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ARBITRARILY VERTEX DECOMPOSABLE CATERPILLARS WITH FOUR OR FIVE LEAVES

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Abstract

A graph G of order n is called arbitrarily vertex decomposable if for each sequence $(a_1, \ldots, a_k)$ of positive integers such that $a_1+\ldots+a_k = n$ there exists a partition $(V_1, \ldots, V_k)$ of the vertex set of G such that for each $i \in \{1, \ldots, k\}$, V_i induces a connected subgraph of G on a_i vertices.

D. Barth and H. Fournier showed that if a tree T is arbitrarily vertex decomposable, then T has maximum degree at most 4. In this paper we give a complete characterization of arbitrarily vertex decomposable caterpillars with four leaves. We also describe two families of arbitrarily vertex decomposable trees with maximum degree three or four.

Keywords: arbitrarily vertex decomposable graphs, trees, caterpillars, star-like trees.

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1. Introduction

Let $G = (V, E)$ be a graph of order n. A sequence $\tau = (a_1, \ldots, a_k)$ of positive integers is called *admissible for G* if it adds up to n. If $\tau = (a_1, \ldots, a_k)$ is an admissible sequence for G and there exists a partition $(V_1, \ldots, V_k)$ of the vertex set V such that for each $i \in \{1, \ldots, k\}$, $|V_i| = a_i$ and a subgraph induced by V_i is connected then τ is called *realizable in G* and the sequence $(V_1, \ldots, V_k)$ is said to be *a G-realization of τ* or *a realization of τ in G*. A graph G is *arbitrarily vertex decomposable* (avd for short) if for each admissible sequence τ for G there exists a G-realization of τ.

The problem of deciding whether a given graph is arbitrarily vertex decomposable has been considered in several papers (see for example [1]–[4]). Generally, this problem is NP-complete [1] but we do not know if this problem is NP-complete when restricted to trees.

However, it is obvious that each path and each traceable graph is avd. The investigation of avd trees is motivated by the fact that a connected graph is avd if its spanning tree is avd. In [4] M. Horňák and M. Woźniak conjectured that if T is a tree with maximum degree $\Delta(T)$ at least five, then T is not avd. This conjecture was proved by D. Barth and H. Fournier [2].

Theorem 1. *If a tree T is arbitrarily vertex decomposable, then $\Delta(T) \leq 4$. Moreover, every vertex of degree four of T is adjacent to a leaf.*

In [1] D. Barth, O. Baudon and J. Puech studied a family of trees each of them being homeomorphic to $K_{1,3}$ (they call them tripodes) and showed that determining if such a tree is avd can be done using a polynomial algorithm.

There is an interesting motivation for investigation of avd graphs. Consider a network connecting different computing resources; such a network is modeled by a graph. Suppose there are k different users, where i-th one needs n_i resources in our graph. The subgraph induced by the set of resources attributed to each user should be connected and a resource should be attributed to at most one user. So we have the problem of seeking a realization of the sequence $(n_1, \ldots, n_k)$ in this graph. Note also that one can find in [4] some references concerning arbitrarily edge decomposable graphs. The aim of this article is a characterization of avd trees with maximum degree at most four that have a very simple structure. Namely, we consider caterpillars or trees which are homeomorphic to a star $K_{1,q}$, where q is three or four.

2. Terminology and Results

In this paper, we deal with finite, simple and undirected graphs.

Let $T = (V, E)$ be a tree. A vertex $x \in V$ is called *primary* if $d(x) \geq 3$. A *leaf* is a vertex of degree one. A path P of T is an *arm* if one of its endvertices is a leaf in T, the other one is primary and all internal vertices of P have degree two in T. A tree T is called *primary* if it contains a primary vertex.

A graph T is a *star-like tree* if it is a tree homeomorphic to a star $K_{1,q}$ for some $q \geq 3$. Such a tree has one primary vertex (let us denote it by c) and q arms (let us denote them by A_i, $i \in \{1, \ldots, q\}$). For each A_i let α_i be the order of A_i. The structure of a star-like tree is (up to a isomorphism) determined by this sequence $(\alpha_1, \ldots, \alpha_q)$ of orders of its arms. Since the ordering of this sequence is not important, we will always assume that $2 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_q$ and will denote the above defined star-like tree by $S(\alpha_1, \ldots, \alpha_q)$. Notice that an order of this star-like tree is equal to $1 + \sum_{i=1}^{q}(\alpha_i - 1)$.

A tree T is *a caterpillar* if the set of vertices of degree at least two induces a path. Let T be a caterpillar such that $\Delta(T) \leq 4$. Let us note that if there are two or more vertices of degree four in T, then the sequences $(2, 2, \ldots, 2)$ if n is even or $(1, 2, 2, \ldots, 2)$ if n is odd are not realizable in T, hence T is not avd. Clearly, these particular sequences are realizable in T if there is a perfect matching or a quasi-perfect matching in T. According to the above remark we will consider only caterpillars of maximum degree at most four having at most one vertex of degree four.

Let T be a caterpillar with $\Delta(T) = 3$ and let $\{y_1, \ldots, y_s\}$ be the set of primary vertices of T. We call T a *caterpillar with s single legs attached at* $y_1, \ldots, y_s$.

