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HEREDITARY DOMINATION AND INDEPENDENCE PARAMETERS

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Abstract

For a graphical property $\mathcal{P}$ and a graph G, we say that a subset S of the vertices of G is a $\mathcal{P}$-set if the subgraph induced by S has the property $\mathcal{P}$. Then the $\mathcal{P}$-domination number of G is the minimum cardinality of a dominating $\mathcal{P}$-set and the $\mathcal{P}$-independence number the maximum cardinality of a $\mathcal{P}$-set. We show that several properties of domination, independent domination and acyclic domination hold for arbitrary properties $\mathcal{P}$ that are closed under disjoint unions and subgraphs.

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1. Introduction

There are many variations on the domination number of a graph. In the book [6], it is proposed that a type of domination is "fundamental" if
(i) every connected nontrivial graph has a domination set of this type; and
(ii) this type of dominating set S is defined in terms of some "natural" property of the subgraph induced by S. Examples include domination, total domination, independent domination, connected domination and paired domination. More recently, Hedetniemi et al. [9] proposed acyclic domination, and Haynes and Henning [8] proposed path-free domination.

Formally, a *property* is a set of graphs (closed under isomorphism). We define four parameters of a graph G for any property $\mathcal{P}$. We say that a set S of vertices is a $\mathcal{P}$-set if the subgraph $\langle S\rangle$ induced by S is in $\mathcal{P}$.

- $\gamma_{\mathcal{P}}(G)$ is the minimum cardinality of a dominating set that is a $\mathcal{P}$-set;
- $\beta_{\mathcal{P}}(G)$ is the maximum cardinality of a $\mathcal{P}$-set;
- $i_{\mathcal{P}}(G)$ is the minimum cardinality of a maximal $\mathcal{P}$-set;
- $\Gamma_{\mathcal{P}}(G)$ is the maximum cardinality of a minimal dominating set that is a $\mathcal{P}$-set.

The generalized independence number $\beta_{\mathcal{P}}(G)$ has been studied before (for example in [2]) with various notations. The other general definitions do not appear in the literature, though recently Michalak [10] has independently introduced the same parameters. Hedetniemi et al. [9] studied the case where $\mathcal{P}$ is the set of acyclic graphs, while Haynes and Henning [8] studied the case where $\mathcal{P}$ is the set of all P_k-free graphs for some fixed k.

We note that if the property $\mathcal{P}$ contains all edgeless graphs, then every maximal independent set is a $\mathcal{P}$-set and thus the four parameters exist for all graphs. We will call such a property *nondegenerate*.

There are two special properties: we will denote by *all* the property of all graphs and by 0 the property of being edgeless. Thus the four standard parameters are γ_{all}, i_0, β_0 and Γ_{all}. It is well known that, in fact, $i_0 = \gamma_0$. (For graphical parameters μ and ν, we will write $\mu \geq \nu$ if $\mu(G) \geq \nu(G)$ for all G, and so on.) In an attempt to reduce clutter, we will drop the subscript when the property is all graphs; so γ and Γ have their normal meanings (and we will avoid using β or i).

In the spirit of independent domination and acyclic domination, we focus on a general property which is additive and hereditary. A property is *hereditary* if it closed under subgraphs, and *additive* if it is closed under disjoint unions. For example, edgeless graphs, acyclic graphs, or graphs with maximum degree at most k are additive and hereditary. Michalak [10] has considered these parameters where the property is additive and induced-hereditary, where *induced-hereditary* means closed under induced subgraphs. (This is more general then hereditary.) Note that an additive induced-hereditary property is always nondegenerate.

A key aspect of hereditary properties is that, unless $\mathcal{P}$ is all graphs, there is a largest complete graph $K_{M_\mathcal{P}}$ in $\mathcal{P}$, and no graph has clique number exceeding $M_\mathcal{P}$. Further,

if $\gamma(G) \leq M_\mathcal{P}$, then $\gamma_\mathcal{P}(G) = \gamma(G)$.

In general, we will say that a property $\mathcal{P}$ is *clique-bounded* if there is a number $M_\mathcal{P}$ such that $K_{M_\mathcal{P}}$ is the largest complete graph in $\mathcal{P}$, and no graph in $\mathcal{P}$ has clique number exceeding $M_\mathcal{P}$.

The relationship between generalized colorings and hereditary properties, and hereditary properties themselves, are now well explored. See, for example, [3] or [1].

