)^{t-1} - 1).$$

2. Distance Local Connectivity of a Graph in k-Connected Graphs

The concept of the local connectivity of a graph was introduced in 1970's. Ryjáček used the concept of the local connectivity of a vertex in [10] for local completing in his closure concept for claw-free graphs. This closure concept gave a solution for several hamiltonian problems. A degree condition is one of the easily verified conditions. Chartrand and Pippert in [3] proved a degree condition for the local connectivity of connected graphs (see Theorem A). The same degree condition can guarantee the local connectivity of any vertex of a connected locally connected graph. In this chapter, degree conditions for the local connectivity of a k-connected graph with a large diameter will be presented as a strengthening of the result of Chartrand and Pippert. Holub and Xiong in [5] proved degree conditions for distance local connectivity of 2-connected graphs. As a strengthening of this condition, degree conditions for distance local connectivity of a k-connected graph with a large diameter will be shown, too.

Theorem 1. *Let $k \geq 2$ be an integer, G be a k-connected graph of order n. Let d be the diameter of G, let $d \geq 5$. If*

$$d_G(u) + d_G(v) > \frac{4}{3}(n - kd + 5k - 3)$$

for every pair of vertices $u, v \in V(G)$, then G is locally connected.

Theorem 2. *Let $k \geq 2$ be an integer, G be a k-connected graph of order n. Let d be the diameter of G, m be an integer such that $2 \leq m \leq \frac{1}{2}(d-7)$. If*

(1) $\sigma_t \geq n - kd + 2mk + 6k - t$, *where* $t = \frac{2}{3}m + 1$ *if* $m \equiv 0 (\mathrm{mod}\, 3)$,

(2) $\sigma_t \geq n - kd + 2mk + 6k - 2 - t$, *where* $t = \frac{2}{3}(m-1) + 3$ *if* $m \equiv 1 (\mathrm{mod}\, 3)$,

(3) $\sigma_t \geq n - kd + 2mk + 4k - 1 - t$, *where* $t = \frac{2}{3}(m-2) + 3$ *if* $m \equiv 2 (\mathrm{mod}\, 3)$.

then G is N_1^m-locally connected.

Before proofs of these two theorems, some auxiliary statements will be shown.

Lemma 1. *Let $k \geq 2$ be an integer, G be a k-connected graph and x be an arbitrary vertex of G. Let d be the diameter of G, let $d \geq 5$. If x does not belong to any diameter-path in G, then there are at least $kd - 5k + 2$ vertices y such that $dist_G(x, y) > 2$.*

Proof. Let P denote a diameter-path in G, let u, v be the end vertices of P. Since G is k-connected, there are at least k vertex-disjoint u, v-paths in G by Menger's theorem. Choose $P_1, \ldots, P_k$ with minimum sum of their lengths. Note that $|E(P_i)| \geq d$, $i = 1, \ldots, k$. Now it will be shown that there are at least $d - 3$ vertices at the required distance from x on each of P_i, $i = 1, \ldots, k$. Let $M_j = \{y \in P_j | \operatorname{dist}_G(x, y) \leq 2\}$, $j = 1, \ldots, k$. For each path of P_i, $i = 1, \ldots, k$, there are two following cases:

Case 1. If $M_j = \emptyset$, then there are at least $d+1$ vertices at the required distance from x on P_j.

Case 2. If $M_j \neq \emptyset$, then let $a_j \in M_j$ such that $\operatorname{dist}_G(a_j, u) = \min_{m \in M_j} \operatorname{dist}_G(m, u)$ and let $b_j \in M_j$ such that $\operatorname{dist}_G(b_j, v) = \min_{m \in M_j} \operatorname{dist}_G(m, v)$. Since x does not belong to any diameter path, we have

$$\operatorname{dist}_G(u, a_j) + \operatorname{dist}_G(a_j, x) + \operatorname{dist}_G(x, b_j) + \operatorname{dist}_G(b_j, v) \geq d + 1.$$

Since $\mathrm{dist}_G(a_j, x) \le 2$ and $\mathrm{dist}_G(b_j, x) \le 2$, we obtain

$$\mathrm{dist}_G(u, a_j) + \mathrm{dist}_G(b_j, v) \ge d - 3.$$

Hence there are at least $d-3$ vertices at the required distance from x on P_j.

On the paths P_i, $i = 1, \ldots, k$, there are at least $k(d-3)$ vertices at the required distance from x in G. Since u and v can be counted only once, there are at least $k(d-5)+2$ different vertices at the required distance from x in G. ■

***Proof of Theorem* 1.** Suppose G is not locally connected. Then there is a vertex x such that x is not locally connected in G. There are at least two components of $\langle N_G(x)\rangle$. Let G_1 denote a smallest component of $\langle N_G(x)\rangle$ and let G_2 be the union of all the other components of $\langle N_G(x)\rangle$. Let $g_1 = |V(G_1)|$, let $g_2 = |V(G_2)|$. Let $Z = \{y \in V(G);\ \mathrm{dist}_G(x, y) = 2\}$, let $z = |Z|$. Let $p = |\{y \in V(G);\ \mathrm{dist}_G(x, y) > 2\}|$.

