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CONNECTIVITY OF PATH GRAPHS

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Abstract

We prove a necessary and sufficient condition under which a connected graph has a connected P_3-path graph. Moreover, an analogous condition for connectivity of the P_k-path graph of a connected graph which does not contain a cycle of length smaller than $k+1$ is derived.

Keywords: connectivity, path graph, cycle.

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1 Introduction

Let G be a graph, $k \geq 1$, and let $\mathcal{P}_k$ be the set of all paths of length k (i.e., with $k+1$ vertices) in G. The vertex set of a *path graph* $P_k(G)$ is the set $\mathcal{P}_k$. Two vertices of $P_k(G)$ are joined by an edge if and only if the edges in the intersection of the corresponding paths form a path of length $k-1$, and their union forms either a cycle or a path of length $k+1$. It means that the

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vertices are adjacent if and only if one can be obtained from the other by "shifting" the corresponding paths in G.

Path graphs were investigated by Broersma and Hoede in [2] as a natural generalization of line graphs, since $P_1(G)$ is the line graph of G. We have to point out that, in the pioneering paper [2] the number k in $P_k(G)$ denotes the number of vertices of the paths and not their length. However, in some applications our notation is more consistent, see e.g., [3]. Traversability of P_2-path graphs is studied in [9], and a characterization of P_2-path graphs is given in [2] and [7]. Distance properties of path graphs are studied in [1], [4] and [5], and [6] and [8] are devoted to isomorphisms of path graphs.

Let $V = V(G)$ be a set of n distinct symbols. Consider strings of length $k+1$ of these symbols, in which all $k+1$ symbols are mutually distinct. Let G be a graph on vertex set V, edges of which correspond to pairs of symbols which can be neighbours in our strings. If we do not distinguish between a string and its reverse, then $P_k(G)$ is connected if and only if every string can be obtained from any other one sequentially, by removing a symbol from one of its ends and adding a symbol to the other end.

Let G be a connected graph. It is well-known (and trivial to prove) that $P_1(G)$, i.e., the line graph of G, is a connected graph. However, this is not the case for P_k-path graphs if $k \geq 2$. This causes some problems, especially when studying distances in path graphs. For example, in [1] the authors give an upper bound for the diameter of every component of a P_k-path graph, as the whole graph can be disconnected. By [4, Theorem 1], we have:

Theorem A. *Let G be a connected graph. Then $P_2(G)$ is disconnected if and only if G contains two distinct paths A and B of length two, such that the degrees of both endvertices of A are* 1 *in G.*

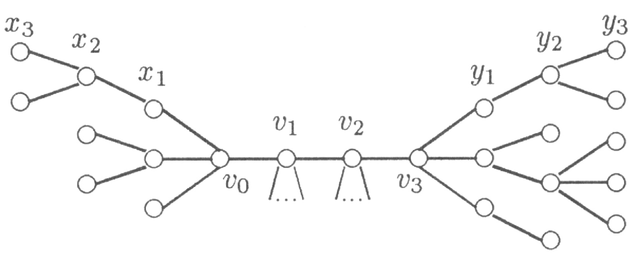
In this paper we generalize Theorem A to P_k-path graphs when G does not contain a cycle of length smaller than $k+1$. Moreover, we completely solve the case of P_3-path graphs.

We use standard graph-theoretic notation. Let G be a graph. The vertex set and the edge set of G, respectively, are denoted by $V(G)$ and $E(G)$. For two subgraphs, H_1 and H_2 of G, by $H_1 \cup H_2$ we denote the union of H_1 and H_2, and $H_1 \cap H_2$ denotes their intersection. Let u and v be vertices in G. By $d_G(u, v)$ we denote the distance from u to v in G, and by $deg_G(u)$ the degree of u is denoted. For the vertex set of a component of G containing u we use $Co(u)$. A path and a cycle, respectively, of length l are denoted by P_l and C_l.

The outline of the paper is as follows. In Section 2 we give a (necessary and sufficient) condition for a connected graph (under some restrictions) to have a connected P_k-path graph, and Section 3 is devoted to an analogous condition for P_3-path graphs of general graphs.

2 P_k-Path Graphs

Let G be a graph, $k \geq 2$, $0 \leq t \leq k-2$, and let A be a path of length k in G. By $P^*_{k,t}$ we denote an induced subgraph of G which is a tree of diameter $k+t$ with a diametric path $(x_t, x_{t-1}, \ldots, x_1, v_0, v_1, \ldots, v_{k-t}, y_1, y_2, \ldots, y_t)$, such that all endvertices of $P^*_{k,t}$ have distance $\leq t$ either to v_0 or to v_{k-t} and the degrees of $v_1, v_2, \ldots, v_{k-t-1}$ are 2 in $P^*_{k,t}$. Moreover, no vertex of $V(P^*_{k,t}) - \{v_1, v_2, \ldots, v_{k-t-1}\}$ is joined by an edge to a vertex in $V(G) - V(P^*_{k,t})$. The path $(v_0, v_1, \ldots, v_{k-t})$ is a *base* of $P^*_{k,t}$, and we say that A *lies in* $P^*_{k,t}$, $A \in P^*_{k,t}$, if and only if the base of $P^*_{k,t}$ is a subpath of A.



In Figure 1 a $P^*_{6,3}$ is pictured. Note that this graph contains also two $P^*_{6,0}$ and one $P^*_{6,1}$, but it does not contain $P^*_{6,2}$. We remark that by thin halfedges are outlined possible edges joining vertices of $P^*_{6,3}$ to vertices in $V(G)-V(P^*_{6,3})$.

