Discussiones Mathematicae
Graph Theory 20 (2000) 143–153

Unique factorization theorem

Peter Mihók
Mathematical Institute, Slovak Academy of Sciences
Grešákova 6, 040 01 Košice, Slovak Republic
and
Faculty of Economics
Technical University Košice, Slovakia
e-mail: mihok@kosice.upjs.sk

Abstract

A property of graphs is any class of graphs closed under isomorphism. A property of graphs is induced-hereditary and additive if it is closed under taking induced subgraphs and disjoint unions of graphs, respectively. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. A graph G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable ($G$ has property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$) if the vertex set $V(G)$ of G can be partitioned into n sets $V_1, V_2, \ldots, V_n$ such that the subgraph $G[V_i]$ of G induced by V_i belongs to $\mathcal{P}_i$; $i = 1, 2, \ldots, n$. A property $\mathcal{R}$ is said to be reducible if there exist properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$; otherwise the property $\mathcal{R}$ is irreducible. We prove that every additive and induced-hereditary property is uniquely factorizable into irreducible factors. Moreover the unique factorization implies the existence of uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs for any irreducible properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$.

Keywords: induced-hereditary, additive property of graphs, reducible property of graphs, unique factorization, uniquely partitionable graphs, generating sets.

1991 Mathematics Subject Classification: 05C15, 05C75.

1. Motivation and Main Results

A *property of graphs* is any nonempty class of graphs closed under isomorphism. A property of graphs is called *induced-hereditary* (hereditary) and *additive* if it is closed under taking induced subgraphs (subgraphs) and disjoint unions of graphs, respectively. Induced-hereditary (hereditary) properties are called also hereditary (monotone) (see [3]). Obviously, any hereditary property of graphs is induced-hereditary, too. On the other hand, many well-known induced-hereditary classes of graphs (e.g., complete graphs, line-graphs, claw-free graphs, interval graphs, perfect graphs, etc.) are not hereditary. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. A graph G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*partitionable* ($G$ has property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$) if the vertex set $V(G)$ of G can be partitioned into n sets $V_1, V_2, \ldots, V_n$ such that the subgraph $G[V_i]$ of G induced by V_i belongs to $\mathcal{P}_i$, $i = 1, 2, \ldots, n$. An induced-hereditary property $\mathcal{R}$ is said to be *reducible* if there exist induced-hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$, otherwise the property $\mathcal{R}$ is *irreducible.* The notion of reducible properties have been introduced in connection with generalized graph colouring and the existence of uniquely partitionable graphs (see [6, 10, 8]).

The problem: *"Is the factorization of every property into irreducible properties unique?"* have been stated in the book [8] of Jensen and Toft "Graph Coloring Problems". Partial results for some subclasses of induced-hereditary properties may be found in [11, 12, 9, 13]. In May 1995 (see [11]) we proved *the unique factorization theorem* (UFT) for the additive hereditary properties with completeness at most 3, in June 1996 (see [9]) we proved UFT. The aim of this paper is to prove the unique factorization in the whole class of additive induced-hereditary properties of graphs.

Theorem 1. *Any reducible additive induced-hereditary property is uniquely factorizable into irreducible factors* (*up to the order of factors*).

Since in general for induced-hereditary properties we cannot use the concept of maximal graphs (used for hereditary properties in [13]), we define new concepts — *the operation* "$*$" and $\mathcal{R}$-*decomposability number of a graph.*

Definition. Let $\mathcal{R}$ be an additive induced-hereditary property. For given graphs $G_1, G_2, \ldots, G_n$, $n \geq 2$, denote by

$$G_1 * G_2 * \ldots * G_n = \{G : \bigcup_{i=1}^{n} G_i \subseteq G \subseteq \sum_{i=1}^{n} G_i\},$$

where $\bigcup_{i=1}^{n} G_i$ denotes the disjoint union and $\sum_{i=1}^{n} G_i$ the join of the graphs $G_1, G_2, \ldots, G_n$, respectively.

Let $dec_{\mathcal{R}}(G) = \max\{n :$ there exist a partition$(V_1, V_2, \ldots, V_n), V_i \neq \emptyset$, of $V(G)$ (called $\mathcal{R}$-decomposition of G) such that for each $k \geq 1, k.G[V_1] * k.G[V_2] * \ldots * k.G[V_n] \subseteq \mathcal{R}\}$. If $G \notin \mathcal{R}$ we set $dec_{\mathcal{R}}(G)$ to be zero.