Case 1. Suppose that x does not belong to any diameter-path in G. By Lemma 1, the number $p \ge kd - 5k + 2$. Clearly $n = g_1 + g_2 + z + p + 1$. Choose arbitrary vertices u and v such that $u \in V(G_1)$ and $v \in V(G_2)$. By the assumptions of Theorem 1

$$d_G(x) + d_G(u) > \frac{4}{3}(n - kd + 5k - 3).$$

Since $d_G(x) = g_1 + g_2$ and $d_G(u) \le g_1 + z = n - 1 - p - g_2 \le n - 1 - g_2 - kd + 5k - 2$, we obtain

$$g_1 + g_2 + n - g_2 - 1 - kd + 5k - 2 > \frac{4}{3}(n - kd + 5k - 3).$$

Clearly $g_1 > \frac{1}{3}(n - kd + 5k - 3)$ and $g_2 > \frac{1}{3}(n - kd + 5k - 3)$ since $g_2 \ge g_1$. Therefore

$$z < \frac{1}{3}(n - kd + 5k - 3).$$

For vertices u and v

$$d_G(u) + d_G(v) \le g_1 + z + g_2 + z < \frac{4}{3}(n - kd + 5k - 3),$$

a contradiction.

Case 2. Suppose that x belongs to a diameter-path P. Let e, f be the end vertices of P. Since G is k-connected, there are at least k vertex-disjoint e, f-paths in G. Choose $P_1, \dots, P_k$ with a minimum sum of their lengths. For each of P_i, $i = 1, \dots, k$ the following cases can happen.

Subcase 2.1. $V(P_i) \cap Z = \emptyset$. Then there are at least $d+1$ vertices on P_i at distance at least 3 from x in G.

Subcase 2.2. $V(P_i) \cap (V(G_1) \cup V(G_2)) = \emptyset$, but $V(P_i) \cap Z \neq \emptyset$. Let $d_i = |V(P_i) \cap Z|$. If $d_i \leq 4$, then there are at least $d-3$ vertices on P_i at distance at least 3 from x in G.

Now suppose that $d_i \geq 5$. If there is a vertex $w \in V(G_1) \cup V(G_2)$ such that w is adjacent to every vertex of $V(P_i) \cap Z$, then there are at least $d-2$ vertices at distance at least 3 from x in G since $\mathrm{dist}_G(e, f) \geq d$. If none of the vertices of $V(G_1) \cup V(G_2)$ is adjacent to every vertex of $V(P_i) \cap Z$, then

$$d_G(u) \leq g_1 + z - (d_i - 3) \leq g_1 + z - 1, \quad \forall u \in V(G_1),$$
$$d_G(v) \leq g_2 + z - (d_i - 3) \leq g_2 + z - 1, \quad \forall v \in V(G_2).$$

Subcase 2.3. $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset$. Let $d_i^1 = |V(P_i) \cap V(G_1)|$, $d_i^2 = |V(P_i) \cap V(G_2)|$ and $d_i = |V(P_i) \cap Z|$. Note that $d_i \geq 2$. The following two possibilities have to be considered.

(i) $d_i^1 = 0$ or $d_i^2 = 0$. Up to symmetry, suppose that d_i^2=0. If $d_i^1 = 1$ and $d_i = 2$, then there are at least $d-3$ vertices on P_i at distance at least 3 from x in G.

Now suppose that $d_i^1 = 1$ and $d_i > 2$. If there is a vertex $w \in V(G_1)$ such that w is adjacent to every vertex of $V(P_i) \cap Z$, then there are at least $d-2$ vertices at distance at least 3 from x in G since $\mathrm{dist}_G(e, f) \geq d$. If there is no vertex $w \in V(G_1)$ adjacent to every vertex of $V(P_i) \cap Z$, then

$$d_G(u) \leq g_1 + z - (d_i - 2) \leq g_1 + z - 1, \forall u \in V(G_1).$$

Now suppose that $d_i^1 > 1$. If there is a vertex $w \in V(G_1)$ such that w is adjacent to every vertex of $V(P_i) \cap Z$, then there are at least $d-2$ vertices at distance at least 3 from x in G since $\mathrm{dist}_G(e, f) \geq d$. If there is no vertex $w \in V(G_1)$ adjacent to every vertex of $V(P_i) \cap Z$, then

$$d_G(u) \leq g_1 + z - (d_i^1 - 2) - (d_i - 1) \leq g_1 + z - 1, \forall u \in V(G_1).$$

(ii) $d_i^1 > 0$ and $d_i^2 > 0$. If P_i is a diameter-path containing x, then there are at least $d-4$ vertices on P_i at distance at least 3 from x in G. If P_i does not contain x, then $d_i \geq 3$. If there is a vertex $w \in V(G_1) \cup V(G_2)$ such that w is adjacent to every vertex of $V(P_i) \cap Z$, then there are at least $d-2$ vertices at distance at least 3 from x in G since $\mathrm{dist}_G(e,f) \geq d$. If there is no vertex $w \in V(G_1) \cup V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$, then

$$d_G(u) \leq g_1 + z - (d_i - 2) \leq g_1 + z - 1, \quad \forall u \in V(G_1),$$
$$d_G(v) \leq g_2 + z - (d_i - 2) \leq g_2 + z - 1, \quad \forall v \in V(G_2).$$

Let l_1 denote the number of such the paths $P_1, \ldots, P_k$, for which one of the following conditions is satisfied

- $V(P_i) \cap V(Z) \neq \emptyset, V(P_i) \cap (V(G_1) \cup V(G_2)) = \emptyset$, $d_i \geq 5$ and there is no vertex $w \in V(G_1) \cup V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset$, $d_i^1 = 1$, $d_i^2 = 0$, $d_i > 2$ and there is no vertex $w \in V(G_1)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset$, $d_i^1 > 1$, $d_i^2 = 0$ and there is no vertex $w \in V(G_1)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset$, $d_i^1 d_i^2 \neq 0$, $x \notin V(P_i)$ and there is no vertex $w \in V(G_1) \cup V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$.