In this section we prove the following theorem.

Theorem 1. *Let G be a connected graph without cycles of length smaller than $k+1$. Then $P_k(G)$ is disconnected if and only if G contains $P^*_{k,t}$, $0 \leq t \leq k-2$, and a path A of length k such that $A \notin P^*_{k,t}$.*

For easier handling of paths of length k in G (i.e., the vertices of $P_k(G)$) we adopt the following convention. We denote the vertices of $P_k(G)$ (as well as the vertices of G) by small letters a, b, $\ldots$, while the corresponding paths of length k in G will be denoted by capital letters A, B, $\ldots$. It means that if A is a path of length k in G and a is a vertex in $P_k(G)$, then a must be the vertex corresponding to the path A.

Lemma 2. *Let G be a connected graph without cycles of length smaller than $k+1$. Moreover, let $A = (x_0, x_1, \dots, x_k)$ be a path of length k in G which is not in $P^*_{k,t}$, $0 \le t \le k-2$. Then for every i, $0 \le i \le k$, there is an $a_i \in Co(a)$ such that x_i is an endvertex of A_i and the edge of A_i incident with x_i lies in A.*

Proof. Observe that if there is a vertex $a_i \in Co(a)$ such that x_i is an endvertex of A_i, then choosing a_i with $d_{P_k(G)}(a, a_i)$ smallest possible, the endedge of A_i incident with x_i is in A.

Thus, suppose that for some i, $0 < i < k$, there is no $a_i \in Co(a)$ such that x_i is an endvertex of A_i. Let H be a subgraph of G formed by the vertices and edges of paths A', where $a' \in Co(a)$. Clearly, $(x_{i-1}, x_i, x_{i+1}) \subseteq A'$ for every $a' \in Co(a)$. Let $R = (v_0, v_1, \dots, v_{k-t})$ be the longest path that share all A', $a' \in Co(a)$. As $k-t \ge 2$, we have $t \le k-2$. Further, $deg_H(v_1) = deg_H(v_2) = \dots = deg_H(v_{k-t-1}) = 2$, and every endvertex of H has distance $\le t$ either to v_0 or to v_{k-t}. Since H does not contain cycles (recall that the length of every cycle in G is at least $k+1$), H is $P^*_{k,t}$, $0 \le t \le k-2$. As $R \subseteq A$ we have $A \in P^*_{k,t}$, a contradiction. ■

Let A and B be two paths of length k in G. If one endvertex of B, say x, lies in A, but the edge of B incident with x is not in A, then we say that the pair (A, B) *forms T with a touching vertex x*.

Note that if (A, B) forms T in G, then $A \cup B$ is not necessarily a tree even if G does not contain a cycle of length $\le k$.

Lemma 3. *Let G be a graph without cycles of length smaller than $k+1$. Moreover, suppose G does not contain $P^*_{k,t}$, $0 \le t \le k-2$, and let (A, B) form T in G. Then $b \in Co(a)$.*

Proof. Let (A, B) form T with a touching vertex x. By Lemma 2, there is $a' \in Co(a)$ such that x is an endvertex of A' and the edge of A' incident with x lies in A. As G does not contain a cycle of length smaller than $k+1$, we have $d_{P_k(G)}(a', b) \le k$, and hence $b \in Co(a)$. ■

Now we are able to prove Theorem 1.

***Proof of Theorem* 1.** We arrange the proof into three steps.

(i) First suppose that G contains some $P^*_{k,t}$, $0 \le t \le k-2$, with a base $R = (v_0, v_1, \dots, v_{k-t})$, and a path A of length k such that $A \notin P^*_{k,t}$. Since the diameter of $P^*_{k,t}$ is $k+t$, there is a path B of length k in G such that $B \in P^*_{k,t}$, i.e., $R \subseteq B$. By the structure of $P^*_{k,t}$, for every vertex b' of $P_k(G)$

which is adjacent to b we have $R \subseteq B'$, too. Hence, for every $b' \in Co(b)$ it holds $R \subseteq B'$. Since A does not contain R, we have $a \notin Co(b)$, so that $P_k(G)$ is a disconnected graph.

(ii) Now suppose that G contains some $P^*_{k,t}$, $0 \leq t \leq k-2$, such that for every $a \in V(P_k(G))$ it holds $A \in P^*_{k,t}$. We show that either $P_k(G)$ is a connected graph, or G contains $P^*_{k,t'}$, $0 \leq t' < t$, and a path B of length k such that $B \notin P^*_{k,t'}$.

Let $R = (v_0, v_1, \ldots, v_{k-t})$ be the base of $P^*_{k,t}$, and let b be a vertex of $P_k(G)$ such that $B \in P^*_{k,t}$ and v_0 is an endvertex of B (e.g., choose B as a part of a diametric path of $P^*_{k,t}$). Let a be a vertex of $P_k(G)$, $A \in P^*_{k,t}$. If there is $a' \in Co(a)$ such that either v_0 or v_{k-t} is an endvertex of A', then either $d_{P_k(G)}(a', b) \leq 2t$ or $d_{P_k(G)}(a', b) = t$ (by the structure of $P^*_{k,t}$ we have $R \subseteq A'$). Hence, $a \in Co(b)$.

