

ARC-TRANSITIVE AND s -REGULAR CAYLEY GRAPHS OF VALENCY FIVE ON ABELIAN GROUPS

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Abstract

Let G be a finite group, and let $1_G \notin S \subseteq G$. A *Cayley di-graph* $\Gamma = \text{Cay}(G, S)$ of G relative to S is a di-graph with a *vertex* set G such that, for $x, y \in G$, the pair (x, y) is an arc if and only if $yx^{-1} \in S$. Further, if $S = S^{-1} := \{s^{-1} | s \in S\}$, then Γ is undirected. Γ is connected if and only if $G = \langle S \rangle$. A Cayley (di)graph $\Gamma = \text{Cay}(G, S)$ is called *normal* if the right regular representation of G is a normal subgroup of the automorphism group of Γ . A graph Γ is said to be arc-transitive, if $\text{Aut}(\Gamma)$ is transitive on an arc set. Also, a graph Γ is s -regular if $\text{Aut}(\Gamma)$ acts regularly on the set of s -arcs.

In this paper, we first give a complete classification for arc-transitive Cayley graphs of valency five on finite Abelian groups. Moreover, we classify s -regular Cayley graph with valency five on an abelian group for each $s \geq 1$.

Keywords: Cayley graph, normal Cayley graph, arc-transitive, s -regular Cayley graph.

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1. INTRODUCTION

For a group G , and a subset S of G such that $1_G \notin S$, a Cayley graph $\text{Cay}(G, S)$ of G relative to S is defined as a graph with a vertex set G and edge set E consisting of those ordered pairs (x, y) from G for which $yx^{-1} \in S$. If S is symmetric, that is, if $S^{-1} = \{s^{-1} : s \in S\}$ is equal to S ,

then (x, y) is an edge if and only if (y, x) is an edge, and $\text{Cay}(G, S)$ is said to be undirected. For a finite, simple and undirect graph Γ , we use $V(\Gamma)$, $E(\Gamma)$, and $\text{Aut}(\Gamma)$ to denote its vertex set, edge set and full automorphism group respectively. Γ is said to be vertex-transitive and edge-transitive, if $\text{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, and $E(\Gamma)$, respectively. Moreover, for a positive integer s , an s -arc of Γ is an $(s + 1)$ -tuple (v_1, v_2, \dots, v_s) of vertices such that $\{v_{i-1}, v_i\} \in E(\Gamma)$ for $1 \leq i \leq s$ and if $s \geq 2$, then $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$. We call Γ s -arc-transitive, if $\text{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ and on the set of s -arcs; and Γ is called an s -transitive graph if Γ is s -arc-transitive but not $(s + 1)$ -arc-transitive. For the case $s = 1$, we simply use $A(\Gamma)$ to denote its 1-arc set and call 1-arc-transitive graphs arc-transitive. An arc-transitive graph Γ is said to be s -regular if for any two s -arcs in Γ , there is a unique automorphism of Γ mapping one to the other. Also, an arc-transitive graph Γ is said to be one regular if $|\text{Aut}(\Gamma)| = |A(\Gamma)|$.

In [11] Ming-Yao Xu and Jing Xu classified all arc-transitive Cayley graphs of valency at most four on Abelian groups and in [12], M.Y. Xu classified all one-regular circulant graphs of valency 4. Ming-Yao Xu, Hyo-Seob Sim and Young- Gheel Baik [13] classified all arc-transitive circulant graphs and digraphs of order p^m , where p is an odd prime. For the case $m = 1$, that is, for the group $G = Z_p$, C.Y. Chao [5] gave such a classification for undirected case in 1971. In 1972 Berggen [4] simplified Chao's proof; also Chao and Wells [6] did the same thing for the directed case in 1973. On the other hand, Alspach Conder, Marusic [1] classified all 2-arc-transitive circulant graphs. The purpose of this paper is to investigate arc-transitive Cayley graphs of valency five on an Abelian group, that is, the arc-transitive graphs whose automorphism groups have an Abelian regular subgroup.

The groups- and graph-theoretic notation and terminology are standard; see [1, 2, 7, 10], for example.

We will denote the semi-directed product of group H by K with $H.K$.