A graph G is said to be $\mathcal{R}$-*decomposable* if $dec_{\mathcal{R}} \geq 2$; otherwise G is $\mathcal{R}$-*indecomposable.*

These new concepts are motivated by the following observation.

Let us suppose that $G \in \mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ and let (V_1, V_2) be a $(\mathcal{P}, \mathcal{Q})$-partition of G. Then by additivity of $\mathcal{P}$ and $\mathcal{Q}$ $k.G[V_1] * k.G[V_2] \subseteq \mathcal{R}$ for every positive integer k. Thus if the property $\mathcal{R}$ is reducible, every graph $G \in \mathcal{R}$ with at least two vertices is $\mathcal{R}$-decomposable.

We shall prove that for any additive reducible induced-hereditary property also the converse assertion holds.

Theorem 2. *An induced-hereditary additive property $\mathcal{R}$ is reducible if and only if all graphs in $\mathcal{R}$ with at least two vertices are $\mathcal{R}$-decomposable.*

The problem of unique factorization have been from the beginning related to the investigation of the existence of uniquely partitionable graphs.

A graph G is said to be *uniquely* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partitionable* if G has exactly one (unordered) $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition $(V_1, V_2, \ldots, V_n)$. Let us denote by $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n)$ the class of all uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs. In the case $\mathcal{P}_1 = \mathcal{P}_2 = \ldots = \mathcal{P}_n = \mathcal{P}$ we write $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n = \mathcal{P}^n$ and we say that G belonging to $\boldsymbol{U}(\mathcal{P}^n)$ is uniquely $(\mathcal{P}, n)$-partitionable.

It turned out that the existence of uniquely partitionable graphs follows from proofs of UFT's. In this paper we prove the conjecture presented in [12].

Theorem 3. *Let $\mathcal{P}$ be an additive induced-hereditary property of graphs. Then for $n \geq 2, \boldsymbol{U}(\mathcal{P}^n) \neq \emptyset$ if and only if $\mathcal{P}$ is irreducible.*

Analogously as for hereditary properties (see [12, 5]) we prove that every reducible additive induced-hereditary property $\mathcal{R}$ can be generated by graphs which are uniquely partitionable with respect to its irreducible factors.

Theorem 4. *Let $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$, $n \geq 2$ be the factorization of a reducible additive induced-hereditary property $\mathcal{R}$ into irreducible factors. Then every graph $G \in \mathcal{R}$ is an induced subgraph of a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph G^*.*

Using the result of A. Berger [2], who proved that every reducible additive induced-hereditary property $\mathcal{P}$ has infinitely many minimal forbidden induced subgraphs, we have the following generalization of the Theorem 1 of [1].

Corollary 5. *Let $\mathcal{P}$ be any induced-hereditary property of graphs defined by a finite set of connected forbidden subgraphs. Then for every positive integer $n > 2$ there exist infinitely many uniquely $(\mathcal{P}, n)$-partitionable graphs.*

The notation and technical preliminary results are presented in Section 2. The proofs of the main Theorems are given in Section 3.

2. Notation and Preliminary Results

All graphs considered in this paper are finite and simple (without multiple edges or loops), the class of all graphs is denoted by $\mathcal{I}$. We use the standard notation (see e.g. [7, 8]. In particular, K_n denotes the complete graph on n vertices, $G \cup H$ denotes the disjoint union of graphs G and H and $k.G$ denotes the disjoint union of k isomorphic copies of G. The symbols $\leq$ and $\subseteq$ stand for the relations "to be an induced subgraph" and "to be a subgraph", respectively. The *join* $\sum_{i=1}^{n} G_i = G_1 + G_2 + \cdots + G_n$ of n graphs $G_1, G_2, \ldots, G_n$ is the graph consisting of the disjoint union of G_i's and all the edges between $V(G_i)$ and $V(G_j)$ for any $1 \leq i < j \leq n$.

A graph $G \in \mathcal{P}$ is said to be $\mathcal{P}$*-maximal* if $G + e \notin \mathcal{P}$ for each $e \in E(\overline{G})$. The structure of graphs maximal with respect to reducible hereditary properties played an important role in the proof of unique factorization of additive and hereditary properties. However for non-hereditary induced-hereditary properties we have to find another way. Let us define the related notion of $\mathcal{P}$-strict graphs using the operation $*$ introduced in Section 1.