Thus, suppose that there is a vertex a in $P_k(G)$, $A \in P^*_{k,t}$, such that for every $a' \in Co(a)$ neither v_0 nor v_{k-t} is an endvertex of A'. Let H be a subgraph of G formed by the vertices and edges of paths A', for which $a' \in Co(a)$. Clearly, $R \subseteq A'$ for every $a' \in Co(a)$. Let $R' = (v'_0, v'_1, \ldots, v'_{k-t'})$ be the longest path that share all A', $a' \in Co(a)$. Since $R \subset R'$, by the choice of A we have $v_0 = v'_i$, $v_1 = v'_{i+1}$, $\ldots$, $v_{k-t} = v'_{i+k-t}$, where $i > 0$ and $i+k-t < k-t'$, i.e., $t' < t - i$. Further, $deg_H(v'_1) = deg_H(v'_2) = \ldots = deg_H(v'_{k-t-1}) = 2$, and every endvertex of H has distance $\leq t'$ either to v'_0 or to $v'_{k-t'}$. Since H does not contain cycles, H is $P^*_{k,t'}$, $0 \leq t \leq k-2$. As $R' \nsubseteq B$, we have $B \notin P^*_{k,t'}$.

(iii) Finally, suppose that G does not contain $P^*_{k,t}$, $0 \leq t \leq k-2$. We show that $P_k(G)$ is a connected graph.

Let $a, b \in V(P_k(G))$. First suppose that $A \cap B$ does not contain an edge. Let $P = (y_0, y_1, \ldots, y_l)$ be a shortest path in G joining a vertex of A with a vertex of B (i.e., $y_l \in V(B)$). By Lemma 2, there is $b' \in Co(b)$ such that y_l is an endvertex of B' and the edge of B' incident with y_l lies in B. Let $B' = (b'_0, b'_1, \ldots, b'_{k-1}, y_l)$. Then $P' = (b'_0, b'_1, \ldots, b'_{k-1}, y_l, y_{l-1}, \ldots, y_0)$ is a walk of length $k + l$. Since G does not contain a cycle of length $\leq k$, every subwalk of P' of length k is a path. Let B'' be a subpath of length k of P' with endvertex y_0. Then $d_{P_k(G)}(b', b'') \leq l$, and hence $b'' \in Co(b)$. As (A, B'') forms T in G, we have $b \in Co(a)$, by Lemma 3.

Now suppose that $A \cap B$ contains an edge. Let $P = (y_0, y_1, \ldots, y_l)$ be a longest path that is shared by A and B. By Lemma 2, for every i, $0 \leq i \leq l$, there is $b_i \in Co(b)$ such that y_i is an endvertex of B_i, and the edge of B_i incident with y_i lies in B. If B_0 does not contain the edge y_0y_1, then (A, B_0) forms T in G, so that $b \in Co(a)$, by Lemma 3. Analogously, if

B_l does not contain $y_{l-1}y_l$, then $b \in Co(a)$. Thus, suppose that B_0 contains the edge y_0y_1 and B_l contains $y_{l-1}y_l$. Then there is some i, $0 \leq i < l$, such that both B_i and B_{i+1} contain the edge y_iy_{i+1}. By Lemma 2, there is $a' \in Co(a)$ such that y_i is an endvertex of A' and the edge of A' incident with y_i lies in A. If A' contains the edge y_iy_{i+1}, then $d_{P_k(G)}(a', b_{i+1}) \leq k-1$, and hence $b \in Co(a)$. On the other hand, if A' does not contain y_iy_{i+1}, we have $d_{P_k(G)}(a', b_i) \leq k$, and hence $b \in Co(a)$ as well. ■

3 P_3-Path Graphs

Let G be a graph and let A be a path of length three in G. By P_3° we denote a subgraph of G induced by vertices of a path of length 3, say (v_0, v_1, v_2, v_3), such that neither v_0 nor v_3 has a neighbour in $V(G) - \{v_1, v_2\}$. We say that the path A is in P_3°, $A \in P_3^\circ$, if $A = (v_0, v_1, v_2, v_3)$.

By P_4° we denote an induced subgraph of G with a path (x, v_0, v_1, v_2, y), in which every neighbour of v_0 (and analogously every neighbour of v_2), except v_0, v_1 and v_2, has degree 1, or it has degree 2 and in this case it is adjacent to v_1. Moreover, no vertex of $V(P_4^\circ) - \{v_1\}$ is joined by an edge to a vertex of $V(G) - V(P_4^\circ)$ in G. The path (v_0, v_1, v_2) is a *base* of P_4°, and we say that the *path A lies in* P_4°, $A \in P_4^\circ$, if the base of P_4° is a subpath of A.

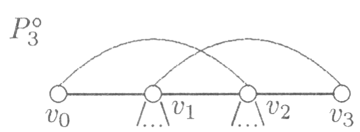


Figure 2

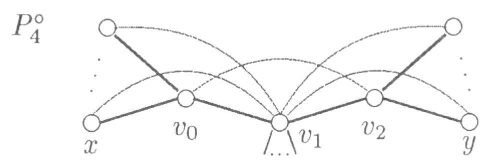


Figure 3

On example of a graph P_3° is pictured in Figure 2 and a graph P_4° in Figure 3. The edges that must be in G are painted thick, while edges, that are not necessarily in G, are painted thin.