Similarly, if T is a caterpillar and $\{x, y_1, \ldots, y_s\}$ the set of primary vertices of T such that $d(x) = 4$ and $d(y_i) = 3$ for all $i \in \{1, \ldots, s\}$, then

T is called a *caterpillar with one double leg attached at* x *and* s *single legs attached at* $y_1, \ldots, y_s$. For simplicity of notation we say sometimes that we have a caterpillar with s single legs or a caterpillar with one double leg and s single legs. We present two examples of such caterpillars in Figure 1 and Figure 4.

Here and subsequently, we assume that every admissible sequence for a graph G is non-decreasing and we write d^λ for the sequence $(\underbrace{d, \ldots, d}_{\lambda})$ and $d^\lambda \cdot g^\mu$ for the sequence $(\underbrace{d, d, \ldots, d}_{\lambda}, \underbrace{g, g, \ldots, g}_{\mu})$, the concatenation of λ times d and μ times g. We will note $d \cdot g^\mu$ and $d^\lambda \cdot g$ instead of $d^1 \cdot g^\mu$ and $d^\lambda \cdot g^1$.

We denote by (a, b) the greatest common divisor of two positive integers a and b and we write $t(i, j)$ for the transposition of the elements i and j of the set $\{1, 2, \ldots, k\}$. Note that if $i = j$, then by transposition $t(i, j)$ we mean the identity.

Let T be a tree, and let (V_1, V_2) and (V_1', V_2') be two partitions of $V(T)$ such that each V_i and each V_i' induces a tree in T. We say that we can *transpose* V_1 and V_2 (into V_1' and V_2') if $|V_i'| = |V_i|$ $(i = 1, 2)$.

Let $P = y_1, \ldots, y_q$ be a subpath of a tree T and U, W two disjoint subsets of $V(T)$. We shall say that U and W are *neighbouring* in P if for some $j \in \{1, \ldots, q-1\}$, $y_j \in U$ and $y_{j+1} \in W$ or $y_j \in W$ and $y_{j+1} \in U$.

The first result characterizing avd star-like trees (i.e., caterpillars with one single leg) was found by D. Barth, O. Baudon and J. Puech [1] and, independently, by M. Horňák and M. Woźniak [3].

Proposition 2. *The star-like tree* $S(2, a, b)$*, with* $2 \leq a \leq b$ *is avd if and only if* $(a, b) = 1$*. Moreover, each admissible and non-realizable sequence in* $S(2, a, b)$ *is of the form* d^λ*, where* $a \equiv b \equiv 0 \pmod{d}$ *and* $d > 1$*.*

In [1] D. Barth, O. Baudon and J. Puech proved the following proposition. In the statement of this result the sequence $(3, a, b)$ is not assumed to be non-decreasing.

Proposition 3. *Each star-like tree* $S(2, 2, a, b)$*, with* $2 \leq a \leq b$ *is avd if and only if*

1^0 *the star-like tree* $S(3, a, b)$ *is avd;*

2^0 *a, b are odd;*

3^0 $a \neq 2 \pmod{3}$ *or* $b \neq 2 \pmod{3}$*.*

The next result due to D. Barth and H. Fournier [2] shows that the structure of avd caterpillars is not obvious.

Theorem 4. *For every* $s \geq 1$ *there exists an avd caterpillar with* s *single legs.*

The main results of this paper are Theorems 5 and 6 of Sections 3 and 4 which give a complete characterization of avd caterpillars with two single legs and avd star-like trees $S(3, a, b)$. In Section 4 we also give a necessary and sufficient condition for a star-like tree $S(2, 2, a, b)$ to be avd. Thus, we describe the family of avd caterpillars with four leaves. In Section 5 we describe an infinite family of avd caterpillars with one double and one single leg (Proposition 9).

3. Arbitrarily Vertex Decomposable Caterpillars with Two Single Legs

Every caterpillar T of order n with two single legs attached at x and y can be obtained by taking a path $P = x_1, \ldots, x_{n-2}$, where $x = x_i$ and $y = x_j$ $(i < j)$ are two internal vertices of P, adding two vertices u and v, and joining u to x and v to y (see Figure 1). For such a graph let us define $l_x(T) := i$, $r_x(T) := n - i$ and, analogously, $l_y(T) := j + 1$ and $r_y(T) := n - j - 1$.

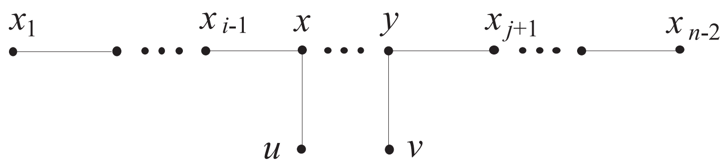

Figure 1. A caterpillar with two single legs.

Theorem 5. *Let* $T = (V, E)$ *be a caterpillar of order* n *with two single legs attached at* x *and* y*. Then* T *is avd if and only if the following conditions hold:*

1^0 $(l_x(T), r_x(T)) = 1$;

2^0 $(l_y(T), r_y(T)) = 1$;

3^0 $(l_x(T), r_y(T)) = 1$;

4^0 $(l_y(T), r_x(T)) < l_y(T) - l_x(T)$ *or* $n \equiv 1 \pmod{(l_y(T), r_x(T))}$;

5^0 $n \neq \alpha l_x(T) + \beta l_y(T)$ *for any* $\alpha, \beta \in \mathbf{N}$;

6^0 $n \neq \alpha r_x(T) + \beta r_y(T)$ *for any* $\alpha, \beta \in \mathbf{N}$.

Proof. For abbreviation we write $l_x = l_x(T)$, $r_x = r_x(T)$, $l_y = l_y(T)$ and $r_y = r_y(T)$. Observe first that $n = l_x + r_x = l_y + r_y$ and there is no loss of generality in assuming $l_x \leq r_y$.