2. Comparable and Incomparable Parameters

For any nondegenerate property $\mathcal{P}$, from the definitions,

$$\gamma_\mathcal{P} \leq \Gamma_\mathcal{P} \leq \beta_\mathcal{P},$$

and

$$i_\mathcal{P} \leq \beta_\mathcal{P}.$$

If $\mathcal{P}$ is also closed under disjoint union with K_1, then

$$\gamma_\mathcal{P} \leq i_\mathcal{P}.$$

For, if S is a set such that $\langle S \rangle$ in $\mathcal{P}$ but S does not dominate vertex z, then $S \cup \{z\}$ is also in $\mathcal{P}$.

The above comments generalize to:

Theorem 1. *For any nondegenerate properties $\mathcal{P}$ and $\mathcal{Q}$ with $\mathcal{P} \subseteq \mathcal{Q}$:*

(a) $\gamma_\mathcal{Q} \leq \gamma_\mathcal{P} \leq \Gamma_\mathcal{P} \leq \Gamma_\mathcal{Q}$.

(b) *If $\mathcal{P}$ is closed under union with K_1, then $\gamma_\mathcal{Q} \leq i_\mathcal{P} \leq \beta_\mathcal{Q}$.*

(Aside: a nondegenerate property can indeed fail to be closed under union with K_1—consider for example the property of every component being isomorphic.)

This establishes the Hasse diagram shown in Figure (which was obtained for additive induced-hereditary properties by Michalak [10]). It is important to note that the Hasse diagram holds for each nondegenerate property $\mathcal{P}$ that is closed under disjoint union with K_1. The dashed edge is used to represent the fact that there are properties $\mathcal{P}$ such that $i_{\mathcal{P}} \geq i_0$ and there are properties $\mathcal{P}$ for which $i_{\mathcal{P}}$ and i_0 are incomparable (see below). That there are no more inequalities for any additive (induced-) hereditary property $\mathcal{P}$ is established by the following theorem. Hedetniemi et al. [9] showed that there were no more inequalities in the special case $\mathcal{P}$ being the acyclic graphs.

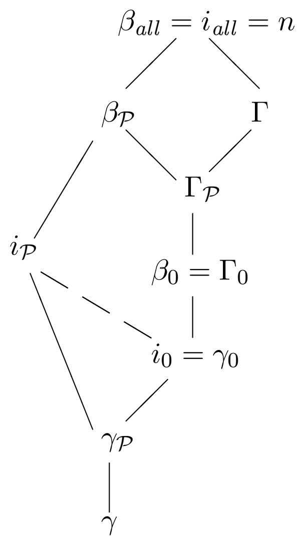

Figure 1. Relationships between parameters for a fixed property $\mathcal{P}$.

Theorem 2. *For any nondegenerate clique-bounded property $\mathcal{P}$ with $M_{\mathcal{P}} \geq 2$,*

(a) *There exists a graph with $i_{\mathcal{P}}(G) > \Gamma(G)$.*

(b) *There exists a graph with $i_{\mathcal{P}}(G) < \beta_0(G)$.*

(c) *There exists a graph with $\beta_{\mathcal{P}}(G) < \Gamma(G)$.*

Proof. (a) The complete graph K_s has $\Gamma(K_s) = 1$ and $i_{\mathcal{P}}(K_s) = M_{\mathcal{P}}$ for $s \geq M_{\mathcal{P}}$.

(b) Consider the join $G = K_{M_{\mathcal{P}}} + sK_1$. This has $i_{\mathcal{P}}(G) \leq M_{\mathcal{P}}$ (the clique is a maximal $\mathcal{P}$-set), while $\beta_0(G) = s$ and so can be made arbitrarily large.

(c) Consider the prism (cartesian product) $G = K_a \times K_2$. Then $\beta_{\mathcal{P}}(G) \leq 2M_{\mathcal{P}}$ (can take at most $M_{\mathcal{P}}$ vertices from each copy of K_a), while $\Gamma(G) = a$ and so can be made arbitrarily large. ∎

In general, $i_{\mathcal{P}}$ and $i_{\mathcal{Q}}$ are not comparable. Indeed, as mentioned above, there are properties where there exist graphs G with $i_{\mathcal{P}}(G) < i_0(G)$ and there are properties where $i_{\mathcal{P}} \geq i_0$. Hedetniemi et al. [9] showed that the former is true when $\mathcal{P}$ is the acyclic graphs. The latter is true for $\mathcal{P}$ the k-colorable graphs, as we now show.

Following [3], we say that property $\mathcal{P}$ is the product of properties $\mathcal{Q}$ and $\mathcal{R}$ if for every graph $G \in \mathcal{P}$, the vertices of G can be two-colored so that one color class induces a subgraph in $\mathcal{Q}$ and the other color class induces a subgraph in $\mathcal{R}$.