Theorem 1.1. *Let G be an Abelian group and let S be a subset of G such that $1_G \notin S$. Suppose that $\Gamma = \text{Cay}(G, S)$ is a connected undirected Cayley graph of group G on S .*

(a) *Let Γ be non-normal. Then all arc-transitive Cayley graphs Γ with valency five are as follows:*

- (1) $G = Z_4 \times Z_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{a, a^{-1}, b, c, d\}$, $\Gamma = K_2 \times Q_4 = Q_5$, $\text{Aut}(\Gamma) = S_2 \text{wr} S_5$.

- (2) $G = Z_4^2 \times Z_2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{a, a^{-1}, b, b^{-1}, c\}$, $\Gamma = C_4 \times Q_3 = Q_5$, $\text{Aut}(\Gamma) = S_2 \text{wr} S_5$.
- (3) $G = Z_4 \times Z_2^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{a, a^{-1}, b, c, a^2bc\}$, $\Gamma = Q_4^d$, $\text{Aut}(\Gamma) = S_2^4.S_5$.
- (4) $G = Z_6 = \langle a \rangle$, $S = \{a, a^2, a^3, a^4, a^5\}$, $\Gamma = C_3[K_2] = K_6$, $\text{Aut}(\Gamma) = S_6$.
- (5) $G = Z_{10} = \langle a \rangle$, $S = \{a, a^3, a^7, a^9, a^5\}$, $\Gamma = K_{5,5}$, $\text{Aut}(\Gamma) = S_5 \text{wr} S_2$.
- (6) $G = Z_6 \times Z_2 = \langle a \rangle \times \langle b \rangle$, $S = \{a, a^{-1}, a^2b, a^{-2}b, b\}$, $\Gamma = K_{6,6} - 6K_2$, $\text{Aut}(\Gamma) = S_6 \times S_2$.
- (b) *Let Γ be normal. Then Γ is arc-transitive if one of the following holds:*
 - (1) $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{a, b, c, d, abcd\}$, $\Gamma = Q_4^d$, $\text{Aut}(\Gamma) = S_2^4.S_5$.
 - (2) $G = Z_2^5 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, $S = \{a, b, c, d, e\}$, and $\Gamma = Q_5$, $\text{Aut}(\Gamma) = S_2 \text{wr} S_5$.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and in Section 3, we prove Theorem 1.1. In the last section, we will classify s -regular Cayley graphs with valency five on an Abelian group for each $s \geq 1$.

2. PRIMARY ANALYSIS

For a graph Γ , we denote the automorphism group of Γ by $\text{Aut}(\Gamma)$. The following propositions are basic.

Proposition 2.1. *Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph of G on S .*

- (1) *$\text{Aut}(\Gamma)$ contains the right regular representation G , so Γ is vertex-transitive.*
- (2) *Γ is connected if and only if $G = \langle S \rangle$.*
- (3) *Γ is undirected if and only if $S^{-1} = S$.*

Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph of G on S , and let

$$\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}.$$

Obviously, $\text{Aut}(\Gamma) \geq G\text{Aut}(G, S)$ write $A = \text{Aut}(\Gamma)$. We have,

Proposition 2.2 [8, 11].

- (1) $N_A(G) = G.\text{Aut}(G, S)$.
- (2) $A = G.\text{Aut}(G, S)$ is equivalent to $G \triangleleft A$.

Proposition 2.3 [9]. *A graph Γ is arc-transitive if it is vertex-transitive and the stabilizer G_u of a vertex u acts transitively on the neighborhood $\Gamma_1(u)$ of u in Γ .*

Definition 2.4. A Cayley graph $\Gamma = \text{Cay}(G, S)$ is called normal if $G \trianglelefteq \text{Aut}(\Gamma)$.

Proposition 2.5. *Let $\Gamma = \text{Cay}(G, S)$ be a normal Cayley graph on G relative to S , Then Γ is arc-transitive if and only if $\text{Aut}(G, S)$ acts transitively on the neighborhood $\Gamma_1(1)$ of 1 in Γ .*

For the normality of Cayley graphs of valency five on Abelian groups we have the following:

Theorem 2.6 [3]. *Let $\Gamma = \text{Cay}(G, S)$ be a connected undirected Cayley graph of an Abelian group G on S with valency 5. Then Γ is normal except when one of the following cases holds:*

