

### 3-CONSECUTIVE C-COLORINGS OF GRAPHS\*

CSILLA BUJTÁS<sup>1</sup>, E. SAMPATHKUMAR<sup>2</sup>, ZSOLT TUZA<sup>1,3</sup>

M.S. SUBRAMANYA<sup>2</sup> AND CHARLES DOMINIC<sup>2</sup>

<sup>1</sup>*Department of Computer Science*  
*University of Pannonia*  
*H-8200 Veszprém, Egyetem u. 10, Hungary*

<sup>2</sup>*Department of Mathematics*  
*University of Mysore, Mysore, India*

<sup>3</sup>*Computer and Automation Institute*  
*Hungarian Academy of Sciences*  
*H-1111 Budapest, Kende u. 13-17, Hungary*

#### Abstract

A 3-consecutive C-coloring of a graph  $G = (V, E)$  is a mapping  $\varphi : V \rightarrow \mathbb{N}$  such that every path on three vertices has at most two colors. We prove general estimates on the maximum number  $\bar{\chi}_{3CC}(G)$  of colors in a 3-consecutive C-coloring of  $G$ , and characterize the structure of connected graphs with  $\bar{\chi}_{3CC}(G) \geq k$  for  $k = 3$  and  $k = 4$ .

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## 1. INTRODUCTION

Motivated by various new developments in the theory of graph and hypergraph coloring, in this note we introduce the notion of 3-consecutive  $C$ -coloring of graphs. For a given graph  $G = (V, E)$ , a mapping

$$\varphi : V \rightarrow \mathbb{N}$$

is called a *3-consecutive  $C$ -coloring* — abbreviated as *3CC-coloring* for short — if there exists no 3-colored path on three vertices; that is, among every three consecutive vertices there exist two having the same color.

Obviously, the trivial coloring that assigns the same color to all vertices of  $G$  is 3-consecutive. Therefore, we are interested in the *maximum* number of colors that can occur in a 3CC-coloring of  $G$ . This number will be called the *3-consecutive upper chromatic number* of  $G$ , denoted by  $\bar{\chi}_{3CC}(G)$ .

It is immediate by definition that, in any 3CC-coloring with more than one color, replacing two color classes with their union results in a 3CC-coloring and the number of colors decreases by precisely one. Consequently, for each integer  $k$  between 1 and  $\bar{\chi}_{3CC}(G)$  there exists a 3CC-coloring of  $G$  with exactly  $k$  colors. Moreover, by the pigeon-hole principle, every coloring with only one or two colors is a 3CC-coloring; that is,  $\bar{\chi}_{3CC}(G) \geq 2$  holds for each graph having at least two vertices. Note further that  $\bar{\chi}_{3CC}(G) = |V(G)|$  if and only if each connected component of  $G$  is an isolated vertex or an isolated edge.

If  $H$  is a spanning subgraph of a graph  $G$ , then any three consecutive vertices of  $H$  are also consecutive in  $G$ . Hence any 3CC-coloring of  $G$  is also a 3CC-coloring of  $H$ , thus

$$\bar{\chi}_{3CC}(G) \leq \bar{\chi}_{3CC}(H).$$

But a maximum 3CC-coloring of  $G$  need not be a maximum 3CC-coloring of  $H$ , therefore strict inequality may hold.

Moreover,  $\bar{\chi}_{3CC}$  is additive with respect to vertex-disjoint union; that is, the 3-consecutive upper chromatic number of a disconnected graph  $G$  is equal to the sum of  $\bar{\chi}_{3CC}$  over the connected components of  $G$ .

**Previous models.** Our new coloring model is closely related to the following earlier ones, which also served as motivation for the present study.