Lemma 1. *If property $\mathcal{P} = \mathcal{Q} \cdot \mathcal{R}$ with $\mathcal{Q}$ and $\mathcal{R}$ hereditary, then $i_{\mathcal{P}} \geq i_{\mathcal{Q}}$.*

Proof. Let G be any graph and consider a smallest maximal $\mathcal{P}$-subgraph; say with vertex set S. Let (T, U) be a partition of S such that $\langle T \rangle \in \mathcal{Q}$ and $\langle U \rangle \in \mathcal{R}$, with T as large as possible. By the maximality of S, no vertex in $V - S$ can be added to $\langle T \rangle$ and still have a subgraph in $\mathcal{Q}$; and by the maximality of T, no vertex in U can be added to $\langle T \rangle$ and still have a subgraph in $\mathcal{Q}$. Thus $i_{\mathcal{Q}}(G) \leq |T| \leq |S| = i_{\mathcal{P}}(G)$. ∎

It follows that if $\mathcal{P} = 0^k$ (the property of being k-colorable), then $i_{\mathcal{P}} \geq i_0$.

On the other hand, many properties have $i_{\mathcal{P}}$ incomparable with i_0. Consider, for example, any hereditary property $\mathcal{P}$ which does not contain the triangle but does contain some odd cycle C_n. For positive integer s, define G_n^s by taking C_n and for every pair of consecutive vertices on the cycle, introducing s vertices adjacent to them only. Then $i_{\mathcal{P}}(G_n^s) \leq n$ (the cycle is maximal), while $i_0(G_n^s) = (n-1)/2 + s$.

3. Calculations

We start with the values for the complete graph.

Observation 1. For any nondegenerate property $\mathcal{P}$,

(a) $\gamma_{\mathcal{P}}(K_n) = \Gamma_{\mathcal{P}}(K_n) = 1$ and

(b) if $\mathcal{P}$ is hereditary then $i_{\mathcal{P}}(K_n) = \beta_{\mathcal{P}}(K_n) = \min(M_{\mathcal{P}}, n)$.

Next we consider the star with s leaves.

Observation 2. For any nondegenerate property $\mathcal{P}$, $\gamma_{\mathcal{P}}(K_{1,s}) = 1$, $\Gamma_{\mathcal{P}}(K_{1,s}) = s$, and $\beta_{\mathcal{P}}(K_{1,s}) \in \{s, s+1\}$.

Next we look at the path on n vertices. Let $Z_{\mathcal{P}}$ denote the length of the longest path in $\mathcal{P}$ (possibly $Z_{\mathcal{P}} = \infty$). If nondegenerate $\mathcal{P} \neq 0$, then $Z_{\mathcal{P}} \geq 2$.

Observation 3. For any additive hereditary property $\mathcal{P} \neq 0$,

(a) $\gamma_{\mathcal{P}}(P_n) = \lceil n/3 \rceil$ and $\Gamma_{\mathcal{P}}(P_n) = \lceil 2n/3 \rceil$.

(b) If $Z_{\mathcal{P}} = \infty$ then $\beta_{\mathcal{P}}(P_n) = n$; otherwise $\beta_{\mathcal{P}}(P_n) = \lceil Z_{\mathcal{P}} n/(Z_{\mathcal{P}}+1) \rceil$.

(c) If $Z_{\mathcal{P}} = \infty$ then $i_{\mathcal{P}}(P_n) = n$. Otherwise let $n = r(Z_{\mathcal{P}}+2) + s$ for $0 \leq s < Z_{\mathcal{P}} + 2$. Then $i_{\mathcal{P}}(P_n) = rZ_{\mathcal{P}} + \min(s, Z_{\mathcal{P}})$.

Proof. (a) A minimal dominating set cannot contain three consecutive vertices; thus any minimal dominating set is a $\mathcal{P}$-set and so $\gamma_{\mathcal{P}}(P_n) = \gamma(P_n)$ and $\Gamma_{\mathcal{P}}(P_n) = \Gamma(P_n)$.

(b) A $\mathcal{P}$-set S must miss (not contain) at least one vertex in every subpath of length Z_p+1. Thus $|S| \leq \lceil Z_{\mathcal{P}} n/(Z_{\mathcal{P}}+1) \rceil$, with equality possible by taking the first $Z_{\mathcal{P}}$ vertices, skipping one vertex, taking the next $Z_{\mathcal{P}}$ vertices, and so on.