Necessity. Suppose that $(l_x, r_x) = d > 1$ ($(l_y, r_y) = d' > 1$, resp.). Then $n = \lambda \cdot d$ ($n = \lambda' \cdot d'$, resp.) for some $\lambda \in \mathbf{N}$ ($\lambda' \in \mathbf{N}$, resp.). It can be easily seen that the sequence d^λ ($d^{\lambda'}$, resp.) is not realizable in T, so the conditions 1^0 and 2^0 are necessary for T to be avd.

Suppose now $l_x = \alpha \cdot d$, $r_y = \beta \cdot d$ for some integers $\alpha, \beta \geq 1$ and $d > 1$. Hence $n = (\alpha+\beta)\cdot d + r$ and, by 1^0, d does not divide r. Let us consider the sequence $r \cdot d^\lambda$ if $r \leq d$ or $d^\lambda \cdot r$ otherwise. Let S be a subtree of T of order r. It can be easily seen that the graph $T - S$ has a connected component C being a star-like tree $S(2, a, b)$ with $(a, b) = \mu d$ for some integer $\mu \geq 1$ or a path of length which is not divisible by d or else a caterpillar T' with two single legs attached at x and y such that d divides $(l_y(T'), r_y(T'))$ or $(l_x(T'), r_x(T'))$. Thus, using the previous argument or Proposition 2 we may deduce that such a sequence is not realizable in C and this implies the necessity of the condition 3^0.

Assume then $(l_y, r_x) = d \geq l_y - l_x \geq 2$ and n is not congruent to 1 modulo d. If $d = l_y - l_x$, then $l_x \equiv 0 \pmod d$ and we can show as above that T is not avd. Assume $d > l_y - l_x$ and let λ and $r \in \{1, \ldots, d-1\}$ be two integers such that $l_x = \lambda d + r$. Thus, $r_x = \alpha d$, $l_y = \beta d$ for some integers α, β and $n = \lambda d + \alpha d + r$. Hence $r \geq 2$ and, because $l_y - l_x < d$, $\beta = \lambda + 1$. Consider now the sequence $\tau = r \cdot d^{\alpha+\lambda}$. Taking the graph $T - S$, where S is a subtree of T on r vertices and using a similar argument as in the previous situation we deduce that τ is not realizable in T, so the condition 4^0 is necessary for T to be avd.

Finally, if $n = \alpha l_x + \beta l_y$ for some $\alpha, \beta \in \mathbf{N}$ (or $n = \alpha r_x + \beta r_y$), then the sequence $l_x^\alpha \cdot l_y^\beta$ (or $r_y^\beta \cdot r_x^\alpha$, resp.) is not realizable in T and this implies the necessity of the conditions 5^0 and 6^0.

Sufficiency. Suppose the conditions 1^0–6^0 hold and let $\tau = (a_1, \ldots, a_k)$ be an admissible sequence for T. We first show that if $a_1 = 1$, then there exists a T-realization of τ. Indeed, consider a caterpillar $T' = T - u$ i.e., a caterpillar with one leg attached at y and an admissible sequence $\tau' =$

$(a_2, a_3, \ldots, a_k)$ for T'. Obviously, if τ' is a realizable sequence for T', then τ is realizable for T. Suppose then, that τ' is not realizable for T'. It follows from Proposition 2 that $(l_y - 1, r_y) = d$ for some integer $d > 1$ and $\tau' = (d, \ldots, d)$. Thus d divides r_y and, by 3^0, l_x is not divisible by d, so τ' is realizable in the tree $T'' = T - v$. It follows that $\tau = (1, d, \ldots, d)$ is realizable in T as claimed.

From now on we will assume that $a_1 \geq 2$, i.e., for every $i = 1, \ldots, k$, $a_i \geq 2$.

Observe that T is avd if and only if for any admissible sequence $\tau = (a_1, \ldots, a_k)$ for T there exists a permutation $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ such that for all $s \in \{1, \ldots, k\}$

$$(*) \qquad \sum_{i=1}^{s} a_{\sigma(i)} \notin \{l_x, l_y\}.$$

Let m be the minimum number $j \in \{1, .., k\}$ such that $a_1 + \ldots + a_j \geq l_x$. Thus, for $m > 1$ we get $a_1 + \ldots + a_{m-1} < l_x$.

Case 1. $a_1 + \ldots + a_m = l_x$. If $a_j = a_1$ for all $j \in \{1, \ldots, k\}$, then we have a contradiction with condition 1^0. Therefore, there exists $j_0 \geq m + 1$ such that $a_{j_0} > a_1$. We may assume that j_0 is minimal with this property. Let σ be the product of three transpositions: $t(1, m)$, $t(m+1, j_0)$ and $t(m, m+1)$ taken in this order. It can be easily seen that $a_{\sigma(1)} + \ldots + a_{\sigma(m)} > l_x$ and $a_{\sigma(1)} + \ldots + a_{\sigma(m-1)} = a_2 + \ldots + a_m < l_x$ for $m > 1$.