- A *3-consecutive coloring* of a graph  $G = (V, E)$  is a coloring of vertices of  $G$  such that if  $uvw$  is a path on 3 vertices, then  $v$  receives the color of  $u$  or  $w$ . The *3-consecutive coloring number*  $\chi_{3c}(G)$  is the maximum number of colors which can be used in such a coloring. This invariant was introduced in [7] and studied in some detail in [8]. Clearly, any 3-consecutive coloring is a 3CC-coloring, and hence

$$\chi_{3c}(G) \leq \bar{\chi}_{3CC}(G).$$

- A hypergraph is a pair  $\mathcal{H} = (V, \mathcal{E})$ , where  $\mathcal{E}$  is a set system over  $V$ , and  $\emptyset \notin \mathcal{E}$ . The elements of  $V$  and  $\mathcal{E}$  are called vertices and edges of  $\mathcal{H}$ , respectively. A *C-coloring* of  $\mathcal{H}$  is a mapping  $\varphi : V \rightarrow \mathbb{N}$  such that every edge  $E \in \mathcal{E}$  contains at least two vertices with a common color; that is,  $|\varphi(E)| < |E|$ . The *upper chromatic number*, denoted by  $\bar{\chi}(\mathcal{H})$ , is the largest possible number of colors that can be used in a C-coloring of  $\mathcal{H}$ .

The roots of this notion date back to the early 1970's in the works of Berge (unpublished, cf. [1, p. 151]) and Sterboul [11]; moreover, C-coloring is a particular case of Voloshin's *mixed hypergraph* model (introduced in [12]) where it exactly means coloring of C-hypergraphs.

The 3CC-colorings of a graph  $G = (V, E)$  can be interpreted as C-colorings of the hypergraph  $\mathcal{H} = (V, \mathcal{E})$  where  $\mathcal{E}$  consists of all 3-element sets  $\{u, v, w\} \subseteq V$  inducing a connected subgraph of  $G$ .

- A *color-bounded hypergraph*  $\mathcal{H} = (V, \mathcal{E}, \mathbf{s}, \mathbf{t})$  is a hypergraph with  $\mathcal{E} = \{E_1, \dots, E_m\}$  where each edge  $E_i$  is associated with two integers  $s_i$  and  $t_i$  such that  $1 \leq s_i \leq t_i \leq |E_i|$ . A vertex coloring  $\varphi : X \rightarrow \mathbb{N}$  is considered to be proper if for each edge  $E_i$  there occur at least  $s_i$  and at most  $t_i$  different colors on it; that is, the inequalities  $s_i \leq |\varphi(E_i)| \leq t_i$  are satisfied, for every  $E_i \in \mathcal{E}$ , [2, 3].

If  $G = (V, E)$  is a graph with  $V = \{v_1, \dots, v_n\}$ , then its 3CC-colorings are precisely the colorings of the color-bounded hypergraph  $\mathcal{H} = (V, \mathcal{E}, \mathbf{s}, \mathbf{t})$  with  $\mathcal{E} = \{E_1, \dots, E_n\}$ , where each hyperedge  $E_i$  ( $1 \leq i \leq n$ ) is the closed neighborhood of vertex  $v_i$  in  $G$ , and the color bounds are  $\mathbf{s}(E_i) = 1$  and  $\mathbf{t}(E_i) = 2$ .

**Our results.** In this paper we study 3CC-colorings of connected graphs. In Section 2 we prove upper bounds on  $\bar{\chi}_{3CC}(G)$  in terms of several parameters of  $G$  and, particularly, we obtain tight bounds for trees and unicyclic graphs. In Section 3 we give characterizations for graphs admitting proper 3CC-colorings with exactly 3 and exactly 4 colors. These theorems also yield

necessary conditions for graphs having 3-consecutive colorings with exactly 3 and 4 colors, respectively, because of the inequality  $\chi_{3c}(G) \leq \bar{\chi}_{3CC}(G)$ .

**Standard notation.** As usual, we write  $N[v]$  for the *closed neighborhood* of vertex  $v$ , and  $d(x, y)$  for the *distance* of vertices  $x$  and  $y$ . In the latter, we sometimes put subscript as  $d_G(x, y)$ , if the graph under consideration has to be emphasized.

## 2. BOUNDS

**Theorem 1.** *For any graph  $G = (V, E)$  of order  $p$  and minimum degree  $\delta$ , we have  $\bar{\chi}_{3CC}(G) \leq \lfloor \frac{2p}{\delta+1} \rfloor$ .*

**Proof.** Consider a 3CC-coloring of  $G$  with exactly  $k$  colors. Let us call a color class or its color “small” if it contains fewer than  $\frac{\delta+1}{2}$  vertices, otherwise call it “big”. If all colors are big, then we immediately obtain that the number of colors is at most  $2p/(\delta+1)$ .

