

## A LOWER BOUND FOR THE IRREDUNDANCE NUMBER OF TREES

MICHAEL POSCHEN AND LUTZ VOLKMANN

*Lehrstuhl II für Mathematik*  
*RWTH Aachen University*  
*52056 Aachen, Germany*

**e-mail:** volkm@math2.rwth-aachen.de

### Abstract

Let  $\text{ir}(G)$  and  $\gamma(G)$  be the irredundance number and domination number of a graph  $G$ , respectively. The number of vertices and leafs of a graph  $G$  is denoted by  $n(G)$  and  $n_1(G)$ . If  $T$  is a tree, then Lemańska [4] presented in 2004 the sharp lower bound

$$\gamma(T) \geq \frac{n(T) + 2 - n_1(T)}{3}.$$

In this paper we prove

$$\text{ir}(T) \geq \frac{n(T) + 2 - n_1(T)}{3}$$

for an arbitrary tree  $T$ . Since  $\gamma(T) \geq \text{ir}(T)$  is always valid, this inequality is an extension and improvement of Lemańska's result.

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### 1. TERMINOLOGY AND INTRODUCTION

We consider finite, undirected, and simple graphs  $G$  with the vertex set  $V(G)$  and the edge set  $E(G)$ . The number of vertices  $|V(G)|$  of a graph  $G$  is called the *order* of  $G$  and is denoted by  $n = n(G)$ . The *open neighborhood*  $N(v) = N(v, G)$  of the vertex  $v$  consists of the vertices adjacent to  $v$ , and the *closed neighborhood* of  $v$  is  $N[v] = N[v, G] = N(v) \cup \{v\}$ . For a subset

$X \subseteq V(G)$ , we define  $N(X) = N(X, G) = \bigcup_{v \in X} N(v)$  and  $N[X] = N[X, G] = N(X) \cup X$ . In addition, let  $G[X]$  be the subgraph induced by  $X$ , and let  $e(X)$  be the number of edges in  $G[X]$ . The vertex  $v$  is a *leaf* of  $G$  if  $d(v, G) = 1$ , and an *isolated vertex* if  $d(v, G) = 0$ , where  $d(v) = d(v, G) = |N(v)|$  is the degree of  $v \in V(G)$ . Let  $n_1 = n_1(G)$  be the number of leafs in a graph  $G$ . By  $\delta = \delta(G)$  and  $\Delta = \Delta(G)$ , we denote the *minimum degree* and *maximum degree* of the graph  $G$ , respectively. If  $X$  and  $Y$  are two disjoint subsets of  $V(G)$ , then let  $e(X, Y)$  be the number of edges with one end in  $X$  and the other in  $Y$ .

A set  $D \subseteq V(G)$  is a *dominating set* of the graph  $G$  if  $N[D, G] = V(G)$ . The *domination number*  $\gamma = \gamma(G)$  of  $G$  is the cardinality of any smallest dominating set.

Let  $I \subseteq V(G)$  and  $v \in I$ . A vertex  $u \in V(G) - I$  is an *I-external private neighbor* of  $v$  if  $N(u) \cap I = \{v\}$ . The set of all *I-external private neighbors* of  $v$  is denoted by  $EPN(v, I)$  and

$$PN(v, I) = \begin{cases} EPN(v, I) \cup \{v\} & \text{if } v \text{ is isolated in } G[I] \\ EPN(v, I) & \text{otherwise.} \end{cases}$$

A subset  $I \subseteq V(G)$  is *irredundant* if  $PN(v, I) \neq \emptyset$  for all  $v \in I$ . An irredundant set  $I$  is *maximal irredundant* if for every vertex  $u \in V(G) - I$ , the set  $I \cup \{u\}$  is not irredundant. The minimum cardinality taken over all maximal irredundant sets of  $G$  is the *irredundance number*  $\text{ir}(G)$  of  $G$ .

For detailed information on domination, irredundance, and related topics see the comprehensive monograph [3] by Haynes, Hedetniemi, and Slater.

Let  $T$  be a tree of maximum degree  $\Delta(T) \geq 3$ . If  $T$  is not isomorphic to the star  $K_{1, \Delta(T)}$ , then Cockayne [1] recently proved that

$$\text{ir}(T) \geq \frac{2(n(T) + 1)}{2\Delta(T) + 3}.$$

In this note we will present the following lower bound of the irredundance number of a tree. If  $T$  is a tree, then

$$\text{ir}(T) \geq \frac{n(T) + 2 - n_1(T)}{3}.$$

Since  $\gamma(G) \geq \text{ir}(G)$  is valid for an arbitrary graph  $G$ , this lower bound is an improvement of Lemańska's [4] inequality

$$\gamma(T) \geq \frac{n(T) + 2 - n_1(T)}{3}.$$

## 2. PRELIMINARY RESULTS

The following partition of  $V(G)$  induced by the vertex subset  $I$  will be involved in the proof of the desired bound.