Let l_2 denote the number of such the paths $P_1, \ldots, P_k$, for which one of the following conditions is satisfied

- $V(P_i) \cap V(Z) \neq \emptyset, V(P_i) \cap (V(G_1) \cup V(G_2)) = \emptyset$, $d_i \geq 5$ and there is no vertex $w \in V(G_1) \cup V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset$, $d_i^1 = 0$, $d_i^2 = 1$, $d_i > 2$ and there is no vertex $w \in V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset$, $d_i^2 > 1$, $d_i^1 = 0$ and there is no vertex $w \in V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$,
- $V(P_i) \cap (V(G_1) \cup V(G_2)) \neq \emptyset$, $d_i^1 d_i^2 \neq 0$, $x \notin V(P_i)$ and there is no vertex $w \in V(G_1) \cup V(G_2)$ adjacent to every vertex of $V(P_i) \cap Z$.

Let $l = l_1 + l_2$. Then there are at least $kd - 5k + 2 - l - 1$ vertices at distance at least 3 from x in G and

$$d_G(u) \leq g_1 + z - l_1, \quad \forall u \in V(G_1),$$
$$d_G(v) \leq g_2 + z - l_2, \quad \forall v \in V(G_2).$$

Suppose that $l_2 \geq l_1$. By the assumptions, for every $u \in V(G_1)$

$$d_G(x) + d_G(u) > \frac{4}{3}(n - kd + 5k - 3).$$

Since $d_G(x) = g_1 + g_2$ and $d_G(u) \leq g_1 + z - l_1 \leq n - 1 - g_2 - l_1 - kd + 5k - 2 + l$, we have

$$g_1 + g_2 + n - g_2 - l_1 - kd + 5k - 3 + l > \frac{4}{3}(n - kd + 5k - 3).$$

Clearly

$$g_1 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l \text{ and}$$
$$g_2 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l,$$

since $g_2 \geq g_1$. Thus

$$z < \frac{1}{3}(n - kd + 5k - 3) + 2l_1 - l.$$

For vertices u and v, it holds that

$$d_G(u) + d_G(v) \leq g_1 + g_2 + 2z - l_1 - l_2 < \frac{4}{3}(n - kd + 5k - 3) + l_1 - l_2,$$

a contradiction, since $l_2 \geq l_1$. Hence suppose that $l_1 > l_2$. Then we get

$$g_1 > \frac{1}{3}(n - kd + 5k - 3) + l_1 - l \geq \frac{1}{3}(n - kd + 5k - 3) - l_1.$$

Thus

$$\begin{aligned} g_2 &> \frac{1}{3}(n - kd + 5k - 3) + l_1 - l, \\ z &< \frac{1}{3}(n - kd + 5k - 3) + l - l_1 + l_1 - l = \frac{1}{3}(n - kd + 5k - 3). \end{aligned}$$

Then

$$d_G(u) + d_G(v) \leq g_1 + g_2 + 2z - l_1 - l_2 < \frac{4}{3}(n - kd + 5k - 3) + 1.$$

Hence

$$d_G(u) + d_G(v) \leq \frac{4}{3}(n - kd + 5k - 3),$$

a contradiction. ■

The following example shows that the conditions of Theorem 1 are sharp.

Example. Let $K_1, \ldots, K_{k_1}$ be k_1 cliques of order k, let $L_1, \ldots, L_{k_2}$ be k_2 cliques of order k. Let K_0, L_0 be two cliques of order $l_1 > 2k - 1$, let M be a clique of order $l_1 - k$. All considered cliques K_i, L_i are vertex-disjoint. Construct a graph G by joining a new vertex x with each vertex of $K_0 \cup L_0$, a new vertex u with each vertex of K_{k_1} and a new vertex v with each vertex of L_{k_2}. Now join each vertex of K_i with each vertex of K_{i-1} for $i = 1, \ldots, k_1$, each vertex of L_i with each vertex of L_{i-1} for $i = 1, \ldots, k_2$ and each vertex of $K_0 \cup L_0$ with each vertex of M. Clearly the prescribed graph G is k-connected and the vertex x is not locally connected. The diameter of G is $d = k_1 + k_2 + 4$. It holds that

$$n = 1 + 2l_1 + l_1 - k + (k_1 + k_2)k + 2 = 3l_1 + (d - 5)k + 3.$$

Thus

$$3l_1 = n - kd + 5k - 3.$$

Furthermore

$$\begin{aligned} d_G(x) &= 2l_1, \\ d_G(y) &= 2l_1, \quad \forall y \in K_0, \\ d_G(z) &= 2l_1, \quad \forall z \in L_0. \end{aligned}$$

Hence for every pair a, b of vertices of $N_G[x]$ holds that

$$d_G(a) + d_G(b) = 4l_1 = \frac{4}{3}(n - kd + 5k - 3).$$

and x is not locally connected.