Definition. A graph $G \in \mathcal{P}$ is said to be $\mathcal{P}$*-strict* if $G * K_1 \not\subseteq \mathcal{P}$. The class of all $\mathcal{P}$-strict graphs is denoted by $\boldsymbol{S}(\mathcal{P})$.

A set $\mathcal{G} \subset \mathcal{I}$ is said to be a *generating set of* $\mathcal{P}$ if $G \in \mathcal{P}$ if and only if G is an induced subgraph of some graph from $\mathcal{G}$. The fact that $\mathcal{G}$ is a generating set of $\mathcal{P}$ will be written as $[\mathcal{G}] = \mathcal{P}$. The members of $\mathcal{G}$ are called *generators* of $\mathcal{P}$.

Let us show that every graph $G \in \mathcal{P}$ is an induced subgraph of a $\mathcal{P}$-strict graph and hence the class $\boldsymbol{S}(\mathcal{P})$ forms a generating set of $\mathcal{P}$.

Obviously for any property $\mathcal{P} \neq \mathcal{I}$ there exists a graph $F \notin \mathcal{P}$. For a property $\mathcal{P}$ we can therefore define $f(\mathcal{P})$ to be the least number of vertices of a forbidden subgraph of $\mathcal{P}$, i.e. $f(\mathcal{P}) = \min\{|V(F)| : F \notin \mathcal{P}\}$. Now it is easy to see, that for every $G \in \mathcal{P}$ the class $G * K_1 * \ldots * K_1 \not\subseteq \mathcal{P}$ if the number of the K_1's is $f(\mathcal{P}) - 1$ which means that if G is not $\mathcal{P}$-strict, then repeating the operation $*$ with K_1's after less than $f(\mathcal{P})$ steps we will obtain a $\mathcal{P}$-strict graph G' such that $G \leq G'$.

Since $dec_{\mathcal{R}}(G) < f(\mathcal{R})$, this fact allows us to define the decomposability number $dec(\mathcal{G})$ of a generating set $\mathcal{G}$ of $\mathcal{R}$ by

$$dec(\mathcal{G}) = \min\{dec_{\mathcal{R}}(G) : G \in \mathcal{G})\}.$$

Put $dec(\mathcal{R}) = dec(\boldsymbol{S}(\mathcal{R}))$.
The next simple Lemma will be used.

Lemma 6. *Let G be an $\mathcal{R}$-strict graph and G^* be an induced supergraph of G i.e., $G \leq G^*$. Then G^* is $\mathcal{R}$-strict and $dec_{\mathcal{R}}(G) \geq dec_{\mathcal{R}}(G^*)$.*

Proof. The fact that $G^* \in \boldsymbol{S}(\mathcal{R})$ is evident. Suppose, in contrary, that $n = dec_{\mathcal{R}}(G) < dec_{\mathcal{R}}(G^*) = m$ and $d = (V_1, V_2, \ldots, V_m)$ be an $\mathcal{R}$-decomposition of G^*. Then at most $n < m$ sets V_i of d have nonempty intersections with $V(G)$ (otherwise $dec_{\mathcal{R}}(G) > n$) and there is a vertex $z \in V_j$ with $V_j \cap V(G) = \emptyset$, in contradiction with assumption: G be $\mathcal{R}$-strict. ■

We are going to show that there exists a generating set $\mathcal{G}^* \subseteq \boldsymbol{S}(\mathcal{R})$ of $\mathcal{R}$ which contains only graphs G with decomposability number $dec_{\mathcal{R}}(G) = dec(\mathcal{R}) = n$ which are uniquely $\mathcal{R}$-decomposable (i.e., there exist exactly one $\mathcal{R}$-decomposition $(V_1, V_2, \ldots, V_n), V_i \neq \emptyset$, such that for each $k \geq 1, k.G[V_1] * k.G[V_2] * \ldots * k.G[V_n] \subseteq \mathcal{R})$.

The final step of the proof of Theorem 2 and Theorem 1 will consist of the construction of corresponding irreducible factors. Analogously as in [13], by the construction it follows that if $dec(\mathcal{R}) = n$, then $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ where the irreducible factors $\mathcal{P}_i, i = 1, 2, \ldots, n$ are uniquely determined by the structure of the generating set $\mathcal{G}^*$.

Our consideration requires the definitions of appropriate generating sets of $\mathcal{R}$ derived from the set of $\mathcal{R}$-strict graphs. Let us present the simple Lemmas on the properties of generating sets. We omit their simple proofs analogous to those given for maximal graphs in [13] (see also [14]).

