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ON A PERFECT PROBLEM

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Abstract

We solve Open Problem (xvi) from *Perfect Problems* of Chvátal [1] available at ftp://dimacs.rutgers.edu/pub/perfect/problems.tex:

Is there a class $\mathcal{C}$ of perfect graphs such that

(a) $\mathcal{C}$ does not include all perfect graphs and

(b) every perfect graph contains a vertex whose neighbors induce a subgraph that belongs to $\mathcal{C}$?

A class $\mathcal{P}$ is called locally reducible if there exists a proper subclass $\mathcal{C}$ of $\mathcal{P}$ such that every graph in $\mathcal{P}$ contains a local subgraph belonging to $\mathcal{C}$. We characterize locally reducible hereditary classes. It implies that there are infinitely many solutions to Open Problem (xvi). However, it is impossible to find a hereditary class $\mathcal{C}$ of perfect graphs satisfying both (a) and (b).

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1. Locally Reducible Classes

A class of graphs $\mathcal{P}$ is *hereditary* if $H \in \mathcal{P}$ for each induced subgraph H of every graph $G \in \mathcal{P}$. As usual, $N(u) = N_G(u)$ is the neighborhood of a vertex u in a graph G. A *local subgraph* in a graph G is a subgraph induced by $N(u)$, where u is a vertex of G. If u is an isolated vertex [i.e., $N(u) = \emptyset$], then the corresponding local subgraph is K_0, the vertexless graph. Let $\mathcal{P}$ be a hereditary class of graphs. If there is a proper subclass $\mathcal{C}$ of $\mathcal{P}$ such that every graph in $\mathcal{P}$ with at least one vertex contains a local subgraph belonging to $\mathcal{C}$, then $\mathcal{P}$ is called a *locally reducible class.*

Problem 1. Characterize locally reducible hereditary classes.

Not all hereditary classes are locally reducible. For example, let us consider the class $\mathcal{K} = \{K_n : n \geq 0\}$, of all complete graphs. Let $\mathcal{C}$ be an arbitrary proper subclass of $\mathcal{K}$. Since $\mathcal{C} \neq \mathcal{K}$, there exists m such that $K_m \notin \mathcal{C}$. The graph K_{m+1} belongs to $\mathcal{K}$. However, all local subgraphs in K_{m+1} are K_m, and therefore they are not in $\mathcal{C}$. By definition, $\mathcal{K}$ is not locally reducible.

Theorem 1. *A non-empty hereditary class $\mathcal{P}$ is locally reducible if and only if $\mathcal{P} \neq \mathcal{K}$.*

Proof. Necessity was shown above.
Sufficiency. As usual, the star $K_{1,n}$ has $n+1$ vertices $v_0, v_1, \ldots, v_n$ and n edges $v_0v_1, v_0v_2, \ldots, v_0v_n$, the vertex v_0 being the *center* of the star.

Claim 1. For a fixed $n \geq 2$, there is no graph G such that the neighborhood of each vertex of G induces $K_{1,n}$.

Proof. Suppose that there exists a graph G such that the neighborhood of each vertex induces $K_{1,n}$. We consider an arbitrary vertex u of G. Its neighborhood induces the subgraph H isomorphic to $K_{1,n}$. We denote $V(H) = \{v_0, v_1, \ldots, v_n\}$, where v_0 is the center, see Figure 1.

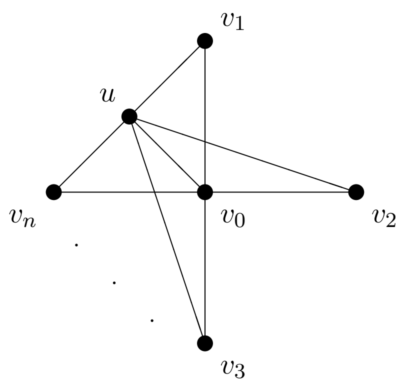

Figure 1. An illustration

The set $N_G(v_0) = \{u, v_1, v_2, \ldots, v_n\}$ induces $K_{1,n}$ centered at u. The vertex v_1 is adjacent to both u and v_0, and v_1 is non-adjacent to all the vertices $v_2, v_3, \ldots, v_n$. It follows that $\{u, v_0\}$ is a connected component of the induced subgraph $G(N(v_1))$. Since $n \geq 2$, $N(v_1)$ cannot induce $K_{1,n}$, a contradiction. ■

First suppose that the path P_3 belongs to $\mathcal{P}$. Then $\mathcal{C} = \mathcal{P} \setminus \{P_3\}$ is a proper subclass of $\mathcal{P}$. We consider an arbitrary graph $G \in \mathcal{P}$. Claim 1 implies that there exists a vertex $x \in V(G)$ such that $N_G(x)$ does not induce $P_3 \cong K_{1,2}$. By the definition of $\mathcal{C}$, $G(N(x)) \in \mathcal{C}$, as required.

It remains to consider the case, where $P_3 \notin \mathcal{P}$. Since P_3 is a forbidden induced subgraph, each graph $G \in \mathcal{P}$ is a disjoint union of complete subgraphs. Clearly, all local subgraphs of G are complete graphs.