- (1) $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{a, b, c, d, abc\}$ and $\Gamma = K_2 \times K_{4,4}$.
- (2) $G = Z_4 \times Z_2^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{a, a^{-1}, a^2, b, c\}$ and $\Gamma = C_4 \times K_4$.
- (3) $G = Z_4 \times Z_2^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{a, a^{-1}, b, c, a^2b\}$ and $\Gamma = K_2 \times K_{4,4}$.
- (4) $G = Z_4 \times Z_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{a, a^{-1}, b, c, d\}$ and $\Gamma = K_2 \times Q_4 = Q_5$.
- (5) $G = Z_6 \times Z_2^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{a, a^{-1}, a^3, b, c\}$ and $\Gamma = K_{3,3} \times C_4$.
- (6) $G = Z_m \times Z_2^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ with $m \geq 3$, $S = \{a, a^{-1}, ab, a^{-1}b, c\}$ and $\Gamma = K_2 \times C_m[2K_1]$.
- (7) $G = Z_{4m} \times Z_2 = \langle a \rangle \times \langle b \rangle$ with $m \geq 3$, $S = \{a, a^{-1}, a^{2m-1}, a^{2m+1}, b\}$ and $\Gamma = K_2 \times C_m[2K_1]$.
- (8) $G = Z_{10} = \langle a \rangle$, $S = \{a^2, a^4, a^6, a^8, a^5\}$ and $\Gamma = K_2 \times K_5$.
- (9) $G = Z_{10} \times Z_2 = \langle a \rangle \times \langle b \rangle$, $S = \{a, a^{-1}, a^3, a^7, b\}$, $\Gamma = K_2 \times (K_{5,5} - 5K_2)$.
- (10) $G = Z_m \times Z_4 = \langle a \rangle \times \langle b \rangle$ with $m \geq 3$, $S = \{a, a^{-1}, b, b^{-1}b^2\}$ and $\Gamma = C_m \times K_4$.

- (11) $G = Z_m \times Z_6 = \langle a \rangle \times \langle b \rangle$ with $m \geq 3$, $S = \{a, a^{-1}, b, b^{-1}, b^3\}$ and $\Gamma = C_m \times K_{3,3}$.
- (12) $G = Z_m \times Z_4 \times Z_2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ with $m \geq 3$, $S = \{a, a^{-1}, b, b^{-1}, c\}$ and $\Gamma = C_m \times Q_3$.
- (13) $G = Z_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{a, b, c, ab, ac\}$ and $\Gamma = K_2[2K_2]$.
- (14) $G = Z_4 \times Z_2 = \langle a \rangle \times \langle b \rangle$, $S = \{a, a^{-1}, b, a^2, a^2b\}$ and $\Gamma = K_2[2K_2]$.
- (15) $G = Z_4 \times Z_2^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{a, a^{-1}, b, c, a^2bc\}$ and $\Gamma = Q_4^d$.
- (16) $G = Z_{2m} = \langle a \rangle$ with $m \geq 3$, $S = \{a, a^{-1}, a^{m+1}, a^{m-1}, a^m\}$ and $\Gamma = C_m[K_2]$.
- (17) $G = Z_{2m} \times Z_2 = \langle a \rangle \times \langle b \rangle$ with $m \geq 2$, $S = \{a, a^{-1}, ab, a^{-1}b, b\}$ and $\Gamma = C_{2m}[K_2]$.
- (18) $G = Z_{2m} \times Z_2 = \langle a \rangle \times \langle b \rangle$ with $m \geq 2$, $S = \{a, a^{-1}, ab, a^{-1}ba^m\}$ and $\Gamma = C_{2m}^d[2K_1]$.
- (19) $G = Z_{10} = \langle a \rangle$, $S = \{a, a^3, a^7, a^9, a^5\}$ and $\Gamma = K_{5,5}$.
- (20) $G = Z_6 \times Z_2 = \langle a \rangle \times \langle b \rangle$, $S = \{a, a^{-1}, a^2b, a^{-2}b, b\}$ and $\Gamma = K_{6,6} - 6K_2$.
- (21) $G = Z_{2m} \times Z_4 = \langle a \rangle \times \langle b \rangle$ with $m \geq 2$, $S = \{a, a^{-1}, b, b^{-1}, a^mb^2\}$ and $\Gamma = Q_3 \times C_m$.
- (22) $G = Z_{6m} = \langle a \rangle$ with m odd and $m \geq 3$, $S = \{a^2, a^{-2}, a^m, a^{5m}, a^{3m}\}$ and $\Gamma = K_{3,3} \times_c C_m$.
- (23) $G = Z_{6m} \times Z_2 = \langle a \rangle \times \langle b \rangle$ with $m \geq 2$, $S = \{a, a^{-1}, ba^m, ba^{-m}, ba^{3m}\}$ and $\Gamma = K_{3,3} \times_c C_{2m}$.