Lemma 7. *Let $\mathcal{R}$ be an induced-hereditary property of graphs and let $\mathcal{G}$ be a generating set of $\mathcal{R}$. If all graphs belonging to $\mathcal{G}$ are $\mathcal{R}$-decomposable, then all $\mathcal{R}$-strict graphs are $\mathcal{R}$-decomposable, too.*

Lemma 8. *Let $\mathcal{P}$ be an additive induced-hereditary property and $\mathcal{G}$ be a generating set of $\mathcal{P}$. If G is an arbitrary graph with property $\mathcal{P}$ then there exists a generating set $\mathcal{G}' \subseteq \mathcal{G}$ such that each graph $H \in \mathcal{G}'$ contains at least one copy of the graph G.*

Lemma 9. *Let $\mathcal{P}$ be an induced-hereditary property of graphs. Let $\mathcal{G}$ be a generating set of $\mathcal{P}$ such that $\mathcal{G} \subseteq \boldsymbol{S}(\mathcal{P})$. Then $dec(\mathcal{G}) = dec(\mathcal{P})$.*

Lemma 10. *Let $\mathcal{P}$ be an additive induced-hereditary property of graphs. Let $\mathcal{G} \subseteq \boldsymbol{S}(\mathcal{P})$ be any generating set of $\mathcal{P}$. Then there exists a set $\mathcal{G}^*$, $\mathcal{G}^* \subseteq \mathcal{G}$, which is a generating set of $\mathcal{P}$ and contains only graphs of $\mathcal{P}$-decomposability number equal to $dec(\mathcal{P})$.*

Now, let us prove the main Lemma of this paper.

Lemma 11. *For every $\mathcal{R}$-strict graph G with $dec_{\mathcal{R}}(G) = dec(\mathcal{R}) \geq 2$ there is a uniquely $\mathcal{R}$-decomposable graph $G^* \geq G$.*

Proof. Let G be a fixed $\mathcal{R}$-strict graph with $dec_{\mathcal{R}}(G) = n$ and $d_i = (V_{i1}, V_{i2}, \ldots, V_{in})$, $i = 0, 1, \ldots, m$, $m \geq 0$ be all $\mathcal{R}$-decompositions of G. Since G is a finite graph, m is a nonnegative integer.

We shall construct a uniquely $\mathcal{R}$-decomposable graph $G^* = G^*(m)$ taking an appropriate number s of disjoint copies of G so that $V(G^*) = V(s.G)$ and $E(G^*) = E(s.G) \cup E^*(m)$ where new edges $e \in E^*(m)$ are joining vertices of different copies of G only. By Lemma 6 we have $dec_{\mathcal{R}}(G^*(m)) = dec_{\mathcal{R}}(G) = n$.

Every $\mathcal{R}$-decomposition $d = (V_1^*, V_2^*, \ldots, V_n^*)$ of G^* restricted to any copy G gives some $\mathcal{R}$-decomposition d_j of G denoted by $d|G = d_j$. The aim of our construction is to add new edges $E^*(m)$ to $s.G$ so that the obtained graph $G^*(m)$ will have only one $\mathcal{R}$-decomposition d such that $d|G = d_0$ for each copy G of $s.G$.

To proceed we shall use two types of constructions:

Construction 1. $G^i \Leftrightarrow G^j$:
Let G^i, G^j be two different copies of G in $s.G$. Since G is $\mathcal{R}$-strict, $G * K_1 \not\subseteq \mathcal{R}$. Let us fix a graph $F \in G * K_1$, $F \notin \mathcal{R}$ and let $N_F(z)$ be the neighbours of $z \in V(K_1)$ in G. Let us denote by $Z_j = V_{0j} \cap N_F(z), j = 1, 2, \ldots, n$ the neighbours of z in $G[V_{0j}]$ with respect to the $\mathcal{R}$-decomposition d_0 of G. Let $G^i, G^j, i \neq j$ be disjoint copies of G, d_0 be the $\mathcal{R}$-decomposition of G and v be a vertex of $G^i \cup G^j$. Add new edges $E^*(G^i \Leftrightarrow G^j)$ so that every vertex $v \in V_{0k}$ of the corresponding $\mathcal{R}$-decomposition d_0 is adjacent to every vertex of $Z_j, j \neq k$ of the other copy of $G^i \cup G^j$.