The following lemma is a proposition analogous to Lemma 1 for the N_1^m-local connectivity of a vertex of a graph.

Lemma 2. *Let $k \geq 2$ be an integer, G be a k-connected graph. Let d be the diameter of G and $m \leq \frac{1}{2}(d-1)$ be an integer. Then, for each vertex x of G, there are at least $kd - 2km + 2$ vertices at distance at least m from x in G.*

Proof. Let P denote a diameter-path in G, let u, v be the end vertices of P. Since G is k-connected, there are at least k vertex-disjoint u, v-paths in G by Menger's theorem. Choose $P_1, \ldots, P_k$ with minimum sum of their lengths. Note that $|E(P_i)| \geq d$, $i = 1, \ldots, k$. Now it will be shown that there are at least $d - 2m + 2$ vertices at the required distance from x on each of P_i, $i = 1, \ldots, k$. Let $M_j = \{y \in P_j | \operatorname{dist}_G(x, y) \leq m - 1\}$, $j = 1, \ldots, k$. For each path of P_i, $i = 1, \ldots, k$ there are two following cases:

Case 1. If $M_i = \emptyset$, then there are at least $d + 1$ vertices at the required distance from x on P_i.

Case 2. If $M_i \neq \emptyset$, then let $a_i \in M_i$ such that $\operatorname{dist}_G(a_i, u) = \min_{m \in M_i} \operatorname{dist}_G(m, u)$ and let $b_i \in M_i$ such that $\operatorname{dist}_G(b_i, v) = \min_{m \in M_i} \operatorname{dist}_G(m, v)$. Clearly

$$\operatorname{dist}_G(u, a_i) + \operatorname{dist}_G(a_i, x) + \operatorname{dist}_G(x, b_i) + \operatorname{dist}_G(b_i, v) \geq d.$$

Since $\operatorname{dist}_G(a_i, x) \leq m - 1$ and $\operatorname{dist}_G(b_i, x) \leq m - 1$, we have

$$\operatorname{dist}_G(u, a_i) + \operatorname{dist}_G(b_i, v) \geq d - 2m + 2.$$

Hence there are at least $d - 2m + 2$ vertices at the required distance from x on P_i.

On the paths P_i, $i = 1, \ldots, k$ there are at least $k(d - 2m + 2)$ vertices at the required distance from x in G. Since u and v can be counted only once, there are at least $kd - 2km + 2$ different vertices at the required distance from x in G. ■

Let C be a cycle, $x \in V(C)$ and $\vec{C}$ be an orientation of C. Let $x^{-(i)}$ denote the i-th predecessor of x on C and $x^{+(i)}$ denote the i-th successor of x on C in the orientation $\vec{C}$.

Lemma 3 [5]**.** *Let G be a 2-connected graph, $x \in V(G)$, and m be a positive integer. If x is not N_1^m-locally connected, then there is an induced cycle C of length at least $2m + 2$ such that, in an orientation of C,*

- $\operatorname{dist}_G(x^{-(i)}, x) = i$ *and* $\operatorname{dist}_G(x^{+(i)}, x) = i$, $i = 1, \ldots, m$,
- $\operatorname{dist}_G(y, x) > m$, *for every* $y \in V(C) \setminus \{x, x^{-(1)}, \ldots, x^{-(m)}, x^{+(1)}, \ldots, x^{+(m)}\}$.

The following consequence proved by Holub and Xiong we use in the proof of Theorem 2.

Corollary 1 [5]. *Let $m \geq 2$ be an integer, G a 2-connected graph. If $x \in V(G)$ is not N_1^m-locally connected, then there is a set $M \subset V(G)$ such that*

(1) *M is independent in $(G-x)^2$, $M \subset N_1^{m+1}(x)$ and $|M| \geq \frac{2}{3}m+1$, if $m \equiv 0 (\mathrm{mod}\, 3)$,*
(2) *M is independent in $(G-N_G[x])^2$, $M \subset (N_1^{m+1}(x) \setminus N_1^1(x))$ and $|M| \geq \frac{2}{3}(m-1)+1$, if $m \equiv 1 (\mathrm{mod}\, 3)$,*
(3) *M is independent in G^2, $M \subset N_1^m[x]$ and $|M| \geq \frac{2}{3}(m-2)+2$, if $m \equiv 2 (\mathrm{mod}\, 3)$.*

***Proof of Theorem* 2.** Suppose that G is not N_1^m-locally connected. Then there is a vertex $x \in V(G)$ such that x is not N_1^m-locally connected in G. Hence $\langle N_1^m(x) \rangle$ consists of at least two components. Let G_1 denote arbitrary component of $\langle N_1^m(x) \rangle$, let G_2 denote the union of all the other components of $\langle N_1^m(x) \rangle$.