Let K_4 be a complete graph on 4 vertices, and let S be a set (possibly empty) of independent vertices. A graph obtained from $K_4 \cup S$ by joining all vertices of S to one special vertex of K_4 is denoted by K_4^*, see Figure 4. Let $K_{2,t}$ be a complete bipartite graph, $t \geq 1$, and let (X, Y) be the bipartition of $K_{2,t}$, $X = \{v_1, v_2\}$. Join t sets of independent vertices by edges, each to one vertex of Y; further, glue a set of stars (each with at least 3 vertices) by one endvertex, each either to v_1 or to v_2; glue a set of triangles by one vertex, each either to v_1 or to v_2; and finally, join v_1 to v_2 by an edge. The resulting graph is denoted by $K^*_{2,t}$, see Figure 5.

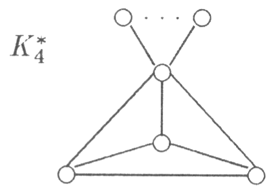


Figure 4

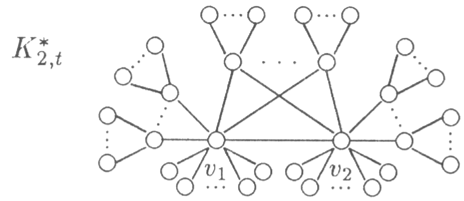


Figure 5

Theorem 4. *Let G be a connected graph such that $P_3(G)$ is not empty. Then $P_3(G)$ is disconnected if and only if one of the following holds:*

(1) *G contains P_t°, $t \in \{3,4\}$, and a path A of length 3 such that $A \notin P_t^\circ$;*

(2) *G is isomorphic to K_4^*;*

(3) *G is isomorphic to $K^*_{2,t}$, $t \geq 1$.*

If $A \in P_3^\circ$ in G, then a is an isolated vertex in $P_3(G)$, and if $A \in P_4^\circ$, then a lies in a complete bipartite graph. Thus, we have the following corollary of Theorem 4.

Corollary 5. *Let G be a connected graph that is not isomorphic to K_4^* or to $K^*_{2,t}$, $t \geq 1$. Then at most one nontrivial component of $P_3(G)$ is different from a complete bipartite graph.*

In the proof of Theorem 4 we use 6 lemmas.

Lemma 6. *Let G be a connected graph, and let a and b be vertices in $P_3(G)$. If neither A nor B is in some P_3° or P_4° in G, then there are vertices c and d in $P_3(G)$, such that $c \in Co(a)$, $d \in Co(b)$ and C and D share an edge in G.*

Proof. Let $A \cap B$ do not contain an edge, and let $P = (y_0, y_1, \ldots, y_l)$ be a shortest path in G joining a vertex of A with a vertex of B (i.e., $y_l \in V(B)$). We show that there is a vertex b' in $Co(b)$, such that y_l is an endvertex of B'.

Suppose that there is no vertex b' with the required property. Then $B = (x_0, x_1, y_l, x_3)$, and since B is not in P_3° in G, there is a vertex $\overline{b}$ in $P_3(G)$ such that $\overline{b}b \in E(P_3(G))$. By our assumption, $\overline{B} = (x_1, y_l, x_3, x_4)$ for some $x_4 \in V(G)$. Moreover, for every neighbour u of b we have $U = (x_1, y_l, x_3, z)$, where z has no neighbours in $V(G) - \{y_l, x_3\}$; and for every neighbour v of $\overline{b}$ we have $V = (z, x_1, y_l, x_3)$, where z has no neighbours in $V(G) - \{x_1, y_l\}$. Hence B is in some P_4°, a contradiction.

Thus, there is a vertex $b' \in Co(b)$, such that y_l is an endvertex of B'. Let b'' be the first vertex on a shortest $b - b'$ path in $P_3(G)$, such that one endvertex of B'' is in P. Assume that $B'' = (b''_3, b''_2, b''_1, y_i)$. Then $P' = (b''_3, b''_2, b''_1, y_i, y_{i-1}, \ldots, y_0)$ is a path of length $i+3 \geq 3$. Let B^* be a subpath of P of length 3, such that y_0 is an endvertex of B^*. Then $d_{P_3(G)}(b'', b^*) = i$, and hence, $b^* \in Co(b)$.

Denote $B^* = (y_0, b^*_1, b^*_2, b^*_3)$, and suppose that $A \cap B^*$ does not contain an edge. Let $A = (a_0, a_1, a_2, a_3)$. Distinguish two cases.

(i) $y_0 = a_1$. Then $b^*_1 \neq a_0$ and $b^*_1 \neq a_2$, so that at least one of a_0 and a_2, say a_0, is different from b^*_2. Since a_0 is not an interior vertex of B^*, $D = (a_0, y_0, b^*_1, b^*_2)$ is a path of length 3 in G. As $b^*d \in E(P_3(G))$, we have $d \in Co(b)$ and $A \cap D$ contains an edge.

(ii) $y_0 = a_0$. If $b^*_1 \neq a_2$ then $C = (b^*_1, y_0, a_1, a_2)$ is a path of length 3 in G, $c \in Co(a)$, $b^* \in Co(b)$, and $C \cap B^*$ contains an edge. On the other hand, if $b^*_1 = a_2$ then $D = (a_1, y_0, a_2, b^*_2)$ is a path of length 3 in G, $d \in Co(b)$, and $A \cap D$ contains an edge. ∎

Lemma 7. *Let G be a connected graph, and let a and b be two vertices in $P_3(G)$ such that $b \notin Co(a)$ and $A \cap B$ contains a path of length two. Moreover, suppose G does not contain P_3° or P_4°. Then G is isomorphic either to K_4^* or to $K_{2,t}^*$ for some $t \geq 1$.*

Proof. Let $A = (x_0, x_1, x_2, x_3)$ and $B = (x_0, x_1, x_2, x_4)$, $x_3 \neq x_4$. Since $b \notin Co(a)$, x_0 has no neighbour in $V(G) - \{x_1, x_2\}$. Thus, both x_3 and x_4 have some neighbours in $V(G) - \{x_1, x_2\}$, as G does dot contain P_3°. Let y be a vertex of G such that $x_1 y \in E(G)$ and $y \notin \{x_0, x_2, x_3, x_4\}$. Then $a' \in Co(a)$ and $b' \in Co(b)$, where $A' = (y, x_1, x_2, x_3)$ and $B' = (y, x_1, x_2, x_4)$. Since $b \notin Co(a)$ we have $b' \notin Co(a')$, and hence, y has no neighbour in $V(G) - \{x_1, x_2\}$.