Let X and Y be two graphs. The *direct product* $X \times Y$ is defined as a graph with a vertex set $V(X \times Y) = V(X) \times V(Y)$ such that for any vertex $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X \times Y)$, $[u, v]$ is an edge in $X \times Y$ whenever $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$ or $y_1 = y_2$ and $[x_1, x_2] \in E(X)$. Two graphs are called relatively prime if they have no nontrivial common direct factor. The *lexicographic product* $X[Y]$ is defined as a graph vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X[Y])$, $[u, v]$ is an edge in $X[Y]$ whenever $[x_1, x_2] \in E(X)$ or $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$. Let $V(Y) = \{y_1, y_2, \dots, y_n\}$. Then there is a natural embedding nX in $X[Y]$, where for $1 \leq i \leq n$, the i th copy of X is a subgraph induced on the vertex subset $\{(x, y_i) | x \in V(X)\}$ in $X[Y]$. The deleted lexicographic product $X[Y] - nX$ is a graph obtained by deleting all the edges of (this natural embedding of) nX from $X[Y]$.

Let Γ be a graph and α a permutation of $V(\Gamma)$, and C_n a circuit of length n . The *twisted product* $\Gamma \times_\alpha C_n$ of Γ by C_n with respect to α is defined by

$$\begin{aligned} V(\Gamma \times_\alpha C_n) &= V(\Gamma) \times V(C_n) = \{(x, i) \mid x \in V(\Gamma), i = 0, 1, \dots, n-1\}, \\ E(\Gamma \times_\alpha C_n) &= \{[(x, i), (x, i+1)] \mid x \in V(\Gamma), i = 0, 1, \dots, n-2\} \\ &\quad \cup \{[(x, n-1), (x^\alpha, 0)] \mid x \in V(\Gamma)\} \\ &\quad \cup \{[(x, i), (y, i)] \mid [x, y] \in E(\Gamma), i = 0, 1, \dots, n-1\}. \end{aligned}$$

Now we introduce some graphs which appear in our main theorem. The graph Q_4^d denotes a graph obtained by connecting all long diagonals of 4-cube Q_4 , that is, connecting all vertex u and v in Q_4 such that $d(u, v) = 4$. The graph $K_{m,m} \times_c C_n$ is a twisted product of $K_{m,m}$ by C_n such that c is a cycle permutation on each part of the complete bipartite graph $K_{m,m}$. The graph $Q_3 \times_d C_n$ is a twisted product of Q_3 by C_n such that d transposes each pair elements on long diagonals of Q_3 . The graph $C_{2m}^d[2K_1]$ is defined by:

$$\begin{aligned} V(C_{2m}^d[2K_1]) &= V(C_{2m}[2K_1]), \\ E(C_{2m}^d[2K_1]) &= E(C_{2m}[2K_1]) \\ &\quad \cup \{[(x_i, y_j), (x_{i+m}, y_j)] \mid i = 0, 1, \dots, m-1, j = 1, 2\}, \end{aligned}$$

where $V(C_{2m}) = \{x_0, x_1, \dots, x_{2m-1}\}$ and $V(2K_1) = \{y_1, y_2\}$.

3. THE PROOF OF THEOREM

In this section, our objective is to show all arc-transitive Cayley graphs of Abelian groups with valency five.

First, we want to show that some cases of Theorem 2.5 are satisfied by Theorem 1.1.

In the following cases we shall assume $G = \text{Aut}(\Gamma)$.

In the cases (1) and (3), let $V(K_2) = \{y_1, y_2\}$ and let $V(K_{4,4}) = \{x_1, x_2, x_3, x_4, x'_1, x'_2, x'_3, x'_4\}$ such that $(x_i, x'_j) \in E(K_{4,4})$ for $1 \leq i, j \leq 4$. We obtain that $f \notin G_{(y_1, x_1)}$ such that $f(y_2, x_1) = (y_1, x'_1)$, so by Proposition 2.3 Γ is not arc-transitive.