Case 1. $m \equiv 0 (\mathrm{mod}\, 3)$. By Corollary 1 case (1), there is a set $M \subset N_1^{m+1}(x)$ such that $|M| = \frac{2}{3}m+1$ and M is independent in $(G-x)^2$. Let $t = |M|$. Using Lemma 3, the set M can be chosen in the following way: $M = \{x_1, x_2, \ldots, x_t\}$, where $x_{2j-1} = x^{-(3j-2)}$, $x_{2j} = x^{+(3j-2)}$, $j = 1, \ldots, \frac{m}{3}$, $x_t = x^{+(m+1)}$. Let $A = \{y \in V(G) | \operatorname{dist}_G(x,y) > m+2\}$, let $a = |A|$. By Lemma 2, the number $a \geq kd - 2(m+3)k + 2$. Since M is independent in $(G-x)^2$, we have, for every pair $u, v \in M \setminus \{x\}$,

$$N_{G-x}(u) \cap N_{G-x}(v) = \emptyset.$$

Since x is adjacent to at most two vertices of M, we obtain

$$\sum_{x_i \in M} d_G(x_i) \leq (n-1) - t - a + 2 = n - t - a + 1.$$

Since $a \geq kd - 2(m+3)k + 2$, we have

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - kd + 2mk + 6k - 1,$$

a contradiction.

Case 2. $m \equiv 1(\text{mod}\,3)$. By Corollary 1 case (2), there is a set $M \subset N_1^{m+1}(x)$ such that $|M| = \frac{2}{3}(m-1)+1$ and M is independent in $(G-N_G[x])^2$. Let $t = |M|$. Using Lemma 3, the set M can be chosen in the following way: $M = \{x_1, x_2, \dots, x_t\}$, where $x_{2j-1} = x^{-(3j-1)}$, $x_{2j} = x^{+(3j-1)}$, $j = 1, \dots, \frac{m-1}{3}$, $x_t = x^{+(m+1)}$. Let $A = \{y \in V(G) |\, \text{dist}_G(x, y) > m+2\}$, let $a = |A|$. By Lemma 2, the number $a \geq kd - 2(m+3)k + 2$. Since M is independent in $(G - N_G[x])^2$, we have, for every pair $u, v \in M$,

$$N_G(u) \cap N_G(v) = \emptyset.$$

Since each vertex of $N_G(x)$ is adjacent to at most one vertex of M, we obtain

$$\sum_{x_i \in M} d_G(x_i) \leq (n-1) - t - a.$$

Since $a \geq kd - 2(m+3)k + 2$, we have

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - kd + 2mk + 6k - 3,$$

a contradiction.

Case 3. $m \equiv 2(\text{mod}\,3)$. By Corollary 1 case (3), there is a set $M \subset N_1^m[x]$ such that $|M| = \frac{2}{3}(m-2)+2$ and M is independent in G^2. Let $A = \{y \in V(G) |\, \text{dist}_G(x, y) > m+1\}$, let $a = |A|$. By Lemma 2, the number $a \geq kd - 2(m+2)k + 2$. Since M is independent in G^2,we have, for every pair $u, v \in M$,

$$N_G(u) \cap N_G(v) = \emptyset.$$

Let $t = |M|$. Hence

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - a.$$

Since $a \geq kd - 2(m+2)k + 2$, we obtain

$$\sum_{x_i \in M} d_G(x_i) \leq n - t - kd + 2mk + 4k - 2,$$

a contradiction. ■

3. Vertex Distance Colouring

There are several results on t-distance chromatic number for planar graphs. In this paragraph, results on t-distance chromatic number in k-connected, not necessary planar, graphs are presented. Moreover, the relations between distance local connectivity and t-distance chromatic number in 2-connected graphs are given. Main results of this section are the following theorems.

Theorem 3. *Let G be a k-connected graph of order n, d be the diameter of G. Let $t < d$ be a positive integer. Then the distance-chromatic number*

$$\chi^{(t)}(G) \leq \begin{cases} n-1 & \text{if } t = d-1, \\ n-(d-t-2)k-2 & \text{if } t < d-1. \end{cases}$$

Theorem 4. *Let G be a 2-connected graph of order n, let t, k be positive integers. If*

$$\chi^{(t)}(G) > n-(2k-1)(t+1),$$

then G is N_1^m-locally connected, where $m = k(t+1)-1$.

Theorem 5. *Let G be a 2-connected graph of order n, k be a positive integer and t be an even positive integer. If*

$$\chi^{(t)}(G) > n-2k(t+1),$$

then G is N_2^m-locally connected, where $m = k(t+1)+\frac{t}{2}-1$.

The distance local connectivity number of a 2-connected graph G, denoted $dlc(G)$, is the smallest positive integer m for which G is N_1^m-locally connected. Since G is 2-connected, the number $dlc(G)$ is well-defined. Note that local connectivity of a graph is the N_1^1-local connectivity. The following statement is a straightforward consequence of Theorem 4.