The resulting graph $G^i \Leftrightarrow G^j$ has the following property: for every $\mathcal{R}$-decomposition $d = (U_1, U_2, \ldots U_n)$ of $G^i \Leftrightarrow G^j$ it holds that $d|G^i = d_0$ if and only if $d|G^j = d_0$. The proof of this fact is simple, suppose that

$d|G^i = d_0$ ($d|G^j = d_0$) and $v \in V_{0k}$ of $G^j(G^i)$ does not belong to U_k. Then d cannot be an $\mathcal{R}$-decomposition of $H = G^i \Leftrightarrow G^j$ since there is a graph in $H[U_1] * H[U_2] * \ldots * H[U_n]$ which contains an induced copy of F (we can add the appropriate edges between v and Z_k.

Construction 2. $n \bullet k(r,s).G$:
Let d_r and d_s be different $\mathcal{R}$-decompositions of G, denote by $A_{ij}(r,s) = V_{ri} \cap V_{sj}, i,j = 1,2,\ldots,n$. Since $d_r \neq d_s$ at least $n+1$ sets $A_{ij}(r,s)$ are nonempty. Because of $dec_{\mathcal{R}}(G) = n$ there exists a positive integer $k(r,s)$ such that $k(r,s).G[A_{11}(r,s)]*k(r,s).G[A_{12}(r,s)]*\ldots*k(r,s).G[A_{nn}(r,s)] \not\subseteq \mathcal{R}$. Let fix a graph $F(r,s) \in k(r,s).G[A_{11}(r,s)] * k(r,s).G[A_{12}(r,s)] * \ldots * k(r,s).G[A_{nn}(r,s)], F(r,s) \notin \mathcal{R}$. Denote by $E_{ij,i'j'}(r,s)$ the set of edges of $F(r,s)$ joining the vertices of $k(r,s).G[A_{ij}(r,s)]$ and $k(r,s).G[A_{i'j'}(r,s)]$.

Let us construct the graph $H^*(r,s) = n \bullet k(r,s).G$ taking n disjoint copies of $H = k(r,s).G$, denoted by $H^j, j = 1,2,\ldots n$. Add new edges joining different copies of $k(r,s).G$ so that the edges $E_{ji,ki}(r,s), i = 1,2,\ldots,n$ be realized between the copies H^j and $H^k, j \neq k$, i.e. for example the edges $E_{11,21}(r,s)$ and $E_{12,22}(r,s)$ etc., of the graph $F(r,s)$ are placed between H^1 and H^2.

The construction 2 gives a graph $H^* = H^*(r,s)$ without an $\mathcal{R}$-decomposition $d = (W_1, W_2, \ldots, W_n)$ such that $d|G = d_s$ for each induced copy of G in H^* because otherwise the graph $F(r,s)$ would appear in $H^*[W_1] * H^*[W_2] * \ldots * H^*[W_n]$.

We are ready to prove the Lemma 11 by constructing G^*:
If $m = 0$, then $G^* = G$ and we are done. In this case $d_0 = (V_{01}, V_{02}, \ldots, V_{0n})$ is the unique $\mathcal{R}$-decomposition of G^*.

If $m \geq 1$ we proceed recurrently:

Universal Step 0. Let $G^0 = G$ be a fixed copy of G and $G(m)$ be a graph consisting of s copies of G (denoted by $G^1, G^2, \ldots, G^s$) (to be described recurrently below). For every $m \geq 1$ add edges between G^0 and $G^i, i = 1,2,\ldots,s$ by Construction 1 so that $U = V(G^0) \cup V(G^i)$ induces in resulting graph G^* the subgraph $G^*[U] = G^0 \Leftrightarrow G^i$. This part of the construction of G^* yields that if for a $\mathcal{R}$-decomposition d of G^* there exists a G^k in $G(m)$ such that $d|G^k = d_0$ then for every $i = 0,1,2,\ldots,s$ $d|G^i = d_0$ implying G^* has unique $\mathcal{R}$-decomposition.

Step 1. Let us denote by $G(1)$ the graph $H^*(0,1) = n \bullet k(0,1).G$ — see Construction 2. Let $G^*(1)$ be obtained from G^0 and $G(1)$ according to the Step 0 ($s = n.k(0,1)$). If $m = 1$, then the graph $G^*(1)$ has unique $\mathcal{R}$-decomposition d since Construction 2 is forcing at least one copy of G of $G(1)$ to have $d|G^j = d_0$ so that by Construction 1 all copies $G^0, G^1, \ldots G^{n.k(0,1)}$ of $G^*(1)$ must have $d|G^i = d_0$.