$V(G) = I \cup B \cup A \cup R$  (disjoint union) where

$$B = \{u \in V(G) - I : |N(u) \cap I| = 1\}$$

$$A = \{u \in V(G) - I : |N(u) \cap I| \geq 2\}$$

$$R = V(G) - N[I].$$

In addition, let  $B = B_0 \cup B_1$  and  $R = R_0 \cup R_1$  such that

$$B_0 = \{u \in B : d(u) \geq 2\}$$

$$B_1 = \{u \in B : d(u) = 1\}$$

$$R_0 = \{u \in R : d(u) \geq 2\}$$

$$R_1 = \{u \in R : d(u) = 1\}.$$

In the following the cardinality of any set (except  $V(G)$ ) denoted by any upper case letter, will be denoted by the corresponding lower case letter i.e.,  $|B| = b$ ,  $|A| = a$  etc. The proof of our main result is based on a useful characterization of maximal irredundant sets by Cockayne, Grobler, Hedetniemi, and McRae [2].

**Theorem 2.1** ([2] 1997). *Let  $I$  be an irredundant set in a graph  $G$ . The set  $I$  is maximal irredundant if and only if for each  $w \in N[R]$ , there exists a vertex  $v \in I$  such that*

$$(1) \quad PN(v, I) \subseteq N[w].$$

*If (1) is satisfied we say that  $w$  annihilates  $v$ .*

Suppose that  $\mathcal{F}(i, n_1)$  is the set of forests of maximum order which have  $n_1$  leafs and a maximal irredundant set of size  $i$ .

**Lemma 2.2.** *Let  $I$  be a maximal irredundant set of size  $i$  of the forest  $G \in \mathcal{F}(i, n_1)$ . For each  $w \in R$ , there exists exactly one  $v \in I$  such that  $w$  annihilates  $v$ .*

**Proof.** In view of Theorem 2.1, there exists a vertex  $v \in I$  such that  $w$  annihilates  $v$ , that means  $PN(v, I) \subseteq N[w]$ . Suppose that there exist two different vertices  $v_1, v_2 \in I$  such that  $w$  annihilates  $v_1$  as well as  $v_2$ . Let  $\{u_1\} = N(w) \cap N(v_1)$  and  $\{u_2\} = N(w) \cap N(v_2)$ . Form the graph  $G_1$  by deleting the edge  $wu_2$  and adding a vertex  $w_2$  to the set  $R$  and the new edges  $ww_2$  and  $u_2w_2$ . Since  $w_2$  annihilates  $v_2$  in  $G_1$ , the set  $I$  is, by Theorem 2.1, furthermore a maximal irredundant set of the tree  $G_1$  with  $n_1$  leafs. This is a contradiction to the hypothesis that  $G \in \mathcal{F}(i, n_1)$ , and the proof of Lemma 2.2 is complete. ■

### 3. MAIN RESULT

**Theorem 3.1.** *If  $T$  is a tree of order  $n$  with  $n_1$  leafs, then*

$$(2) \quad \text{ir}(T) \geq \frac{n + 2 - n_1}{3}.$$

**Proof.** Since the result is immediate for  $n \leq 3$ , we assume in the following that  $n \geq 4$ . It is evident that it is enough to prove inequality (3) for  $T \in \mathcal{F}(i, n_1)$ . Thus let now  $T \in \mathcal{F}(i, n_1)$ , and let  $I$  be a maximal irredundant set of size  $i$ . It is well-known that  $|V(T)| - 1 = |E(T)|$ , and thus we deduce that

$$\begin{aligned} |V(T)| - 1 &= i + b_0 + a + r_0 + b_1 + r_1 - 1 \\ &= e(B_0) + e(B_0, A) + e(A) + e(I) + e(R_0) + e(B_0, R_0) + e(A, R_0) \\ (3) \quad &+ e(I, B_0) + e(I, A) + b_1 + r_1. \end{aligned}$$

Next let  $B_0 = X \cup Y$  and  $R_0 = R'_0 \cup R''_0$  with

$$\begin{aligned} X &= \{u \in B_0 : N(u) \cap R \neq \emptyset\} \\ Y &= \{u \in B_0 : N(u) \cap R = \emptyset\} \\ R'_0 &= \{w \in R_0 : |N(w) \cap B| = 1\} \\ R''_0 &= \{w \in R_0 : |N(w) \cap B| \geq 2\}. \end{aligned}$$

Furthermore, we define the set  $X_0 \subseteq X$  as follows: If  $u \in X$  is adjacent to the vertex  $w \in R$ , then  $u$  is also adjacent to the vertex  $v$  with the property that  $w$  annihilates  $v$ . Finally, let  $X_1 = X - X_0$ .