Corollary 2. *Let G be a 2-connected graph, let t be a positive integer. If $dlc(G) = m$, then $\chi^{(t)}(G) \leq n-(k-1)(t+1)$, where $k = \lfloor \frac{2m}{t+1} \rfloor$.*

Proof of Theorem 3. Let u, v denote the end vertices of a diameter path in G. Since G is k-connected, there are at least k vertex-disjoint u, v-paths $P_1, \ldots, P_k$ in G by Menger's theorem. Since $\mathrm{dist}_G(u,v) = d$, each of P_i, $i = 1, \ldots, k$, has length at least d. Let $u_{i,j}$ denote a vertex on P_i such that $\mathrm{dist}_G(u, u_{i,j}) = j$, $i = 1, \ldots, k$, $j = 1, \ldots, d$. Since $d > t$, there is at least one vertex $u_{i,j}$ on P_i, $i = 1, \ldots, k$, including the end-vertex v, such that $j > t$, $j = t+1, \ldots, d$. If $d - t = 1$, then $u_{i,t+1} = v$ for every $i \in \{1, \ldots, k\}$. We define colouring χ of vertices of G in such a way that $\chi(v) = \chi(u)$ and $\chi(x) \neq \chi(y)$ for all pairs $x, y \in V(G) \setminus \{u, v\}$. Clearly χ is a t-distance colouring of G and

$$\chi^{(t)}(G) \leq n - 1.$$

Suppose that $d - t > 1$. We define a colouring χ of vertices of G in such a way that the vertices of $N_1^{t+1}(u)$ have distinct colours in G, $\chi(u) = \chi(u_{i,t+1})$ and $\chi(v) = \chi(u_{i,d-t-1})$ for some $i \in \{1, \ldots, k\}$. Moreover, if $d - t > 2$, then, for every $i \in \{1, \ldots, k\}$, $\chi(u_{i,j+t+1}) = \chi(u_{i,j})$, since $\mathrm{dist}_G(u_{i,j}, u_{i,j+t+1}) = t + 1$, $j = 1, \ldots, d - t - 2$. Clearly χ is a t-distance colouring of G. Hence there are at least $k(d - t - 2) + 2$ vertices with previously used colours, implying that

$$\chi^{(t)}(G) \leq n - k(d - t - 2) - 2. \qquad \blacksquare$$

For the proofs of Theorem 4 and Theorem 5 we need some auxiliary statements. The following lemma is the analogue of Lemma 3.

Lemma 4. *Let G be a 2-connected graph, $x \in V(G)$ and m be a positive integer. If x is not N_2^m-locally connected, then there is an induced cycle C containing x of length at least $2m + 3$ such that, in an orientation of C,*

$$\mathrm{dist}_G(x^{-(i)}, x) = i \quad \text{and} \quad \mathrm{dist}_G(x^{+(i)}, x) = i, \; i = 1, \ldots, m + 1,$$

Proof. The vertex x is not N_2^m-locally connected. The N_2^m-neighbourhood of a vertex x consists of at least two components G_1, G_2. Since G is 2-connected, there is a cycle C containing x, such that $x^{-(1)} \in G_1$ and $x^{+(1)} \in G_2$ in an orientation of C. Choose C shortest possible with this property. Since x is not N_2^m-locally connected, $|V(C)| \geq 2m + 3$. It is easy to see that C has the required property since otherwise there is a shorter cycle. $\blacksquare$

From the definition of a t-distance colouring we obtain the following clear observation.

Proposition 1. *Let G be a 2-connected graph of order n, let t be a positive integer, let d denote the diameter of G. Then $\chi^{(t)}(G) = n$ if and only if $d \leq t$.*

Corollary 3. *Let G be a 2-connected graph of order n, let t be a positive integer. If $\chi^{(t)}(G) = n$, then G is N_1^t-locally connected.*

Proof. Suppose that G is not N_1^t-locally connected, i.e., there is a vertex $x \in V(G)$ such that x is not N_1^t-locally connected in G. By Proposition 1, $d \leq t$. By Lemma 3, there is an induced cycle C in G of length at least $2t+2$, which contradicts the fact that $d \leq t$. ■

***Proof of Theorem* 4.** Suppose that G is not N_1^m-locally connected, i.e., there is a vertex x which is not N_1^m-locally connected. By Lemma 3 there is an induced cycle C containing x, such that $|V(C)| \geq 2m+2$. Moreover $\text{dist}_G(x, x^{-(i)}) = \text{dist}_G(x, x^{+(i)}) = i$ for $i = 1, \ldots, m$. Since x is not N_1^m−locally connected, the cycle C can be chosen such that $x^{-(1)}$ and $x^{+(1)}$ belong to different components of $\langle N_1^m(x)\rangle$. Clearly $\text{dist}_G(x^{-(i)}, x^{-(j)}) = |i-j|$, for $i, j = 0, \ldots, m$ where $x^{-(0)} = x$.

We define a colouring χ of vertices of G in such a way that all the vertices $x^{-(0)}, \ldots, x^{-(t)}$ have distinct colours, $\chi(x^{-(i)}) = \chi(x^{-(i+t+1)})$, $i = 0, \ldots, t$, since $|V(C)| \geq 2(t+1)$. If $k > 1$, then $\chi(x^{-(i+j(t+1))}) = \chi(x^{-(i+(j-1)(t+1))})$ for $i = 0, \ldots, t$ and $j = 1, \ldots, 2k-1$. All the remaining vertices of G will be coloured with distinct unused colours. Clearly χ is a t-vertex distance colouring in G.