Step *j*. For $j \geq 2$, let $G(j-1)$ be the graph constructed in the Step $j-1$. To construct $G(j)$ let us take $n.k(0,j)$ disjoint copies of $G(j-1)$ and add new edges inserting $H^*(0,j) = n \bullet k(0,j).G$ for every choice of $n.k(0,j)$ copies of G one by one from different copies of $G(j-1)$.

Let the graph $G^*(j)$ be obtained from G^0 and $G(j)$ according to the Step 0. Let d be a $\mathcal{R}$-decomposition of $G^*(j)$. Suppose that there is a G^k such that $d|G^k \neq d_0$. Then $d|G^i = d_j$ for a copy G^i of G from each copy of $G(j-1)$, since otherwise for every G^i $d|G^i = d_0$ by step $j-1$. However if every copy of $G(j-1)$ should have a copy of G with $d|G = d_j$, then a copy of $H^*(0,j)$ is forcing a contradiction.

The uniquely $\mathcal{R}$-decomposable graph $G^* = G^*(m)$ is obtained in the Step m. ■

Let $\mathcal{G}^*(\mathcal{R})$ denotes the class of all uniquely $\mathcal{R}$-decomposable graphs with $\mathcal{R}$-decomposibility number $n = dec(\mathcal{R}) \geq 2$. By Lemma 11 $\mathcal{G}^*$ is a generating set of $\mathcal{R}$. Using Lemma 11 we can proceed the same way as for hereditary properties in [13].

First let us describe the structure of the generators of $\mathcal{G}^*(\mathcal{R})$. Let $\mathcal{G}^* = \mathcal{G}^*(\mathcal{R}) = \{G_i; i \in I\}$ and let $(V_i^1, V_i^2, \ldots, V_i^n)$ be the unique $\mathcal{R}$-decomposition of G_i. The graphs $G_i^j = G_i[V_i^j]$ are called *indecomposable-parts* of the generator G_i. The set of all indecomposable-parts of graphs belonging to $\mathcal{G}^*$ will be denoted by $Ip\ (\mathcal{R})$ so that if $Ip\ (G_i) = \{G_i^j, j = 1, 2, \ldots n\}$ then $Ip\ (\mathcal{R}) = \bigcup_{i \in I} Ip\ (G_i)$. For $F \in Ip\ (\mathcal{R})$ and $G_k \in \mathcal{G}^*$ let us denote by $m(F, G_k)$ the number of different (possibly isomorphic) ind-parts of G_k which $m(F) = \max\{m(F, G_i); G_i \in \mathcal{G}^*\}$. The positive integer $m(F)$ is called *the multiplicity of the ind-part* $F \in Ip\ (\mathcal{R})$ *in* $\mathcal{R}$. Obviously for every $F \in Ip\ (\mathcal{R}) : 1 \leq m(F) \leq n = dec(\mathcal{P})$.

A technical Lemma analogous to Lemma 2.6 from [13] holds.

Lemma 12. *Let $\mathcal{G}^* \subseteq \boldsymbol{S}(\mathcal{R})$ be the generating set of $\mathcal{R}$ consisting of uniquely $\mathcal{R}$-decomposable graphs with decomposibility number $n = dec(\mathcal{R})$. Let G be an arbitrary graph from $\mathcal{G}^*$ and let $(V^1, V^2, \ldots V^n)$ be its unique $\mathcal{R}$-decomposition. If a graph $H \in \mathcal{G}^*$ contains G as an induced-subgraph, then the ind-parts G^j of G, $j \in \{1, 2, \ldots, n\}$, are induced-subgraphs of different ind-parts H^k of H, $k \in \{1, 2, \ldots, n\}$.*

Proof. If some ind-parts G^i, G^j, $i \neq j$ are induced-subgraphs of the same ind-part H^k ($k \in \{1, 2, \ldots, n\}$) of H, then there is at least one ind-part of H which has empty intersection with G. But this contradicts the assumption that G is $\mathcal{R}$-strict.