(c) Consider a maximal $\mathcal{P}$-set S. Then S misses at most two vertices in every subpath of length $Z_{\mathcal{P}}+2$; for, if S were to miss three vertices, the middle vertex of the triad could be added to S. This means that if n is a multiple of $Z_{\mathcal{P}}+2$, the claimed value is a lower bound.

For an actual set S, partition the path into subpaths such that all but possibly the last have length $Z_{\mathcal{P}}+2$. Take for S the middle $Z_{\mathcal{P}}$ vertices from each full subpath. From the incomplete subpath, take all its vertices if $s < Z_{\mathcal{P}}+1$ and $Z_{\mathcal{P}}$ consecutive vertices if $s = Z_{\mathcal{P}}+1$. So the claimed value is an upper bound.

For the lower bound when $s \neq 0$, consider the first vertex. If it is not in S, then at least the next $Z_{\mathcal{P}}$ are, and one can induct on the remainder. Similar ideas work when the first vertex is in S. The details are omitted. ■

Similar results hold for the cycle.

For positive integers $a_1 \geq a_2 \geq \ldots \geq a_s$, we define the generalized *corona* $C(a_1, a_2, \ldots, a_s)$ as the complete graph on s vertices (say $v_1, \ldots, v_s$), together with a_j new end-vertices adjacent to v_j for each j. Obviously $\gamma(C(a_1, a_2, \ldots, a_s)) = s$.

Observation 4. For an additive hereditary property $\mathcal{P}$ and $s \geq M_{\mathcal{P}}$, $\gamma_{\mathcal{P}}(C(a_1, a_2, \ldots, a_s)) = M_{\mathcal{P}} + \sum_{i=M_{\mathcal{P}}+1}^{s} a_i$.

As a consequence, it follows that one can arbitrarily prescribe $\gamma(G)$, $\gamma_{\mathcal{P}}(G)$ and $i_0(G)$ provided $\gamma(G) > M_{\mathcal{P}}$.

Large minimum degree and/or regularity is not enough to force $\gamma(G) = \gamma_{\mathcal{P}}(G)$. The idea is to turn a corona into a regular graph. See Fricke's graph in Figure 6 of [9].

We look now at graphs with small maximum degree. Recall that for a dominating set S, a *private neighbor* of a vertex v is a vertex in $V - S$ adjacent only to v. It is easy to see that if one takes a minimum dominating set S with the minimum number of internal edges, every nonisolated vertex of S has at least two private neighbors. (Otherwise replace by unique private neighbor and contradict minimality of S.) In particular, we obtain the following (which is surely known):

Observation 5. For graph G with maximum degree Δ, there is a minimum dominating set S such that $\langle S \rangle$ has maximum degree at most $\Delta - 2$. In particular, for a cubic graph G, $\gamma(G) = \gamma_{\mathcal{P}}(G)$ if $\mathcal{P}$ is an additive induced-hereditary property with $M_{\mathcal{P}} \geq 2$.

For example, this was observed for $\mathcal{P}$ the P_3-free graphs in [8].

It is interesting to note that for $\mathcal{P} = 0$ the difference between $\gamma(G)$ and $\gamma_{\mathcal{P}}(G)$ can be arbitrarily large for cubic graphs, even if they are required to be 3-connected [11]. If we consider the specific additive hereditary property $\mathcal{P} = 0^k$ (k-colorable), our next result follows directly from Observation 5 and Brooks' theorem:

Observation 6. For an r-regular graph G and $\mathcal{P} = 0^{r-1}$, $\gamma_{\mathcal{P}}(G) = \gamma(G)$.

By a grid, we mean the cartesian product of two paths (though the following result extends to products of cycles).

Theorem 3. *For an additive hereditary property $\mathcal{P}$ and a grid G, $\gamma_{\mathcal{P}}(G) = \gamma(G)$ provided $P_3 \in \mathcal{P}$.*

Proof. We extend the argument in [9] to show that there is a minimum dominating set of G whose components are each a subgraph of P_3. We will use u_{ij} to denote the vertex in row i and column j.

Take a minimum dominating set S with the minimum number of edges. As noted above, every nonisolated vertex of S has at least two private neighbors. Suppose S contains a component not restricted to a single row or single column. Then, without loss of generality, S contains three vertices say u_{11}, u_{12} and u_{21}. Then, because each has two private neighbors, this component is either a P_3, and we are done, or is a 4-cycle, contains u_{22} and is not on the boundary of the grid. Then one can replace these four vertices in S with the four vertices u_{20}, u_{32}, u_{13} and u_{01}, and thus decrease the number of edges in S, a contradiction.