In the cases (2) and (10), let $V(C_m) = \{1, 2, 3, \dots, m\}$ and let $V(K_4) = \{x_1, x_2, x_3, x_4\}$. We also obtain that $f \notin G_{(2, x_1)}$ such that $f(2, x_4) = (3, x_1)$, so by Proposition 2.3 Γ is not arc-transitive.

In the cases (5) and (11), let $V(K_{3,3}) = \{x_1, x_2, x_3, x'_1, x'_2, x'_3\}$ and let $V(C_4) = \{y_1, y_2, y_3, y_4\}$. We have $(x_i, x'_j) \in E(K_{3,3})$, for $1 \leq i, j \leq 4$, and $(y_i, y_{i+1}) \in E(C_4)$. We obtain $f \notin G_{(x_1, y_1)}$ such that $f(x'_1, y_1) = (x_1, y_2)$, so by Proposition 2.3 Γ is not arc-transitive.

In the cases (6) and (7), $\Gamma = K_2 \times C_m[2K_1]$ contains two copies X and Y from $C_m[2K_1]$. Let $V(X) = \{x_1, x_2, \dots, x_m; y_1, y_2, \dots, y_m\}$ and $V(Y) = \{x'_1, x'_2, \dots, x'_m; y'_1, y'_2, \dots, y'_m\}$ such that vertices x_1, x_2, \dots, x_m form a circuit on a copy of X . We find that $f \notin G_{(x_1)}$ such that $f(x_2) = x'_1$, so by Proposition 2.3, Γ is not arc-transitive.

In the case (8), we obtained that for $m \geq 4$, $K_2 \times K_m$ is not arc-transitive. Neither is the graph $K_2 \times K_{4,4}$.

In the case (9), let $V(K_{5,5} - 5K_2) = \{x_1, x_2, \dots, x_5, x'_1, x'_2, \dots, x'_5\}$, $V(K_2) = \{y_1, y_2\}$ such that $(x_i, x'_j) \in E(K_{5,5} - 5K_2)$ for $i \neq j, 1 \leq i, j \leq 5$. We find that $f \notin G_{(y_1, x_1)}$ such that $f(y_1, x'_2) = (x_1, y_2)$, so by Proposition 2.3 Γ is not arc-transitive.

In the cases (12) for $[m \neq 4]$ and (21), let $V(C_m) = \{0, 2, 3, \dots, m-1\}$ and Q_3 contain two circuits C_4, C'_4 with set of vertices $V(C_4) = \{x_1, x_2, x_3, x_4\}$ and $V(C'_4) = \{y_1, y_2, y_3, y_4\}$, respectively. In addition $(x_i, x'_i) \in E(Q_3)$ for $1 \leq i \leq 4$. We obtained that $f \notin G_{(x_1, 0)}$ such that $f(x_1, 0) = (x_1, 0)$, so by Proposition 2.3 Γ is not arc-transitive.

In the cases (13) and (14), let $V(K_2) = \{x, y\}$ and $V(2K_2) = \{1, 2, 3, 4\}$, and also $E(2K_2)$ contain two edges $(1, 2), (3, 4)$. We find that $f \notin G_{(y, 1)}$ such that $f(x, 1) = (y, 2)$, so by Proposition 2.3 Γ is not arc-transitive.

In the case (16) for $[m \neq 3]$, let $V(C_m) = \{1, 2, \dots, m\}$ and $V(K_2) = \{x, y\}$. We find that $f \notin \text{Aut}(\Gamma)$ such that $f([(2, y), (3, x)]) = [(3, x), (3, y)]$.

The case (17) is also the special case of (16), since $2m \neq 3$.

In the case (18), we find that $f \notin G_{(x_0, y_2)}$ such that $f(x_1, y_2) = (x_m, y_2)$, so by Proposition 2.3 Γ is not arc-transitive.

In the cases (22) and (23), let $V(C_m) = \{0, 1, \dots, m-1\}$, $V(K_{3,3}) = \{x_1, x_2, x_3, x'_1, x'_2, x'_3\}$ and also $(x_i, x'_j) \in E(K_{3,3})$ for $1 \leq i, j \leq 3$. We find that $f \notin G_{(x_1, 0)}$ such that $f(x'_1, 0) = (x_1, 1)$, so by Proposition 2.3 Γ is not arc-transitive.