Now, suppose that an ind-part, say G^1, be an induced-subgraph of at least two different ind-parts H^j, H^k, $j \neq k$, of H. Then $G^1[V^1) \cap V(H^j)] * G^1[V^1 \cap V(H^k)] * G^2 * \ldots * G^n \subseteq \mathcal{R}$ which is in contradiction to $dec_{\mathcal{R}}(G) = n$. ■

3. The Proofs of the Main Results

We are prepared to prove the main results. The proof of Theorem 2 is analagous as for additive hereditary properties in [13]. We recall it to present full insight into the structure of irreducible factors.

***Proofs of Theorems* 1 *and* 2.** Let every graph $G \in \mathcal{R}$ with at least two vertices be $\mathcal{R}$-decomposable. We will find the factorization of the property $\mathcal{R}$ into at least two irreducible factors $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$.

Let $\mathcal{G}^* \subseteq \boldsymbol{S}(\mathcal{R})$ be the generating set consisting of of all uniquely $\mathcal{R}$-decomposable graphs of decomposability number $n = dec(\mathcal{R})$ and let $Ip\ (\mathcal{R})$ be the set of all ind-parts of $\mathcal{R}$. We distinguish two cases:

Case 1. Let us suppose that there exists an ind-part $F \in Ip\ (\mathcal{R})$ of multiplicity $m(F) = k$ where $k < dec(\mathcal{P})$. Let $G \in \mathcal{G}^*$ be a generator of $\mathcal{P}$ for which $m(F, G) = k$. Let us consider, in accordance with Lemma 8, the generating set $\mathcal{G}^*_G \subseteq \mathcal{G}^*$ such that $\mathcal{G}^*_G = \{H \in \mathcal{G}^*; G \leq H\}$. By the definition of $\mathcal{G}^*_G$ and by Lemma 12 for every generator $H \in \mathcal{G}^*_G$, $m(F, H) = k$. Let the induced-hereditary property $\mathcal{Q}_1$ ($\mathcal{Q}_2$) be generated by the subgraphs induced by union of vertices of k ($n - k$) ind-parts of generators $H \in \mathcal{G}^*_G$ containing (not containing) the ind-part F.

Let us show that $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2$. It is easy to see that $\mathcal{R} \subseteq \mathcal{Q}_1 \circ \mathcal{Q}_2$. Let $H^* \in \mathcal{Q}_1 \circ \mathcal{Q}_2$. Then $H^* \in H_1^* * H_2^*$ where H_1^* (H_2^*) is the subgraph induced by the union of vertices of $k(n - k)$ ind-parts of some generator $H_1(H_2) \in \mathcal{G}^*_G$ which contain (do not contain) the ind-part F. Let $G^* \in \mathcal{G}^*_G$ be such a graph that $H_1 \cup H_2 \leq G^*$. By Lemma 12 and by the definition of $\mathcal{G}^*_G$ $H_1 * H_2 \subseteq \mathcal{R}$ implying $H^* \in \mathcal{R}$. Hence $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2$.

The additivity of $\mathcal{Q}_1, \mathcal{Q}_2$: suppose that the graphs H_1^* and H_2^* belong to $\mathcal{Q}_1(\mathcal{Q}_2)$. Then H_i^* is a subgraph of the join of $k(n - k)$ ind-part of some generator $H_i \in \mathcal{G}^*_G$ containing (not containing) the ind-part F, $i \in \{1, 2\}$. If $G^* \in \mathcal{G}^*_G$ such that $H_1 \cup H_2 \leq G^*$, then by Lemma 12 and by the definition of $\mathcal{G}_G^*$, both H_1^* and H_2^* are induced subgraphs of $k(n-k)$ ind-parts of G^* which contain (do not contain) the ind-part F as an induced subgraph. Then $H_1^* \cup H_2^* \in \mathcal{Q}_1(\mathcal{Q}_2)$ and hence $\mathcal{Q}_1, \mathcal{Q}_2$ are additive.

Case 2. Suppose that $m(F) = n = dec(\mathcal{R}) \geq 2$ for each $F \in Ip\ (\mathcal{R})$. Let $\mathcal{Q}$ be an induced-hereditary property generated by $Ip\ (\mathcal{R})$. It is easy to see that $\mathcal{R} \subseteq \mathcal{Q}^n$. The converse inclusion, $\mathcal{Q}^n \subseteq \mathcal{R}$, and the additivity of $\mathcal{Q}$ follows analogously as in the Case 1. The proof of Theorem 2 is finished.