Suppose that $x_3 x_4 \in E(G)$ and distinguish three cases.

Case 1. $x_1 x_3, x_1 x_4 \in E(G)$, see Figure 6.
Let G' be a graph obtained from G by joining x_0 to x_2. Then A, (x_1, x_2, x_3, x_4), (x_2, x_3, x_4, x_1), (x_3, x_4, x_1, x_0), (x_4, x_1, x_0, x_2), (x_1, x_0, x_2, x_4), (x_0, x_2, x_4, x_3), (x_2, x_4, x_3, x_1), (x_1, x_2, x_4, x_3), B is a sequence of paths

whose images produce a walk of length 9 from a to b in $P_3(G')$. (We remark that $d_{P_3(G')}(a,b) = 9$.) Thus $b \in Co(a)$, a contradiction. Hence $deg_G(x_0) = 1$.

Let $C_1 = (x_1, x_2, x_3, x_4)$ and $C_2 = (x_1, x_2, x_4, x_3)$ be two cycles of length 4 in G. For every subpath A' of C_1 of length 3 we have $a' \in Co(a)$, and for every subpath B' of C_2 of length 3 we have $b' \in Co(b)$. Let y be a vertex in $V(G) - \{x_1, \ldots, x_4\}$ which is joined to some $x \in \{x_1, \ldots, x_4\}$. Since $C_1 \cap C_2$ contains an edge incident with x, there are paths A'' and B'' of length 3 in G, both containing the edge yx, such that $a'' \in Co(a)$, $b'' \in Co(b)$ and $A'' \cap B''$ contains P_2. Thus, analogously as above it can be shown that $deg_G(y) = 1$. Finally, as G does not contain P_3° we have $x = x_1$, and hence $G \cong K_4^*$.

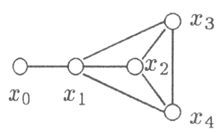


Figure 6

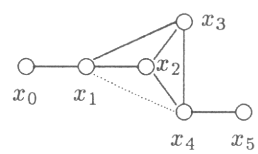


Figure 7

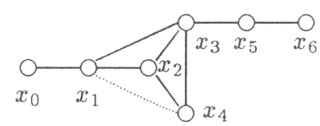


Figure 8

Case 2. $x_1x_3 \in E(G)$ and $x_1x_4 \notin E(G)$, see Figure 7 and Figure 8 (by dotted lines edges that are missing in G are pictured).

Since (x_1, x_2, x_3) is not a base of P_4°, either there is a vertex $y \in V(G) - \{x_0, \ldots, x_4\}$ such that $yx_4 \in E(G)$, or there is a path of length 2 glued by one endvertex to x_3 (the other vertices of the path are not in $\{x_0, \ldots, x_4\}$).

First suppose that there is $x_5 \in V(G) - \{x_0, \ldots, x_4\}$ such that $x_4x_5 \in E(G)$, see Figure 7. Let G' be a graph obtained from G by joining x_0 to x_2. Then A, (x_1, x_2, x_3, x_4), (x_2, x_3, x_4, x_5), (x_0, x_2, x_3, x_4), (x_1, x_0, x_2, x_3), (x_3, x_1, x_0, x_2), (x_4, x_3, x_1, x_0), (x_2, x_4, x_3, x_1), (x_1, x_2, x_4, x_3), B is a sequence of paths whose images produce a walk of length 9 from a to b in $P_3(G')$. Thus $b \in Co(a)$, a contradiction.

Hence $deg_G(x_0) = 1$. Analogously, for every vertex x, such that $xx_2, xx_3 \in E(G)$, every neighbour of x (different from x_2 and x_3) has degree 1 in G.

Let y_1 and y_2 be vertices in $V(G) - \{x_0, \ldots, x_5\}$, such that $x_2y_1, y_1y_2 \in E(G)$. If y_2 is joined by an edge to a vertex, say z, of $V(G) - \{x_2, y_1\}$, then for $C = (x_2, y_1, y_2, z)$ we have $c \in Co(a)$ and $c \in Co(b)$. Hence $b \in Co(a)$, a contradiction. Since G contains P_3° if there is a vertex of degree 1 joined to x_2, we have $G \cong K_{2,t}^*$ for some $t \geq 2$.

Now suppose that there are $x_5, x_6 \in V(G) - \{x_0, \ldots, x_4\}$ such that $x_3x_5, x_5x_6 \in E(G)$, see Figure 8. (Observe that the cases $x_6 \in \{x_0, x_1, x_4\}$ imply $b \in Co(a)$.)