In the case (4), $\Gamma = K_2 \times Q_4 \simeq C_4 \times Q_3$ and Q_3 is arc-transitive, then by combination of functions we conclude that Γ is arc-transitive.

In the cases of (6) and (7) the case (12) for $[m = 4]$ is similarly the case (4).

In the case (15), we will obtain similarly graph $\Gamma = Q_4$.

In the case (16), for $m = 3$ we have $\Gamma \simeq K_6$.

The case (19) is obvious and in the case (20), $\Gamma = K_{6,6} - 6K_2$, and we will obtain the same result in graph $K_{6,6}$. Thus we complete the proof of Theorem 1.1(a).

For the normal case, since $|S| = 5$, S contains at least one element of order 2. Since $\text{Aut}(G, S)$ is transitive on S all five elements in S are of order 2. Then we have one of the following cases:

- (1) $G = Z_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{a, b, c, ab, ac\}$, $\Gamma = K_2[2K_2]$.
- (2) $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{a, b, c, d, abc\}$, $\Gamma = K_2 \times K_{4,4}$.
- (3) $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{a, b, c, d, ab\}$, $\Gamma = K_4 \times C_4$.
- (4) $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{a, b, c, d, abcd\}$, $\Gamma = Q_4^d$.
- (5) $G = Z_2^5 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, $S = \{a, b, c, d, e\}$, $\Gamma = Q_5$.

Note that graphs of the case (1) and (2) are non-normal. In the case (2) of non-normal graphs we showed that graph $\Gamma = K_4 \times C_4$ is not arc-transitive, and in the case (10) of non-normal graphs we showed that $\Gamma = Q_4^d$ is arc-transitive. In the final non-normal case we obtained that the graph Q_5 is arc-transitive.

4. s -REGULAR CAYLEY GRAPH WITH VALENCY FIVE ON ABELIAN GROUPS

Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph on G with respect to S and let $A = \text{Aut}(\Gamma)$. Denote by A_1 the stabilizer of identity 1 of G in A and by $\text{Aut}(G, S)$ the subgroup of A fixing S setwise. Then we have:

Theorem 4.1 [14, Proposition 1.5]. *Γ is normal if and only if $A_1 = \text{Aut}(G, S)$.*

By noting that all Cayley graphs are vertex-transitive, one can easily prove the following lemma.

Lemma 4.2. *All s -regular ($s \geq 1$) Cayley graphs are connected.*

The following theorem gives a classification of s -regular Cayley graphs with valency five on Abelian groups for each $s \geq 1$.

Theorem 4.3. *Let Γ be an s -regular Cayley graph with valency five on an Abelian group for some $s \geq 1$. Then $s = 2$ or 3 . Furthermore, Γ is 2-regular if and only if Γ is isomorphic to Q_4^d , or Q_5 , or K_6 , or $K_{6,6} - 6K_2$; and is 3-regular if and only if Γ is isomorphic to the complete bipartite graph $K_{5,5}$.*

Proof. Let G be an Abelian group. Assume that $\Gamma = \text{Cay}(G, S)$ is an s -regular Cayley graph with valency five for some $s \geq 1$. Then by Lemma 4.2 Γ is connected. By Theorem 1.1, the only non-normal arc-transitive Cayley graphs $\text{Cay}(G, S)$ are the $\Gamma_1 = Q_5$, $\Gamma_2 = Q_4^d$, $\Gamma_3 = K_6$, $\Gamma_4 = K_{5,5}$, and $\Gamma_5 = K_{6,6} - 6K_2$. The graphs Γ_1 , Γ_2 , Γ_3 , and Γ_5 are 2-regular and the graph Γ_4 is 3-regular. Thus, we assume the $\Gamma = \text{Cay}(G, S)$ is normal from now on. Since Γ is of valency 5, $S = S^{-1}$ contains at least one involution in G . As Γ is arc-transitive, so the group $\text{Aut}(G, S)$ acts transitively on S . Hence S consists of five involutions. Since S generates the group G , we have $G = Z_2^3$, or $G = Z_2^4$, or $G = Z_2^5$. By Theorem 1.1, the only normal arc-transitive Cayley graphs $\text{Cay}(G, S)$ with valency 5 are

- (1) $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{a, b, c, d, abcd\}$, $\Gamma = Q_4^d$.
- (2) $G = Z_2^5 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, $S = \{a, b, c, d, e\}$, and $\Gamma = Q_5$,

and each of them is 2-regular.

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