Suppose that $\mathcal{P}$ contains O_2, the graph with two non-adjacent vertices. Clearly, we can define $\mathcal{C} = \mathcal{P} \setminus \{O_2\}$. If $\mathcal{P}$ does not contain O_2, then $\mathcal{P}$ consists of complete graphs only. According to the condition, $\mathcal{P} \neq \mathcal{K}$, i.e., there exists m such that $K_m \notin \mathcal{P}$. Note that the class $\mathcal{P}$ is not empty implying that $m \geq 1$. We may assume that $K_{m-1} \in \mathcal{P}$. Since $\mathcal{P}$ is a hereditary class, $\mathcal{P} = \{K_0, K_1, \ldots, K_{m-1}\}$. We may set $\mathcal{C} = \mathcal{P} \setminus \{K_{m-1}\}$, thus completing the proof. ■

Recall that a graph G is called *perfect* if $\omega(H) = \chi(H)$ for each induced subgraph H of G, where $\omega(H)$ is the clique number of H – the size of the largest complete subgraph in H, and $\chi(H)$ is the chromatic number of H – the minimum number of colors in proper vertex colorings of H, see [3]. If $\mathcal{P} = \mathcal{PERF}$ is the class of all perfect graphs, Problem 1 coincides with Open Problem (xvi) in Chvátal's list [1]. Theorem 1 gives a solution to this problem. Since all stars are perfect graphs, Claim 1 implies a more general fact.

Corollary 1. *There are infinitely many proper subclasses $\mathcal{C}$ of $\mathcal{PERF}$ such that every perfect graph contains a local subgraph belonging to $\mathcal{C}$.*

Proof. We define $\mathcal{C}_n = \mathcal{PERF} \setminus \{K_{1,n}\}$ for each $n \geq 2$ and apply Claim 1. ■

A *Zykov graph* H is defined by the property that there exists a graph G such that neighborhood of each vertex $u \in V(G)$ induces H, see the *Neighborhood Problem* in Zykov [4]. In our proof we used the fact that all stars $K_{1,n}$ with $n \geq 2$ are not Zykov graphs.

Corollary 2. *Let $\mathcal{P}$ be a class of graphs closed under taking local subgraphs. If $\mathcal{P}$ contains a graph H which is not a Zykov graph, then $\mathcal{P}$ is locally reducible.*

Proof. We define $\mathcal{C} = \mathcal{P} \setminus \{H\}$. Since H is not a Zykov graph, an arbitrary graph $G \in \mathcal{P}$ has a local subgraph $L \ncong H$. According to the condition, $L \in \mathcal{P}$. Thus, $L \in \mathcal{P} \setminus \{H\} = \mathcal{C}$. ■

2. Hereditary Subclasses

Now we consider a more complicated problem. A hereditary class $\mathcal{P}$ of graphs is called *locally h-reducible* if there exists a proper hereditary subclass $\mathcal{C}$ of $\mathcal{P}$ such that every graph in $\mathcal{P}$ with at least one vertex contains a local subgraph belonging to $\mathcal{C}$.

Problem 2. Characterize locally h-reducible hereditary classes.

Join of graphs G and H, denoted by $G+H$, is obtained from vertex-disjoint copies of G and H by adding all edges between $V(G)$ and $V(H)$. A class $\mathcal{P}$ of graphs is called *join-closed* if $G + H \in \mathcal{P}$ whenever $G, H \in \mathcal{P}$.

Claim 2. Each join-closed hereditary class $\mathcal{P}$ having a graph H with at least one vertex is not locally h-reducible.

Proof. Suppose that $\mathcal{P}$ is a locally h-reducible class, i.e., there exists a proper hereditary subclass $\mathcal{C}$ of $\mathcal{P}$ such that every graph in $\mathcal{P}$ with at least one vertex contains a local subgraph belonging to $\mathcal{C}$. There exists a graph $H \in \mathcal{P} \setminus \mathcal{C}$. Since the class $\mathcal{C}$ is hereditary, each graph in $\mathcal{C}$ is H-free. We consider the graph $G = H + H \in \mathcal{P}$. We see that each local subgraph L in G contains H as an induced subgraph. It implies that $L \notin \mathcal{C}$, a contradiction to the assumption that $\mathcal{P}$ is a locally h-reducible class. ■

Claim 2 shows that the class $\mathcal{PERF}$ is not locally h-reducible. Indeed, join of perfect graphs G and H always produces a perfect graph: $\omega(G + H) = \omega(G) + \omega(H)$ and $\chi(G + H) = \chi(G) + \chi(H)$. Thus, it is impossible to strengthen Corollary 1 requiring that $\mathcal{C}$ is a hereditary class.

A graph is *chordal* if it does not contain the cycles C_n with $n \geq 4$ as induced subgraphs. Claim 2 does not hold for the class $\mathcal{P} = \mathcal{CHORD}$ of all chordal graphs. Indeed, according to Dirac [2] each chordal graph

$G \neq K_0$ has a *simplicial* vertex — a vertex whose neighborhood induces a complete subgraph. It shows that we can choose $\mathcal{C} = \mathcal{K}$ as a hereditary proper subclass of all chordal graphs. The reason is that the class $\mathcal{CHORD}$ is not join-closed: $C_4 = O_2 + O_2$ is not a chordal graph, while O_2 is. Thus, Problem 2 remains open for all hereditary classes which are not join-closed.

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