Let G' be a graph obtained from G by joining x_0 to x_2. Then A, (x_1, x_2, x_3, x_5), (x_2, x_3, x_5, x_6), (x_0, x_2, x_3, x_5), (x_1, x_0, x_2, x_3), (x_3, x_1, x_0, x_2), (x_4, x_3, x_1, x_0), (x_2, x_4, x_3, x_1), (x_1, x_2, x_4, x_3), B is a sequence of paths whose images produce a walk of length 9 from a to b in $P_3(G')$. Thus $b \in Co(a)$, a contradiction.

Hence $deg_G(x_0) = 1$. Analogously, for every vertex x, such that $xx_2, xx_3 \in E(G)$, every neighbour of x (different from x_2 and x_3) has degree 1 in G. Now analogously as above it can be shown that $G \cong K^*_{2,t}$ for some $t \geq 2$.

Case 3. $x_1x_3, x_1x_4 \notin E(G)$, see Figure 9.
Since neither (x_1, x_2, x_3) nor (x_1, x_2, x_4) is a base of P°_4, there is a vertex $x_5 \in V(G) - \{x_0, \ldots, x_4\}$ which is adjacent either to x_3 or to x_4. Assume that $x_3x_5 \in E(G)$. As $b \notin Co(a)$, x_5 has no neighbour in $\{x_0, x_1, x_4\}$. Since (x_1, x_2, x_3) is not a base of P°_4, there is a vertex $y \in V(G) - \{x_0, \ldots, x_5\}$ such that either $yx_5 \in E(G)$ or $yx_4 \in E(G)$.

First suppose that there is a vertex $x_6 \in V(G) - \{x_0, \ldots, x_5\}$ such that $x_5x_6 \in E(G)$. Then every neighbour of x_4 (different from x_2 and x_3) has degree 1 in G, otherwise $b \in Co(a)$. Analogously, for every vertex x, such that $xx_2, xx_3 \in E(G)$, every neighbour of x (different from x_2 and x_3) has degree 1 in G. Thus, analogously as above we have $G \cong K^*_{2,t}$ for some $t \geq 1$.

If there is $x_6 \in V(G) - \{x_0, \ldots, x_5\}$ such that $x_4x_6 \in E(G)$, then the problem is reduced to the previous case as (x_3, x_2, x_4) is not a base of P°_4.

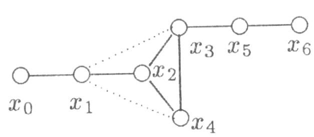


Figure 9

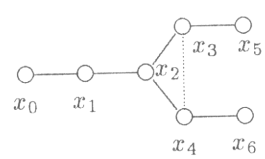


Figure 10

To prove the lemma it remains to consider the case $x_3x_4 \notin E(G)$, see Figure 10.

As $b \notin Co(a)$, there is no cycle $(x_3, x_2, x_4, \ldots)$ of length at least 4 in G. Since neither A nor B is in P°_3 in G, there are $x_5, x_6 \in V(G) - \{x_0, \ldots, x_5\}$, $x_5 \neq x_6$, such that $x_3x_5, x_4x_6 \in E(G)$. Moreover, as G does not contain

P_4° with base (x_1, x_2, x_3), there is $x_7 \in V(G) - \{x_0, \ldots, x_6\}$ such that $x_5x_7 \in E(G)$, and analogously, there is $x_8 \in V(G) - \{x_0, \ldots, x_7\}$ such that $x_6x_8 \in E(G)$. (Observe that $b \in Co(a)$ if $x_7 = x_1$, and the same holds if $x_8 = x_1$.) But now $d_{P_3(G)}(a, b) \le 7$, and hence $b \in Co(a)$, a contradiction. ■

Lemma 8. *Let G be a connected graph, and let a and b be two vertices in $P_3(G)$ such that $b \notin Co(a)$ and $A \cap B$ contains two independent edges. Moreover, suppose G does not contain P_3° or P_4°. Then G is isomorphic either to K_4^* or to $K_{2,t}^*$ for some $t \ge 1$, or there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains a path of length* 2.

Proof. Let $A = (x_0, x_1, x_2, x_3)$. Since $b \notin Co(a)$, $B = (x_0, x_1, x_3, x_2)$. We may assume that x_0 has no neighbour in $V(G) - \{x_0, \ldots, x_3\}$, otherwise there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains P_2.

Distinguish three cases.

Case 1. $x_0x_2, x_0x_3 \in E(G)$. Then both A and B lie in cycles of length 4. If there is a vertex y adjacent to a vertex of $\{x_0, \ldots, x_4\}$, then there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains P_2. Thus, $G \cong K_4$ which is a special K_4^*.

Case 2. $x_0x_2 \in E(G)$ and $x_0x_3 \notin E(G)$, see Figure 11. Since A is not in P_3° in G, there is a vertex $x_4 \in V(G) - \{x_0, \ldots, x_3\}$ such that $x_3x_4 \in E(G)$. But then $a' \in Co(a)$, $b' \in Co(b)$ and $A' \cap B'$ contains P_2, where $A' = (x_1, x_2, x_3, x_4)$ and $B' = (x_0, x_2, x_3, x_4)$.

Case 3. $x_0x_2, x_0x_3 \notin E(G)$, see Figure 12. Since neither A nor B is in P_3° in G, there are vertices $x_4, x_5 \in V(G) - \{x_0, \ldots, x_3\}$ such that $x_2x_4, x_3x_5 \in E(G)$. We may assume that the degree of every neighbour of x_1 (except x_2 and x_3) is 1 in G, as the other possibilities we have already solved.