We have coloured $2k(t+1)$ vertices of C with only $t+1$ colours. Since $m = k(t+1)-1$, we have coloured $2m+2$ vertices of C with only $t+1$ colours, implying that

$$\chi^{(t)}(G) \leq n - (2m+2) + (t+1) = n - (2k-1)(t+1),$$

a contradiction. ■

***Proof of Theorem* 5.** We will use similar arguments as is the proof of Theorem 4. Suppose that G is not N_2^m-locally connected, i.e., there is a vertex x which is not N_2^m-locally connected. By Lemma 4 there is an induced cycle C containing x, such that $|V(C)| \geq 2m+3$. Moreover $\text{dist}_G(x, x^{-(i)}) = \text{dist}_G(x, x^{+(i)}) = i$ for $i = 1, \ldots, m+1$. Since x is not N_2^m−locally connected, the cycle C can be chosen such that $x^{-(1)}$ and $x^{+(1)}$ belong to different components of $\langle N_2^m(x)\rangle$. Clearly $\text{dist}_G(x^{-(i)}, x^{-(j)}) = |i-j|$, for $i, j = 0, \ldots, m+1$ where $x^{-(0)} = x$.

We define a colouring χ of vertices of G in such a way that all the vertices $x^{-(0)}, \ldots, x^{-(t)}$ have distinct colours, $\chi(x^{-(i)}) = \chi(x^{-(i+t+1)})$, $i = 0, \ldots, t$, since $|V(C)| \geq 2(t+1)$. If $k > 1$, then $\chi(x^{-(i+j(t+1))}) = \chi(x^{-(i+(j-1)(t+1))})$ for $i = 0, \dots, t$ and $j = 1, \dots, 2k$, since $|V(C)| \geq 2m+3 = (2k+1)(t+1)$. All the remaining vertices of G will be coloured with distinct unused colours. Clearly χ is a t-vertex distance colouring in G.

Thus we can colour $(2k+1)(t+1)$ vertices of C with only $t+1$ colours. Hence we have

$$\chi^{(t)}(G) \leq n - 2k(t+1),$$

a contradiction. ■

Now we give an example which show that conditions of Theorem 3 are sharp. Let d and $k \geq 2$ be two positive integers. Consider two vertices u and v and $d-1$ cliques $K_1, \dots, K_{d-1}$ of order k. We construct a graph G by joining each vertex of K_1 with u, each vertex of K_{d-1} with v and each vertex of K_i with each vertex of K_{i+1} for each $i \in \{1, \dots, d-2\}$. The diameter of G is d, the graph G is k-connected and the t-distance chromatic number is equal to

$$\begin{cases} n-1 & \text{if } t = d-1, \\ n-(d-t-2)k-2 & \text{if } t < d-1. \end{cases}$$

For the following two examples the conditions of Theorem 3 give better upper bound on the t-distance chromatic number than the conditions of Theorem B and C. Let d be a positive integer. Consider two vertices u, v and $d-1$ cliques $K_1, \dots, K_{d-1}$ of order 3. Construct a graph G by joining each vertex of K_1 with u, each vertex of K_{d-1} with v. Now pair vertices of K_i with vertices of K_{i+1}, for each $i \in \{1, \dots, d-2\}$. The structure of G is shown in Figure 1.

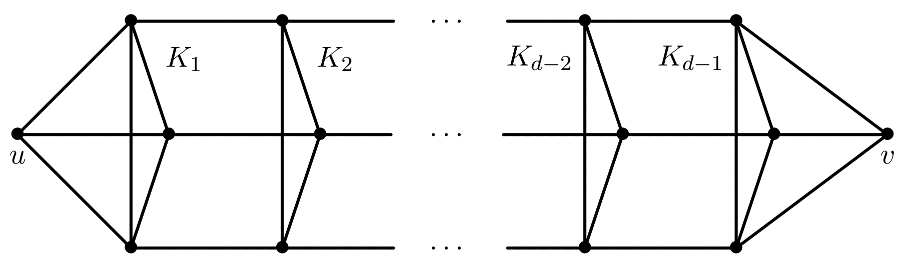

Figure 1

The graph G is 3-connected, the diameter of G is d and G is planar, because the graph on the following picture (Figure 2) is isomorphic with G.

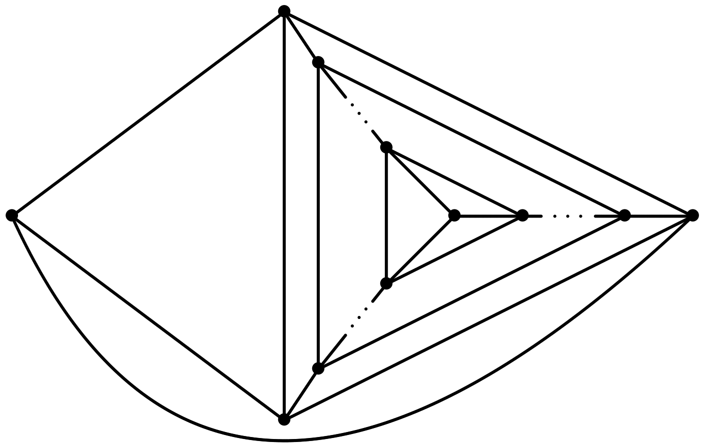

Figure 2

From Theorem 3 we obtain $\chi^{(t)}(G) \leq 3(t+1)$ and from Theorem B we get $\chi^{(t)}(G) \leq \frac{9}{2}((7)^{t-1}-1)+6$. For $t \geq 2$ the upper bound of Theorem 3 is better.