Assume that there exists $m' \geq m$ such that $a_{\sigma(1)} + \ldots + a_{\sigma(m')} = l_y$. Now, if $a_{\sigma(j)} = a_1$ for each $j \in \{m'+1, \ldots, k\}$ then $r_y \geq 2a_1$ $(k - 1 > m')$, because $l_x \leq r_y$ and $(l_x, r_y) = 1$. So $j_0 = k$ and $a_i = a_1$ for each $i < k$. It follows that $l_x = ma_1$ and $l_y = (m' - 1)a_1 + a_k$; consequently $r_y = n - l_y = \alpha a_1$ for some α which contradicts 3^0. Hence, we can also assume there exists $s \in \{m' + 1, \ldots, k\}$ such that $a_{\sigma(s)} > a_1$.

Case 1.1. $m = m'$. Hence $a_{\sigma(m)} \geq l_y - l_x + 1$. If $a_{\sigma(j)} > a_{\sigma(m)}$ for some $j > m$ then we can take the permutation $t(m, m+1) \circ t(m+1, j) \circ \sigma$ satisfying $(*)$. Thus we may assume that if $j > m$ then $a_{\sigma(j)}$ can take only two values: a_1 and $a_{\sigma(m)}$. Moreover, by 5^0, we have $m \geq 2$. Set

$d = a_{\sigma(m)}$,
$r = \sum_{i=2}^{m-1} a_i$ for $m > 2$ and
$r = 0$ for $m = 2$.

Hence $l_x = a_1 + r + a_m$ and $l_y = r + a_m + d$.

Case 1.1.1. $d > a_m$. Suppose first $a_m > a_1$ and take the permutation $\sigma' = t(1, m+1) \circ \sigma$ (recall that $a_{\sigma(1)} = a_m$ and $a_{\sigma(m+1)} = a_1$). We have now $a_{\sigma'(1)} + \ldots + a_{\sigma'(m-1)} = a_1 + r < a_1 + r + a_m = l_x$, $l_y = r + a_m + d > a_{\sigma'(1)} + \ldots + a_{\sigma'(m)} = a_1 + r + d > l_x$ (because $a_m > a_1$ and $d > a_m$), $a_{\sigma'(1)} + \ldots + a_{\sigma'(m+1)} = a_1 + r + d + a_m = l_y + a_1 > l_y$, therefore σ' verifies $(*)$. Suppose then $a_1 = a_m$, i.e., $a_j = a_1$ for all $j \in \{1, \ldots, m\}$ and $l_x = \lambda a_1$ for some integer $\lambda \geq 2$. Therefore, by 3^0, there exists $i_0 \geq m+1$, $i_0 \neq j_0$, such that $a_{i_0} = d$. Consider now the permutation $\sigma' = t(m-1, i_0) \circ \sigma$. We have $a_{\sigma'(1)} + \ldots + a_{\sigma'(m)} = (\lambda - 2)a_1 + 2d > l_y = (\lambda - 1)a_1 + d$. Thus, if $(\lambda - 2)a_1 + d \neq l_x = \lambda a_1$, i.e., $d \neq 2a_1$, then σ' satisfies $(*)$. But if $d = 2a_1$, then r_y is divisible by a_1 and we get a contradiction with 3^0.

Case 1.1.2. $d = a_m$. By construction of our permutation σ, we get $a_j = d$, for all $j \geq m$, so $r_x = (k-m)d$ and $a_1 < d$. Instead of our permutation σ take another permutation ρ given by the following formula: $\rho(i) = k - i + 1$, $i = 1, 2, \ldots, k$. Clearly, $a_{\rho(i)} = a_m = d$ for $i = 1, \ldots, k-m$ and, since $l_y < r_x$, we obtain $\sum_{i=1}^{k-m} a_{\rho(i)} > l_y$. From 1^0, l_x is not divisible by d, therefore the condition $(*)$ does not hold for ρ if $l_y = \gamma d$ for some integer γ. But in this case there are three positive integers w, α', β' such that $(l_y, r_x) = wd \geq d > d - a_1 = l_y - l_x$ and $n = r_x + l_x = r_x + l_y - d + a_1 = \alpha' wd + \beta' wd - d + a_1 = (\alpha' + \beta' - 1)wd + (w-1)d + a_1 \neq 1 (\operatorname{mod} wd)$ (because $d > a_1 \geq 2$) and we obtain a contradiction with 4^0.

Case 1.2. $m < m'$. Suppose that there exists $s_0 \in \{m'+1, \ldots, k\}$ such that $a_{\sigma(s_0)} \neq a_{\sigma(m')}$. Without loss of generality we can assume that $s_0 = m'+1$ (if necessary, we can perform an appropriate transposition). Now taking the transposition $t(m', m'+1)$ we get a permutation that satisfies $(*)$. Assume then $a_{\sigma(s)} = a_{\sigma(m')}$ for all $s \in \{m'+1, \ldots, k\}$.

Now, if $m + 1 < m'$ and for some $i \in \{m+1, m'-1\}$ we have $a_{\sigma(i)} \neq a_{\sigma(m')}$, then we can take the permutation $t(m', m'+1) \circ t(i, m') \circ \sigma$ that verifies $(*)$. Therefore, we can assume that $a_{\sigma(s)} = a_{\sigma(m')}$ for all $s \in \{m+1, \ldots, m'\}$, so $a_{\sigma(s)} = a_1$ for $s \in \{m+1, \ldots, k\}$ and $l_x = ma_1$, which is impossible by 3^0.