If $x_4 \ne x_5$, then there are $x_6, x_7 \in V(G) - \{x_0, \ldots, x_3\}$ such that $x_4x_6, x_5x_7 \in E(G)$, as neither (x_1, x_3, x_2) nor (x_1, x_2, x_3) is a base of P_4°. But then $b \in Co(a)$, a contradiction.

Thus, suppose that $x_4 = x_5$. By previous subcase, we may assume that $deg_G(x_2) = deg_G(x_3) = 3$. As (x_1, x_2, x_3) is not a base of P_4°, there is $x_5 \in V(G) - \{x_0, \ldots, x_4\}$ such that $x_4x_5 \in E(G)$. By our assumptions, $deg_G(x_5) = 1$. Hence, $deg_G(x_0) = deg_G(x_5) = 1$, $deg_G(x_2) = deg_G(x_3) = 3$, and all neighbours of x_1 and x_4 (except x_2 and x_3) have degree 1 in G. Thus, $G \cong K^*_{2,2}$. ■

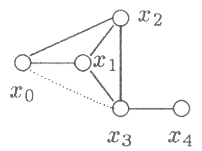


Figure 11

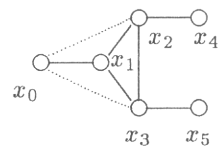


Figure 12

Lemma 9. *Let G be a connected graph, and let a and b be two vertices in $P_3(G)$ such that $b \notin Co(a)$ and $A \cap B$ contains exactly one edge and two vertices outside this edge. Moreover, suppose G does not contain P_3° or P_4°. Then there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains two independent edges.*

Proof. Let $A = (x_0, x_1, x_2, x_3)$. Then either $B = (x_0, x_2, x_1, x_3)$ or $B = (x_1, x_2, x_0, x_3)$.

First suppose that $B = (x_0, x_2, x_1, x_3)$. Since A is not in P_3° in G, either $x_0x_3 \in E(G)$ or $x_3x_4 \in E(G)$ for some $x_4 \in V(G) - \{x_0, \ldots, x_3\}$. In both these cases there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains two independent edges.

Now suppose that $B = (x_1, x_2, x_0, x_3)$. Then for $A' = (x_1, x_2, x_3, x_0)$ we have $a' \in Co(a)$, and $A' \cap B$ contains two independent edges. ■

Lemma 10. *Let G be a connected graph, and let a and b be two vertices in $P_3(G)$ such that $b \notin Co(a)$ and $A \cap B$ contains exactly one edge and one vertex outside this edge. Moreover, suppose G does not contain P_3° or P_4°. Then there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains two edges.*

Proof. Let $A = (x_0, x_1, x_2, x_3)$, and let x_4 be a vertex of B lying outside A. Distinguish four cases.

Case 1. Suppose that x_1x_2 is the middle edge of B. Then $B = (x_3, x_1, x_2, x_4)$. If x_4 has a neighbour in $V(G) - \{x_1, x_2\}$, then for $B' = (x_0, x_1, x_2, x_4)$ we have $b' \in Co(b)$ and $A \cap B' = P_2$. Thus, we may assume that both x_0 and x_4 have no neighbour in $V(G) - \{x_1, x_2\}$. However, then there is some P_3° in G, a contradiction.

Case 2. Suppose that x_1x_2 is an endedge of B.
If $B = (x_1, x_2, x_0, x_4)$, then for $A' = (x_4, x_0, x_1, x_2)$ we have $a' \in Co(a)$ and $A' \cap B$ contains two independent edges.

If $B = (x_1, x_2, x_4, x_0)$ then $b \in Co(a)$; and if $B = (x_1, x_2, x_4, x_3)$, then for $B' = (x_0, x_1, x_2, x_4)$ we have $b' \in Co(b)$ and $A \cap B' = P_2$.

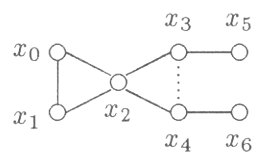


Figure 13

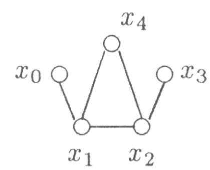


Figure 14

Case 3. Suppose that x_0x_1 is an endedge of B and x_1 is an endvertex of B. If $B = (x_1, x_0, x_4, x_2)$, $B = (x_1, x_0, x_3, x_4)$, or $B = (x_1, x_0, x_4, x_3)$, then $b \in Co(a)$. Thus, suppose that $B = (x_1, x_0, x_2, x_4)$, see Figure 13.

If $deg_G(x_1) > 2$, then for $B' = (x_1, x_0, x_2, x_3)$ we have $b' \in Co(b)$ and $A \cap B'$ contains two independent edges. Thus, suppose that $deg_G(x_0) = deg_G(x_1) = 2$.

If $x_3x_4 \in E(G)$, then analogously as above we have $deg_G(x_3) = deg_G(x_4) = 2$, and hence, there is P_4° with base (x_0, x_2, x_3) in G, a contradiction. Thus, suppose that $x_3x_4 \notin E(G)$.

As $b \notin Co(a)$, there is no cycle $(x_3, x_2, x_4, \ldots)$ of length at least 4 in G. Since neither A nor B is in P_3° in G, there are $x_5, x_6 \in V(G) - \{x_0, \ldots, x_4\}$, $x_5 \neq x_6$, such that $x_3x_5, x_4x_6 \in E(G)$. Moreover, as G does not contain P_4° with base (x_0, x_2, x_3), there is a vertex $x_7 \in V(G) - \{x_0, \ldots, x_6\}$ such that $x_5x_7 \in E(G)$. Thus, for $A' = (x_6, x_4, x_2, x_3)$ and $B' = (x_0, x_2, x_4, x_6)$ we have $a' \in Co(a)$, $b' \in Co(b)$ and $A' \cap B' = P_2$.