For any positive integer d, consider two vertices u, v, and $d-1$ cliques $K_1, \ldots, K_{d-1}$, such that K_1 and K_{d-1} are triangles and $K_1, \ldots, K_{d-2}$ are alternatively cliques of orders 3 and 4. Construct a graph G in such a way that we join each vertex of K_1 with u, each vertex of K_{d-1} with v and we pair vertices of K_i with vertices of K_{i+1}, for all $i \in \{1, \ldots, d-2\}$, in such a way that is shown in Figure 3.

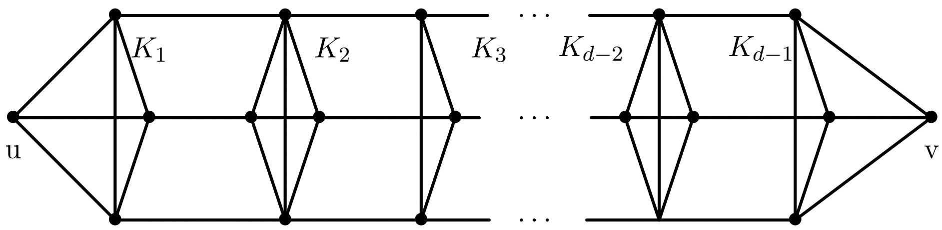

Figure 3

This graph G is 3-connected, the diameter of G is d and G is planar, because the graph on the following picture (Figure 4) is isomorphic with G.

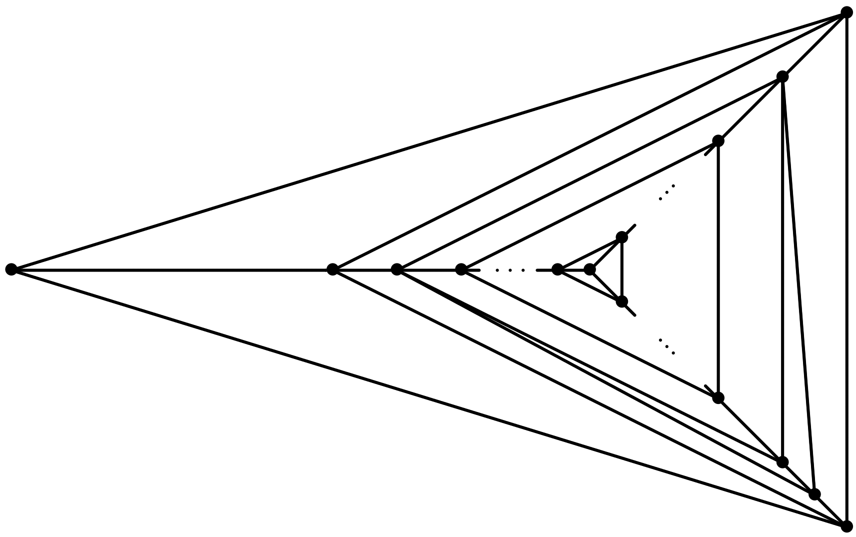

Figure 4

From Theorem 3 we get $\chi^{(t)}(G) \leq 3(t+1)+2+\frac{d-1}{2}$, and, from Theorem B we obtain $\chi^{(t)}(G) \leq \frac{9}{2}((7)^{t-1}-1)+6$. Comparing these two values, the upper bound of Theorem 3 is asymptotically better for $t \geq 2$ and $d \ll 7^t$.

References

[1] P. Baldi, *On a generalized family of colourings*, Graphs Combin. **6** (1990) 95–110.

[2] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, London and Elsevier, 1976).

[3] G. Chartrand and R.E. Pippert, *Locally connected graphs*, Čas. pro pěstování matematiky **99** (1974) 158–163.

[4] A. Chen, A. Gyárfás and R.H. Schelp, *Vertex coloring with a distance restriction*, Discrete Math. **191** (1998) 83–90.

[5] P. Holub and L. Xiong, *On Distance local connectivity and the hamiltonian index*, submitted to Discrete Math.

[6] S. Jendrol' and Z. Skupień, *Local structures in plane maps and distance colourings*, Discrete Math. **236** (2001) 167–177.

[7] F. Kramer and H. Kramer, *Un problème de coloration des sommets d'un graphe*, C.R. Acad. Sci. Paris Sér. A–B **268** (1969) A46–A48.

[8] F. Kramer and H. Kramer H, *On the generalized chromatic number*, Ann. Discrete Math. **30** (1986) 275–284.

[9] T. Madaras and A. Marcinová, *On the structural result on normal plane maps*, Discuss. Math. Graph Theory **22** (2002) 293–303.

[10] Z. Ryjáček, *On a closure concept in claw-free graphs*, J. Combin. Theory (B) **70** (1997) 217–224.

[11] Z. Skupień, *Some maximum multigraphs and edge/vertex distance colourings*, Discuss. Math. Graph Theory **15** (1995) 89–106.

Received 15 September 2005
Revised 18 May 2007
Accepted 18 May 2007