Case 2. $a_1 + \ldots + a_m > l_x$. We may assume that there exists $m' \geq m$ such that $a_1 + \ldots + a_{m'} = l_y$, because otherwise the identity permutation satisfies $(*)$. Now, since $a_i \geq a_{m'}$ for $i > m'$, it is enough to consider only the case where $a_i = a_{m'}$ for $i > m'$, i.e., $r_y = \alpha a_{m'}$ for some integer α. Using the same method as in Case 1.2 we see that if there is no permutation verifying $(*)$, then $a_i = a_{m'}$ for all $i > m$. Notice that if $a_{m+1} > a_m$ then the transposition $\sigma = t(m, m'+1)$ satisfies $(*)$. So assume $a_i = a_{m'}$ for all $i \geq m$. Hence $l_y < r_x < (k-m+1)a_{m'}$. Now take the permutation ρ defined as follows: $\rho(i) = k-i+1$, $i = 1, 2, \ldots, k$. Since $r_y = \alpha a_{m'}$, for some integer α, it follows by 3^0 and 2^0 that the condition $(*)$ holds for ρ and we are done. This finishes the proof of the theorem. ∎

4. Arbitrarily Vertex Decomposable $S(3, a, b)$ and $S(2, 2, a, b)$

Theorem 6. *Let a, b, $3 \leq a \leq b$, be two integers and $T = S(3, a, b)$ a star-like tree with three arms. Then T is avd if and only if the following conditions hold:*

1^0 $(a, b) \leq 2$;
2^0 $(a+1, b) \leq 2$;
3^0 $(a, b+1) \leq 2$;
4^0 $(a+1, b+1) \leq 3$;
5^0 $n \neq \alpha \cdot a + \beta \cdot (a+1)$ *for* $\alpha, \beta \in \mathbf{N}$.

Proof. Let c be the primary vertex of degree three of T and A_1, A_2, A_3 its arms. The vertices of three arms will be denoted as follows:

$$V(A_1) = \{c, x, y\},$$
$$V(A_2) = \{x_1, \ldots, x_a = c\},$$
$$V(A_3) = \{x_a = c, x_{a+1}, \ldots, x_{a+b-1}\},$$

(see Figure 2).

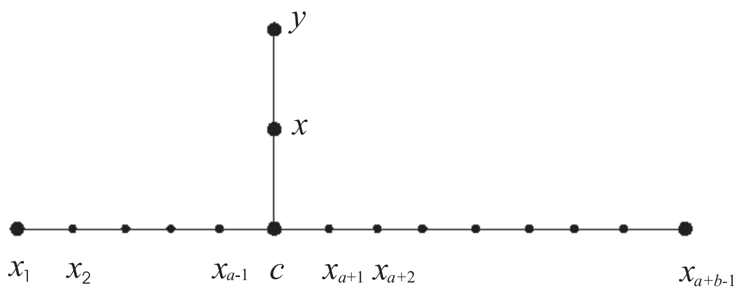

Figure 2. $S(3, a, b)$

Necessity. Suppose that $(a, b) = d > 2$. Then $n = \lambda \cdot d + 1$ for some integer $\lambda \geq 2$, and it can be easily seen that the sequence $d^{\lambda-1} \cdot (d+1)$ is not realizable in T.

Let $(a+1, b) = d \geq 3$ ($(a, b+1) = d' \geq 3$). We have $n = \lambda \cdot d$, $\lambda \in \mathbf{N}, \lambda \geq 2$ ($n = \lambda' \cdot d'$, $\lambda' \in \mathbf{N}, \lambda' \geq 2$, resp.) and it is easy to check that the sequence d^{λ} ($(d')^{\lambda'}$, resp.) is not realizable in T.

Similarly, if $(a+1, b+1) = d > 3$, then $n = \lambda \cdot d - 1$, $\lambda \in \mathbf{N}$, so the sequence $(d-1) \cdot d^{\lambda-1}$ is not realizable in T.

We now turn to the case $n = \alpha \cdot a + \beta \cdot (a+1)$, $\alpha, \beta \in \mathbf{N}$. This implies that the sequence $a^{\alpha} \cdot (a+1)^{\beta}$ is not realizable in T.

Sufficiency. Suppose that conditions 1^0–5^0 hold and let $\tau = (m_1, \ldots, m_k)$ be an admissible sequence for the tree T. Such a sequence is realizable in T if $m_k = 1$ (because it is ordered in a non-decreasing way), so we will assume $m_k > 1$. Let $\hat{\tau} = (n_1, \ldots, n_k)$ be a non-decreasing ordering of the sequence $(m_1, \ldots, m_{k-1}, m_k - 1)$, with $n_s = m_k - 1$. Consider the tree $\hat{T} = T - y$ which is isomorphic to the star-like tree $S(2, a, b)$. Clearly, the sequence $\hat{\tau}$ is admissible for the tree $\hat{T}$. Suppose $\hat{\tau}$ is not realizable in $\hat{T}$. Then, by *1*^{0} and Proposition 2, $(a, b) = 2$ and $\hat{\tau} = 2^k$. Hence $\tau = 2^{k-1} \cdot 3$ is obviously realizable in T. From now on we will assume that $\hat{\tau}$ is realizable in $\hat{T}$.

Furthermore, since τ is realizable in T if $m_i \in \{1, 2\}$ for some i, we can assume $n_j \geq 3$ for all $j \neq s$ and $n_s \geq 2$. Let $\hat{M} = (V_1, \ldots, V_s, \ldots, V_k)$ be a $\hat{T}$-realization of $\hat{\tau}$ such that $|V_i| = n_i$ for $i = 1, \ldots, k$, and V_p induces the primary tree of $\hat{T}$. Observe that if

$$(*) \qquad |V_p| = m_k - 1,$$

then the sequence M, obtained from $\hat{M}$ by replacing V_p by $V_p \cup \{y\}$, is a T-realization of τ. Therefore, we will assume that the condition $(*)$ does not hold (so $V_p \neq V_s$).