Case 4. Suppose that x_0x_1 is an endedge of B and x_0 is an endvertex of B.

If $B = (x_0, x_1, x_4, x_3)$, then $b \in Co(a)$. Since the cases $B = (x_0, x_1, x_4, x_2)$ and $B = (x_0, x_1, x_3, x_4)$ are equivalent, suppose that $B = (x_0, x_1, x_4, x_2)$, see Figure 14.

We have $x_0x_3 \notin E(G)$, otherwise $b \in Co(a)$. Since A is not in P_3° in G, there is $y \in V(G) - \{x_0, \ldots, x_3\}$ such that either $x_0y \in E(G)$ or $x_3y \in E(G)$. Assume that $x_0y \in E(G)$. If $y \neq x_4$, then for $A' = (y, x_0, x_1, x_2)$ and $B' = (y, x_0, x_1, x_4)$ we have $a' \in Co(a)$, $b' \in Co(b)$ and $A' \cap B' = P_2$. On the other hand, if $y = x_4$, then for $A' = (x_2, x_4, x_0, x_1)$ we have $a' \in Co(a)$ and

$A' \cap B$ contains two independent edges. ∎

Lemma 11. *Let G be a connected graph, and let a and b be two vertices in $P_3(G)$ such that $b \notin Co(a)$ and $A \cap B$ contains exactly one edge and no vertex outside this edge. Moreover, suppose G does not contain P_3° or P_4°. Then there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains two edges.*

Proof. Let $A = (x_0, x_1, x_2, x_3)$, and let x_4 and x_5 be vertices of B lying outside A. If $A' \cap B'$ does not contain P_2 for every $a' \in Co(a)$ and $b' \in Co(b)$, then either $B = (x_0, x_1, x_4, x_5)$ or $B = (x_4, x_1, x_2, x_5)$.

First suppose that $B = (x_0, x_1, x_4, x_5)$, see Figure 15. If there is $y \in V(G) - \{x_1, x_2\}$ such that $yx_3 \in E(G)$, then for $A' = (x_5, x_4, x_1, x_2)$ we have $a' \in Co(a)$ and $A' \cap B = P_2$. Hence, we may assume that x_3 has no neighbour in $V(G) - \{x_1, x_2\}$. Since A is not in P_3° in G, there is $y \in V(G) - \{x_1, x_2\}$ such that $yx_0 \in E(G)$. If $y \neq x_4$, then for $A' = (y, x_0, x_1, x_2)$ and $B' = (y, x_0, x_1, x_4)$ we have $a' \in Co(a)$, $b' \in Co(b)$ and $A' \cap B' = P_2$. On the other hand, if $x_0x_4 \in E(G)$, then for $A' = (x_5, x_4, x_0, x_1)$ we have $a' \in Co(a)$ and $A' \cap B$ contains two edges.

Thus, suppose that $B = (x_4, x_1, x_2, x_5)$. Since A is not in P_3° in G, we may assume that there is $y \in V(G) - \{x_1, x_2\}$ such that $x_0y \in E(G)$. Then for $A' = (x_0, x_1, x_2, x_5)$ we have $a' \in Co(a)$ and $A' \cap B = P_2$. ∎

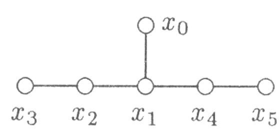


Figure 15

Now we prove Theorem 4.

Proof of Theorem 4. First suppose that G contains P_3° and a path A of length 3 such that $A \notin P_3^\circ$. Then there is a path B of length 3 in G such that $B \in P_3^\circ$. Since b is an isolated vertex in $P_3(G)$, $b \notin Co(a)$. Now suppose that G contains P_4°, and choose $B \in P_4^\circ$. For every vertex $b' \in Co(b)$, B' contains the base of P_4°. Hence, $P_3(G)$ is disconnected if there is a path A of length 3 such that $A \notin P_4^\circ$.

If G is isomorphic to K_4^*, then $P_3(G)$ has three components, each containing C_4. Finally, if G is isomorphic to $K_{2,t}^*$, $t \geq 1$, and $P_3(G)$ is not empty, then some paths of length 3 in G contain the edge v_1v_2, while the

other do not, see Figure 5. Let $a \in V(P_3(G))$ such that $v_1v_2 \in A$. Then $v_1v_2 \in A'$ for every $a' \in Co(a)$, so that $P_3(G)$ is a disconnected graph.
To prove the "only if" part of Theorem 4, first suppose that G contains P_t°, $t \in \{3,4\}$, but no path A of length 3 such that $A \notin P_t^\circ$. If G contains P_3°, then our assumption implies that G is a path of length 3. On the other hand, if G contains P_4° and there is no P_3° in G, then G is a tree of diameter 4 and $P_3(G)$ is a complete bipartite graph. Thus, in what follows we restrict our considerations to graphs which do not contain P_t°, $t \in \{3,4\}$.

Let G be a graph which does not contain P_3° or P_4°, and let a and b be vertices of $P_3(G)$ such that $b \notin Co(a)$. By Lemma 6, there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains an edge. Hence, G is either isomorphic to K_4^* or to $K_{2,t}^*$, $t \geq 1$, by Lemmas 7, 8, 9, 10 and 11. ■

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