Suppose S contains a component with at least four vertices restricted to a single row; say u_{11} to u_{14}. Then because of the private neighbors, this component is in neither the first nor the last row; so one can replace u_{12} and u_{13} in S by u_{02} and u_{23} and thus decrease the number of edges in S, a contradiction. Hence we have shown that each component in S has at most three vertices. ∎

The result actually extends to any induced subgraph of a grid.

4. Extreme Values

We determine the maximum value of $\gamma_{\mathcal{P}}$ for a given order. This depends again on the maximum clique size. The following theorem generalizes Theorem 2 of [8], for example.

Theorem 4. *Let $\mathcal{P}$ be an additive hereditary property. For a graph G with n vertices and no isolates,*

$$\gamma_{\mathcal{P}}(G) \leq n + 2M_{\mathcal{P}} - 2\sqrt{nM_{\mathcal{P}}},$$

and this bound is sharp for all $\mathcal{P}$ and infinitely many $n \geq 4M_{\mathcal{P}}$.

Proof. We extend the proof of Gimbel and Vestergaard [5] for the result for γ_0. Let S be a minimum dominating set $\{v_1, \ldots, v_\gamma\}$ of G such that every vertex has an external private neighbor. (Choose S such that $\langle S \rangle$ has the minimum number of isolates.) The claimed upper bound is at least $M_{\mathcal{P}}$; so we may assume that $\gamma \geq M_{\mathcal{P}}$. For convenience we write M for $M_{\mathcal{P}}$.

Now, partition V into sets N_j, $1 \leq j \leq \gamma$, such that $v_j \in N_j \subseteq N[v_j]$. By the choice of S, each N_j has at least two vertices. Assume $|N_1| \geq |N_2| \geq \ldots \geq |N_\gamma|$.

Now construct a dominating set T by taking the union of $\{v_1, \ldots, v_M\}$ and a maximal independent set of $G - N[\{v_1, \ldots, v_M\}]$. Then T is a $\mathcal{P}$-set. The set T must miss at least one vertex from each N_j for $j > M$. Thus

$$\begin{aligned} \gamma_{\mathcal{P}}(G) &\leq M + (n - \textstyle\sum_{j=1}^{M} |N_j|) - (\gamma - M) \\ &\leq M + n - Mn/\gamma - \gamma + M \\ &= n + 2M - Mn/\gamma - \gamma. \end{aligned}$$

The last expression is maximized as a function of γ at $\gamma = \sqrt{nM}$ in general. Thus we get the stated upper bound. If $M \geq n/4$, then the optimal value of γ is actually $n/2$, the maximum it can be, and the bound $\gamma_{\mathcal{P}}(G) \leq n/2$ is better.

For equality, assume n/M is a perfect square, and consider the generalized corona with a complete graph on $\sqrt{Mn}$ vertices and $\sqrt{n/M} - 1$ end-vertices attached to each vertex. ■

The other extreme values are less interesting. The minimum value of $\gamma_{\mathcal{P}}(G)$ is 1 as it is for $\Gamma_{\mathcal{P}}(G)$ (the complete graph); the maximum value for $\Gamma_{\mathcal{P}}(G)$ is $n-1$ (the star). The maximum value for $\beta_{\mathcal{P}}(G)$ and $i_{\mathcal{P}}(G)$ is n (e.g. union of cliques of cardinality at most $M_{\mathcal{P}}$); and the minimum value is $\min(M_N, n)$ (e.g. the join $K_{M_{\mathcal{P}}} + \bar{K}_{n-M_{\mathcal{P}}}$).

5. Future Work

In [4], it is observed that $\beta_{\mathcal{P}}$ is always NP-hard for an hereditary property. The standard reduction from 3SAT to domination (see [4] or [6]) actually shows that $\gamma_{\mathcal{P}}$ is NP-hard for any additive hereditary property; indeed, by Observation 5, $\gamma_{\mathcal{P}}$ remains NP-hard for cubic graphs. However, we have no idea about the complexity of $\Gamma_{\mathcal{P}}$ or $i_{\mathcal{P}}$.

Commonly studied additive hereditary properties include graphs with maximum degree at most k, graphs which are k-degenerate (any subgraph has minimum degree at most k), k-colorable graphs, graphs with component orders at most k, planar graphs and F-free graphs for a fixed graph F. Maybe one can prove more specific results for particular families of properties (similar to our Observation 6).

Another direction is to consider generalized irredundance, but the results are again likely to mirror those of acyclic domination.

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