Case 1. $V_s \subset V(A_2)$. Suppose $x_{a-1} \in V_p$. Because $A_2 - V_p$ is a path in $\hat{T}$, we can arrange the sets V_i's covering this path in such a way that V_p and V_s are neighboring in A_2. Therefore, the subtree of T induced by $V_p \cup V_s \cup \{y\}$ can be covered by $(V_s \cup \{z\}, V_p \setminus \{z\} \cup \{y\})$, where z is the first vertex of V_p on A_2. Adding the remaining sets V_i we get a T-realization of τ. Thus, let us assume that V_p induces a path in $\hat{T}$ such that $V_p \setminus \{x\} \subset V(A_3)$ (see Figure 3).

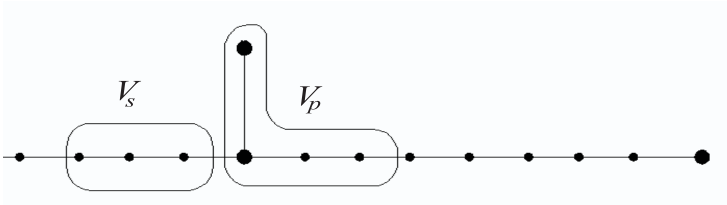

Figure 3. V_p and V_s are neighboring in A_2.

Suppose now $n_s > n_p$. Since $A_2 - V_p$ is a path in $\hat{T}$, we can assume without loss of generality that V_p and V_s are neighboring in A_2 (see Figure 3). Now the subtree of $\hat{T}$ induced by the set $V_s \cup V_p$ can be covered by (V_s', V_p'), where V_s' induces a subpath of A_2 on n_p vertices, and V_p' a star-like tree on $n_s = m_k - 1$ vertices that contains c. Put $V_i' = V_i$ for $i \neq p, s$. It is easy to see that $(V_1', \ldots, V_k')$ is a $\hat{T}$-realization of $\hat{\tau}$ satisfying $(*)$ and we can easily obtain a T-realization of τ. Hence, by the choice of n_s, we can assume that $n_s = n_p - 1 = m_k - 1$. Then $n_i \leq n_p$ for all i's. If for some $i \neq s, p$, $V_i \subset A_2$ and $|V_i| \leq n_p - 2$, then, assuming that V_i and V_p are neighboring in A_2, we can cover $V_i \cup V_p$ by the pair (V_i', V_p'), where V_i' induces a subpath of $A_3 - c$ on n_i vertices and V_p' induces a tree containing c. Applying the same argument as above we get a T-realization of τ. Hence, $n_p - 1 \leq |V_i| \leq n_p$ for all i's such that $V_i \subset V(A_2)$. Suppose that for some j, $V_j \subset V(A_3)$ and $|V_j| < n_p$. Now, because V_p induces a path in $\hat{T}$, we can place this V_j at the beginning of the path $xcx_{a+1} \ldots x_{a+b-1}$ and find a T-realization of τ as in the previous cases. Thus, $|V_i| = n_p$ for all i's such that $V_i \subset V(A_3)$.

Let $q := n_p$. We have now $a = \lambda q + \mu(q-1)$ and $b + 1 = \nu q$, for some integers $\lambda > 0$, $\mu \geq 0$ and $\nu > 0$. Moreover, the sequence τ is of the form

$$(q-1)^{\mu} \cdot q^{\lambda+\nu}.$$

If $\mu = 0$, then, by 3^0, $q \leq 2$, a contradiction with our assumption on n_p. Suppose $\mu = 1$. Then $a+1 = (\lambda+1)q$, hence, by 4^0, $q = 3$, so $\tau = 2 \cdot 3^k$ and this sequence is clearly T-realizable. So consider the case $\mu \geq 2$. Because $a < b$, it follows that $\nu \geq 2$, so the sequence $(q-1)^2 \cdot q^{\nu-2}$ is realizable in $A_3 - c$ and the sequence $(q-1)^{\mu-2} \cdot q^{\lambda+2}$ is realizable in the tree induced by $A_2 \cup \{x, y\}$, hence τ is realizable in T.

Case 2. $V_s \subset V(A_3)$. As in Case 1 we assume that $x_{a+1} \notin V_p$, $q - 1 \leq |V_i| \leq q$ for $V_i \subset V(A_3)$ and $|V_j| = q$ for $V_j \subset V(A_2)$, where $q = n_p$. Now we can write $b = \lambda q + \mu(q-1)$ and $a+1 = \nu q$, for some integers $\lambda > 0$, $\mu \geq 0$ and $\nu > 0$. If $\mu = 0$, then, by 2^0, $q \leq 2$, and we get a contradiction with our assumption on n_p. For $\mu = 1$ we proceed as in Case 1 and show that τ is realizable in T. Suppose then $\mu \geq 2$. If $\nu \geq 2$ we proceed as in Case 1 and we show that τ is realizable in T. If $\nu = 1$ (the essential difference between Case 1 and Case 2), then $q = a+1$ and $n = (\lambda+1)(a+1) + \mu a$, a contradiction. This finishes the proof of the theorem. ■

Corollary 7. *Let a, b, $3 \leq a \leq b$ be two integers and $T = S(2,2,a,b)$ a star-like tree on n vertices. Then T is avd if and only if the following conditions hold:*

$1'$ $(a,b) = 1$;

$2'$ $(a+1,b) = 1$;

$3'$ $(a,b+1) = 1$;

$4'$ $(a+1,b+1) = 2$;

$5'$ $n \neq \alpha \cdot a + \beta \cdot (a+1)$ *for* $\alpha, \beta \in \mathbf{N}$.

