

## SOME CONJECTURES ON PERFECT GRAPHS

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The complement of a graph  $G$  is denoted by  $\overline{G}$ .  $\chi(G)$  denotes the chromatic number and  $\omega(G)$  the clique number of  $G$ . The cycles of odd length at least five are called *odd holes* and the complements of odd holes are called *odd anti-holes*.

### 1.

A graph  $G$  is called *perfect* if, for each induced subgraph  $G'$  of  $G$ ,  $\chi(G') = \omega(G')$ . Classical examples of perfect graphs consist of bipartite graphs, chordal graphs and comparability graphs. Examples of *nonperfect* graphs are the odd holes and odd anti-holes. The most important result on perfect graphs is the following one, due to L. Lovász.

**The Perfect Graph Theorem** ([6]). *The complement of a perfect graph is perfect.*

The Perfect Graph Theorem used to be the so-called weak perfect graph conjecture posed by C. Berge around 1960s. His stronger conjecture on perfect graphs, which is still open, is as follows.

**The Strong Perfect Graph Conjecture.** *Graphs without induced odd holes and odd anti-holes are perfect.*

Clearly, if the Strong Perfect Graph Conjecture is true, it implies the Perfect Graph Theorem. See [1, 2, 3, 4] for more information on perfect graphs.

## 2.

The edge-version of graph perfection has been considered by L.E. Trotter [8]. It can be formulated in terms of line graphs as follows. The well-known *line graph*  $L(G)$  of a graph  $G$  has the edge set of  $G$  as its vertex set, and two distinct edges of  $G$  are adjacent in  $L(G)$  if and only if they have an endvertex in common; see Figure 1. A graph  $H$  is a *line graph* if there exists a graph  $G$  such that  $H$  is (isomorphic to)  $L(G)$ . It is well-known that line graphs can be recognized in linear time.

A graph  $G$  is called *line perfect* if its line graph  $L(G)$  is perfect. It is well-known that  $L(G)$  is perfect if and only if  $L(G)$  contains no induced odd holes. Trotter proved that a graph is line perfect if and only if it has no (not necessary induced) odd holes. It then follows easily that line perfect graphs are perfect.

By definition,  $L(G)$  is perfect if and only if, for all induced subgraphs  $H$  of  $L(G)$ ,  $\chi(H) = \omega(H)$ . Since the induced subgraphs of  $L(G)$  are in one-to-one correspondence with the line graphs of subgraphs of  $G$ ,  $L(G)$  is perfect if and only if, for all subgraphs  $G'$  of  $G$ ,  $\chi(L(G')) = \omega(L(G'))$ .

Call a graph  $G$  *weakly line perfect* if, for all *induced* subgraphs  $G'$  of  $G$ ,  $\chi(L(G')) = \omega(L(G'))$ . Clearly, line perfect graphs are weakly line perfect. The graph  $G$  in Figure 1 is weakly line perfect but not line perfect ( $L(G)$  contains an induced odd hole of length five). It is easy to see that weakly line perfect graphs cannot have induced odd holes and odd anti-holes. Thus, the Strong Perfect Graph Conjecture implies the following

**Conjecture A.** *Weakly line perfect graphs are perfect.*

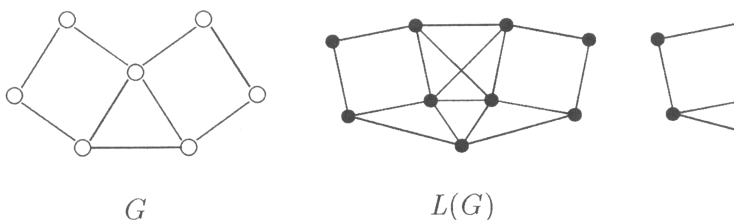


Figure 1. The line graph and the Gallai graph of a graph

## 3.

The *Gallai graph*  $\Gamma(G)$  of a graph  $G$  has the edge set of  $G$  as its vertex set, and two distinct edges of  $G$  are adjacent in  $\Gamma(G)$  if, and only if, they have

an endvertex in common and the two other endvertices are nonadjacent in  $G$ ; see Figure 1. Thus,  $\Gamma(G)$  is a spanning subgraph of the line graph  $L(G)$ .

A graph  $H$  is a *Gallai graph* if there exists a graph  $G$  such that  $H$  is (isomorphic to)  $\Gamma(G)$ . Note that, in contrast to line graphs, not every induced subgraph of  $\Gamma(G)$  is again a Gallai graph.

**Problem B.** *Recognize Gallai graphs in polynomial time, or prove that recognizing Gallai graphs is NP-complete.*

A nice connection between Gallai graphs and perfect graphs is the following: Call the graph  $G$  *Gallai perfect* if its Gallai graph  $\Gamma(G)$  has no induced odd holes. L. Sun proved

**Theorem ([7]).** *Gallai perfect graphs are perfect.*

#### 4.

It is shown in [5] that if  $\Gamma(G)$  contains an induced odd anti-hole, it also contains an induced odd hole. Thus, the Strong Perfect Graph Conjecture implies the following

**Conjecture C ([5]).** *Gallai graphs without induced odd holes are perfect.*

If Conjecture C is true, it implies Sun's Theorem (see [5]). Also, it is easy to see that Conjecture C implies a theorem of D. König, saying that line graphs of bipartite graphs are perfect. In [5], Conjecture C is proved for Gallai graphs  $\Gamma(G)$  of graphs  $G$  with  $\chi(\overline{G}) \leq 4$ .

#### 5.

Sun's Theorem implies, in particular, that  $G$  is perfect if  $\Gamma(G)$  is perfect. By definition,  $\Gamma(G)$  is perfect if and only if, for all induced subgraphs  $H$  of  $\Gamma(G)$ ,  $\chi(H) = \omega(H)$ .

Recall that not every induced subgraph of a Gallai graph is again a Gallai graph. This motivates the following definition: Call the graph  $G$  *weakly Gallai perfect* if, for all induced subgraphs  $G'$  of  $G$ ,  $\chi(\Gamma(G')) = \omega(\Gamma(G'))$ .

The graph  $G$  in Figure 1 is weakly Gallai perfect but not Gallai perfect ( $\Gamma(G)$  contains an induced odd hole of length seven). It is not clear whether all Gallai perfect graphs are weakly Gallai perfect. However, if Conjecture C is true, they are.

It is easy to see that weakly Gallai perfect graphs cannot contain odd holes and odd anti-holes (indeed, if  $C$  is an odd hole or an odd anti-hole, then  $\chi(\Gamma(C)) = 3$  while  $\omega(\Gamma(C)) = 2$ ). So, the Strong Perfect Graph Conjecture implies the following

**Conjecture D.** *Weakly Gallai perfect graphs are perfect.*

## 6.

It is shown in [5] that, for all graphs  $G$ ,  $\chi(\Gamma(G)) \leq \chi(\overline{G})$ . Thus, if  $G$  is perfect, then  $\chi(\Gamma(G')) \leq \omega(\overline{G'})$  and  $\chi(\Gamma(\overline{G'})) \leq \omega(G')$  hold for all induced subgraphs  $G'$  of  $G$ . We conjecture that these properties characterize perfect graphs.

**Conjecture E** ([5, 2]). *A graph  $G$  is perfect if and only if, for all induced subgraphs  $G'$  of  $G$ ,  $\chi(\Gamma(G')) \leq \omega(\overline{G'})$  and  $\chi(\Gamma(\overline{G'})) \leq \omega(G')$ .*

This conjecture has a “semi-strong” property: It implies the Perfect Graph Theorem and it is implied by the Strong Perfect Graph Conjecture.

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