To complete the proof of Theorem 1 we use induction on $n = dec(\mathcal{R})$. If $n = 1$, the property $\mathcal{R}$ is irreducible. Let us suppose that every property with decomposability number $1 \leq k < n$ has a unique factorization into irreducible factors and let $\mathcal{R}$ be a property with $dec(\mathcal{R}) = n$.

The structure of the factorization of the property $\mathcal{R}$ depends on the multiplicities of the ind-parts of $\mathcal{R}$ as described above. This factorization is uniquely determined because the generators of $\mathcal{R}$ are uniquely $\mathcal{R}$-decomposable into ind-parts. Suppose there exists an ind-part F of $\mathcal{R}$ with multiplicity $m(F) = k < dec(\mathcal{P}) = n$. Then we consider the properties $\mathcal{Q}_1$ and $\mathcal{Q}_2$ defined in the Case 1. By the induction hypothesis they are uniquely factorizable into irreducible factors. Since the generators of $\mathcal{R}$ are uniquely $(\mathcal{Q}_1, \mathcal{Q}_2)$-partitionable, the proof is complete.

If for every ind-part F of $\mathcal{R}$ its multiplicity $m(F)$ in $\mathcal{R}$ is equal to n, then $\mathcal{R} = \mathcal{Q}^n$ by the Case 2.

***Proofs of Theorems* 3 *and* 4.** Let $\mathcal{R}$ be any reducible, additive induced-hereditary property. We proved above that the property $\mathcal{R}$ can be generated by a class $\mathcal{G}^*$ of graphs with decomposibility number $n \geq 2$ which are uniquely $\mathcal{R}$-decomposable into n indecomposable parts generating the corresponding irreducible factors. It means that if $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$, $n \geq 2$ be the factorization of $\mathcal{R}$ into irreducible factors, then every generator from $\mathcal{G}^*$ is uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable. On the other hand, let a property $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$ be reducible, then obviously there are no uniquely $(\mathcal{P}, n)$-partitionable graphs since the parts belonging to $\mathcal{P}_2$ in any $(\mathcal{P}^n)$-partition of G are interchangeable.

Acknowledgement

The autor is very grateful to M. Borowiecki I. Broere, G. Semanišin, R. Vasky and especially to J. Ivančo for fruitful discussions and their help in the preparation of this paper.

REFERENCES

[1] D. Achlioptas, J.I. Brown, D.G. Corneil and M.S.O. Molloy, *The existence of uniquely $-G$ colourable graphs*, Discrete Math. **179** (1998) 1–11.

[2] A. Berger, *Reducible properties have infinitely many minimal forbidden subgraphs*, manuscript.

[3] B. Bollobás and A.G. Thomason, *Hereditary and monotone properties of graphs*, in: R.L. Graham and J. Nešetřil, eds., The mathematics of Paul Erdős, II, Algorithms and Combinatorics vol. 14 (Springer-Verlag, 1997), 70–78.

[4] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *Survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50.

[5] I. Broere, J. Bucko, *Divisibility in additive hereditary graph properties and uniquely partitionable graphs*, Tatra Mountains Math. Publications **18** (1999) 79–87.

[6] E.J. Cockayne, *Color clasess for r-graphs*, Canad. Math. Bull. **15** (3) (1972) 349–354.

[7] R.L. Graham, M. Grötschel and L. Lovász, Handbook of combinatorics (Elsevier Science B.V., Amsterdam, 1995).

[8] T.R. Jensen and B. Toft, Graph colouring problems (Wiley-Interscience Publications, New York, 1995).

[9] J. Kratochvíl, P. Mihók, *Hom-properties are uniquely factorizable into irreducible factors*, Discrete Math. **213** (2000) 189–194.

[10] P. Mihók, *Additive hereditary properties and uniquely partitionable graphs*, in: M. Borowiecki and Z. Skupien, eds., Graphs, hypergraphs and matroids (Zielona Góra, 1985), 49–58.

[11] P. Mihók and R. Vasky, *On the factorization of reducible properties of graphs into irreducible factors*, Discuss. Math. Graph Theory **15** (1995) 195–203.

[12] P. Mihók, *Reducible properties and uniquely partitionable graphs*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Volume **49** (1999) 213–218.

[13] P. Mihók, G. Semanišin and R. Vasky, *Additive and Hereditary Properties of Graphs are Uniquely Factorizable into Irreducible Factors*, J. Graph Theory **33** (2000) 44–53.

[14] G. Semanišin, *On generating sets of induced-hereditary properties*, manuscript.

Received 22 March 2000
Revised 4 May 2000