Proof. **Necessity.** Assume that T is avd. Hence, from Proposition 3, $S(3,a,b)$ is avd, a, b are odd, and $a \neq 2 \pmod 3$ or $b \neq 2 \pmod 3$.

Therefore, the odd numbers a and b satisfy the conditions 1^0–5^0 of Theorem 6, hence also the conditions $1'$ and $5'$ of our theorem. Since a and b are odd, it follows by 1^0, 2^0 and 3^0 that $(a,b) = 1$, $(a+1,b) = 1$ and $(a,b+1) = 1$. So a and b satisfy $1'$, $2'$ and $3'$. By 4^0, $(a+1,b+1) \in \{2,3\}$ and since $a \neq 2 \pmod 3$ or $b \neq 2 \pmod 3$, we have $(a+1,b+1) \neq 3$ and the condition $4'$ holds.

Sufficiency. If a and b verify the conditions $1'$–$5'$ then the conditions 1^0–5^0 of Theorem 6 are satisfied. Thus $S(3,a,b)$ is avd and, by $1'$–$3'$, a and b are odd.

Suppose that $a \equiv 2 \pmod 3$ and $b \equiv 2 \pmod 3$. Then $a+1 \equiv 0 \pmod 3$ and $b+1 \equiv 0 \pmod 3$, so $(a+1,b+1) \geq 3$, a contradiction. This implies that $a \neq 2 \pmod 3$ or $b \neq 2 \pmod 3$, and, by Proposition 3, T is avd. This finishes the proof. ■

Corollary 8. *There are infinitely many arbitrarily vertex decomposable star-like trees $S(3,a,b)$ and $S(2,2,a,b)$.*

Proof. Let $a \geq 5$ be a prime and $b = a + 2$. It can be easily seen that a and b satisfy the conditions $1'$–$5'$ (and also 1^0–5^0) for $n = 2a + 3$. ■

5. Caterpillars with One Double and One Single Leg

Every caterpillar with one double and one single leg attached at x and y can be constructed in the following way. Take a path $P = x_1, \ldots, x_{n-3}$ where $x = x_a$ and $y = x_j$ $(a < j)$ are two internal vertices of P, add three vertices u,v and z and join u and v to x and v to y (see Figure 4).

Let $L_x = \{x_1, x_2, \ldots, x\}$, $R_x = \{x, x_{a+1}, \ldots, x_{n-3}\} \cup \{z\}$, $L_y = \{x_1, x_2, \ldots, y\} \cup \{u, v\}$, $R_y = \{y, x_{j+1}, \ldots, x_{n-3}\}$ and let $l_x = |L_x|$, $r_x = |R_x|$, $l_y = |L_y|$ and $r_y = |R_y|$.

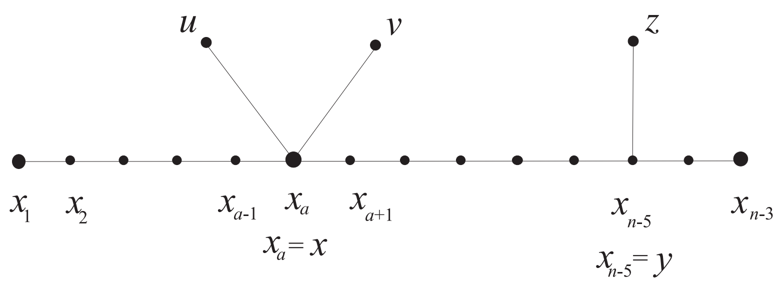

Figure 4. A caterpillar with one double and one single leg.

Proposition 9. *Let T be a caterpillar of order n with one double and one single leg attached at x and y resp. Let $a = l_x$ and $b = r_x$. If $a \equiv 1 \pmod 6$, $b \equiv 0 \pmod 3$, $7 \leq a < b$, $(a - 3, b) = 1$, $n - 1 \neq \alpha a$ $(\alpha \in \mathbf{N})$, $r_y = 3$ and a and b satisfy the conditions $1'$–$5'$ of Corollary 7, then T is avd.*

Proof. Let u and v denote two vertices of degree one adjacent to x and let z be the vertex of degree one adjacent to y (see Figure 4). It follows from our assumptions that $n = a + b + 1 \equiv 2 \pmod 3$. Let $\tau = (a_1, \ldots, a_k)$ be an admissible sequence for the tree T. We will show that it suffices to consider the case where $a_t \geq 2$ for all t. Indeed, the caterpillar $T' = T - v$ with two single legs satisfies $l'_x = a$, $r'_x = b$, $l'_y = a + b - 3 = n - 4 \equiv 1 \pmod 3$, $r'_y = 3$, so the conditions 1^0–3^0 of Theorem 5 are satisfied. We also have $(l'_y, r'_x) = (a + b - 3, b) = (a - 3, b) = 1 < l'_y - l'_x = b - 3$, so the condition

4^0 holds. Furthermore, if $\alpha l'_x + \beta l'_y = \alpha a + (n-4)\beta = n-1$, for some $\alpha, \beta \in \mathbf{N}$, then, since $a \geq 7$, we have $\beta = 0$, which is a contradiction. Assume $n-1 = \alpha r'_x + \beta r'_y = \alpha b + 3\beta \equiv 0 \pmod 3$ ($\alpha, \beta \in \mathbf{N}$). But $n-1 = a+b \equiv 1 \pmod 3$, and we get a contradiction. So also 5^0–6^0 of Theorem 5 hold. Now, if $a_1 = 1$, we can put $V_1 = \{v\}$ and the existence of T-realization of τ is obvious. Therefore, we may assume $a_t \geq 2$ for all t.