If  $u \in Y$ , then  $d(u) \geq 2$ . Because of  $|N(u) \cap I| = 1$  and  $|N(u) \cap R| = 0$ , the vertex  $u$  is adjacent to a vertex of  $A \cup B_0$ . This easily leads to

$$(4) \quad \frac{y}{2} \leq e(B_0) + e(B_0, A).$$

Let  $u \in X_0$  and  $v \in I$  the unique neighbor of  $u$  in  $I$ . By the definition of  $X_0$ , there exists a vertex  $w \in R \cap N(u)$  such that  $PN(v, I) \subseteq N[w]$ . This implies that  $v$  is no isolated vertex in the subgraph  $G[I]$ , because otherwise we would arrive at the contradiction  $\{v\} \subseteq PN(v, I) \not\subseteq N[w]$ . Let  $u_1 \neq u$  be a further vertex in  $X_0$ . Suppose that  $v$  is also the unique neighbor of  $u_1$  in  $I$ . Since  $T$  is a tree, we observe that  $u_1$  and  $w$  are not adjacent. This leads to the contradiction  $\{u_1\} \subseteq PN(v, I) \not\subseteq N[w]$ . Altogether, we conclude that

$$(5) \quad \frac{x_0}{2} \leq e(I).$$

According to Theorem 2.1, each vertex  $w \in R'_0$  annihilates a vertex  $v$  in  $I$ . Hence each vertex  $w \in R'_0$  is adjacent to a vertex  $u \in B_0$ . Moreover, in view of Lemma 2.2, the vertex  $u$  is unique and thus  $|R'_0| \leq e(R'_0, B_0)$ . The condition  $d(w) \geq 2$  shows that  $w$  is adjacent to a further vertex in  $A \cup R$ . Since, by Theorem 2.1,  $R_1$  is not possible,  $w$  is adjacent to a vertex in  $A \cup R_0$ . We obtain the minimum number of edges if each  $w \in R'_0$  is adjacent to exactly one vertex of  $R'_0$  and  $w$  has no neighbor in  $A \cup R''_0$ . This yields

$$\frac{|R'_0|}{2} \leq e(R'_0) + e(R'_0, R''_0) + e(R'_0, A)$$

and thus

$$(6) \quad \begin{aligned} \frac{3|R'_0|}{2} &\leq e(R'_0, B_0) + e(R'_0) + e(R'_0, R''_0) + e(R'_0, A) \\ &\leq e(R'_0, B_0) + e(R_0) + e(R'_0, A). \end{aligned}$$

Assume that  $w \in R''_0$ . Again Theorem 2.1 implies that  $w$  annihilates a vertex  $v$  in  $I$ . Hence  $w$  is adjacent to a vertex  $u \in X_0$ . In view of Lemma 2.2, the vertex  $u$  is unique and thus

$$(7) \quad |R''_0| = e(R''_0, X_0).$$

In addition, the definition of  $R''_0$  shows that  $w$  is adjacent to a further vertex  $u' \in X_1$ , and each vertex  $u'' \in X_1$  is adjacent to a vertex in  $R''_0$ . Hence it

follows that

$$(8) \quad e(R_0'', X_1) \geq \max\{|R_0''|, |X_1|\} \geq \frac{|R_0''|}{2} + \frac{|X_1|}{2}.$$

Combining (5) – (9) with the inequality

$$e(R_0'', X_0) + e(R_0'', X_1) + e(R_0', B_0) \leq e(R_0, B_0)$$

we arrive at

$$\begin{aligned} \frac{b_0}{2} + \frac{3r_0}{2} &= \frac{x_0}{2} + \frac{x_1}{2} + \frac{y}{2} + \frac{3|R_0'|}{2} + \frac{3|R_0''|}{2} \\ &= \frac{x_0}{2} + \frac{y}{2} + \frac{3|R_0'|}{2} + |R_0''| + \frac{x_1}{2} + \frac{|R_0''|}{2} \\ &\leq e(I) + e(B_0) + e(B_0, A) + e(R_0', B_0) + e(R_0) \\ &\quad + e(R_0', A) + e(R_0'', X_0) + e(R_0'', X_1) \\ &\leq e(I) + e(B_0) + e(B_0, A) + e(R_0, B_0) + e(R_0) + e(R_0, A) + e(A). \end{aligned}$$

Now we deduce from (4)

$$\begin{aligned} i + b_0 + a + r_0 - 1 &= e(B_0) + e(B_0, A) + e(A) + e(I) + e(R_0) + e(B_0, R_0) \\ &\quad + e(A, R_0) + e(I, B_0) + e(I, A) \\ &\geq \frac{b_0}{2} + \frac{3r_0}{2} + b_0 + 2a \\ &\geq \frac{3b_0}{2} + \frac{3r_0}{2} + \frac{3a}{2}. \end{aligned}$$

This implies  $2i - 2 \geq b_0 + a + r_0$  and thus

$$2i - 2 + n_1 \geq b_0 + a + r_0 + n_1.$$

Since by definition  $n_1 \geq b_1 + r_1$ , we obtain

$$3i - 2 + n_1 \geq i + b_0 + a + r_0 + b_1 + r_1 = n,$$

and this leads to

$$i \geq \frac{n + 2 - n_1}{3}.$$

Since the last bound is valid for all maximal irredundant sets  $I$  with  $|I| = i$ , the desired inequality (3) is proved. ■

**Remark 3.2.** If  $T$  is a tree, then Lemańska [4] has proved that

$$\gamma(T) = \frac{n(T) + 2 - n_1(T)}{3}$$

if and only if the distance between each pair of distinct leaves in  $T$  is congruent 2 modulo 3. An analyses of the proof of Theorem 3.1 shows that we obtain equality in (3) for exactly the same family of trees.

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