Notice that, by Corollary 7, the star-like tree $\hat{T} = S(2,2,a,b)$ obtained by deleting the edge zy and adding zx_{n-3} is avd. Let $\hat{M} = (V_1, \ldots, V_k)$ be a $\hat{T}$-realization of τ such that V_p (V_s, resp.) induces a primary tree (a primary tree or a subpath, resp.) of $\hat{T}$ containing x (y, resp.). Observe that if V_s contains x_{n-4} (the vertex that follows y in the path $x_1, \ldots, x_{n-3}$) then τ is T-realizable. Indeed, if $z \in V_s$, then $\hat{M}$ is also a T-realization of τ and if $z \in V_{s'}$ for some $s' \neq s$, then $V_{s'} = \{x_{n-3}, z\}$ and replacing in $\hat{M}$ V_s and $V_{s'}$ by the sets $(V_s \setminus \{x_{n-4}\}) \cup \{z\}$ and $(V_{s'} \setminus \{z\}) \cup \{x_{n-4}\}$, we get a T-realization of τ.

Therefore, we shall assume that V_s does not contain x_{n-4}. Hence, because $a_r \geq 2$ for all r, there is g with $V_g = \{x_{n-4}, x_{n-3}, z\}$ (see Figure 5).

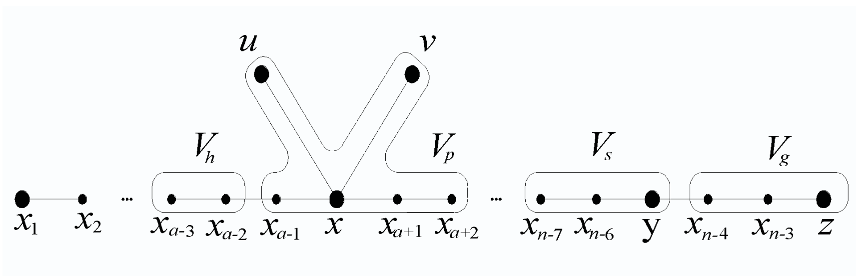

Figure 5. Case 1.1

Notice that for every r such that $V_r \subset R_x$ we have $|V_r| = 3$, for otherwise $g \neq r$ and assuming V_r and V_g are neighboring in R_x we could transpose V_r and V_g into V'_r and V'_g, in such a way that V'_r or V'_g contains the set $\{y, x_{n-4}\}$.

Now, since $|R_x| = b \equiv 0 \pmod 3$, we have $|V_p \cap (R_x \setminus \{x\})| \equiv 2 \pmod 3$, hence $|V_p \cap (R_x \setminus \{x\})| \geq 2$. Furthermore, since $a_r \geq 2$ for all r, we have $u, v \in V_p$ and $|V_p| \geq 5$.

Case 1. There is h such that $V_h \subset L_x$ and $|V_h| \neq 3$. Obviously, we may suppose that V_h and V_p are neighboring in L_x.

Case 1.1. $|V_h| \leq |V_p \cap (R_x \setminus \{x\})|$ (see Figure 5). Now we can transpose V_p and V_h into V'_p and V'_h with $V'_h \subset R_x$. Using the same argument as above, we easily find a T-realization $(V'_1, \ldots, V'_k)$ of τ.

Case 1.2. $|V_h| > |V_p \cap (R_x \setminus \{x\})|$. Let $b = 3q$ and $|V_h| = 3w + r$, where q, w, r are three integers such that $3 \leq q$, $1 \leq w$ and $r \in \{0, 1, 2\}$. We have by assumption $3 < |V_h| = 3w + r < a < b = 3q$, so setting:
$V'_h = \{x_{n-3w-r-1}, x_{n-3w-r}, \ldots, x_{n-3}, z\}$,
$V'_p = \{x_t, x_{t+1}, \ldots, x_a, \ldots, x_{a+2-r}\} \cup \{u, v\}$,
where $|V'_p| = |V_p| = a_p$, we can cover the remaining vertices of R_x by $q - w - 1 \geq 0$ sets of cardinality 3 and the existence of a T-realization $(V'_1, \ldots, V'_k)$ of τ is obvious.

Case 2. $\tau = (3, 3, \ldots, 3, |V_p|)$. Because $a - 1 \equiv 0 \pmod 3$ and $|V_p| > 3$, we can place the set of cardinality $|V_p|$ at the end of the path $x_1, x_2, \ldots, x_{n-3}, z$ and easily construct a realization of τ in T. ■

Theorem 10. *The number of avd caterpillars with one double and one single leg is infinite.*

Proof. Take a such that $b = a + 2 = 3p$, where p is a prime number greater than five. Therefore, $a \equiv 1 \pmod 6$, $(b, a - 3) = 1$, $n = 2a + 3$, $n - 1 = 2a + 2$ and it is easy to check that the assumptions $1'$–$5'$ of Corollary 7 hold. By Proposition 9 our caterpillar is avd. ■

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