Hence, we can assume that there are  $\ell \geq 1$  small color classes. Choose one vertex from each. In this way we have vertices  $v_1, \dots, v_\ell$  with small colors  $c_1, \dots, c_\ell$ , respectively. The closed neighborhood  $N[v_i]$  of each  $v_i$  contains at least  $\delta+1$  vertices, from exactly two colors, namely  $c_i$  and another one, say  $\alpha_i$ . Since  $c_i$  is a small color,  $\alpha_i$  is a big one. Moreover, the sets  $N[v_1], \dots, N[v_\ell]$  are mutually disjoint. Indeed, a common vertex with a small color would imply the identity  $\alpha_i = c_j$  for a big and a small color, whilst a common vertex with a big color would yield a polychromatic  $P_3$ .

Now, the set  $A = \bigcup_{1 \leq i \leq \ell} N[v_i]$  contains at least  $\ell(\delta+1)$  vertices and at most  $2\ell$  different colors. Thus, the average size of color classes intersecting  $A$  is at least  $\frac{\delta+1}{2}$ , and all the remaining classes are big, of size at least  $\frac{\delta+1}{2}$  each. This implies  $\bar{\chi}_{3CC}(G) \leq \lfloor \frac{2p}{\delta+1} \rfloor$ . ■

In a graph  $G = (V, E)$  a set  $S \subseteq V$  is a *neighborhood set* if  $\bigcup_{v \in S} \langle N[v] \rangle = G$ , where  $\langle N(v) \rangle$  is the subgraph induced by  $N[v]$ , the closed neighborhood of  $v$ . The *neighborhood number* of a graph  $G$ , denoted by  $n_0(G)$ , is the minimum cardinality of a neighborhood set in  $G$  (see [9]). For short, we shall write *N-set* for neighborhood set in general, and  *$N_0$ -set* for neighborhood set of minimum cardinality.

**Theorem 2.** *Let  $G$  be a connected graph. Then,  $\bar{\chi}_{3CC}(G) \leq n_0(G) + 1$ . Further, for a tree  $T$ ,  $\bar{\chi}_{3CC}(T) = n_0(T) + 1$ .*

**Proof.** Let  $k = n_0(G)$  and  $S = \{v_1, \dots, v_k\}$  be an  $N_0$ -set. Suppose that the vertices of  $S$  are labeled in such a way that  $(N[v_i] \cap \bigcup_{1 \leq j < i} N[v_j]) \neq \emptyset$  for all  $2 \leq i \leq k$ . Such an order on  $S$  exists because  $G$  is connected.

Since each  $N[v_i]$  can have at most two colors, and at least one of them occurs in  $\bigcup_{1 \leq j < i} N[v_j]$  if  $i \neq 1$ ,  $G$  cannot be colored with more than  $|S| + 1 = n_0(G) + 1$  colors. This completes the proof of the first statement.

To prove the second part, we first fix a root in the tree  $T$  and choose a smallest  $N$ -set  $S^*$  with  $|S^*| = n_0(T)$ . If  $S^*$  contains some vertex  $v$  all of whose children also belong to  $S^*$  (or, in particular, if  $v$  is a leaf) then  $v$  can be replaced by its parent in the  $N$ -set. Repeatedly applying this replacement, an  $N$ -set  $S$  is obtained, in which every vertex has at least one child not contained in  $S$ .

Next, we show a procedure which yields a proper 3CC-coloring of  $T$  with exactly  $n_0(T) + 1$  colors. First, assign color 1 to the root, and then in every step choose a vertex  $v$  which has already got a color but its children have not yet. To color its children, we apply the following rules:

- (i) If  $v \notin S$  then every child of  $v$  receives the color of  $v$ .
- (ii) If  $v \in S$  then we choose precisely one child  $u$  not contained in  $S$ . In the coloring,  $u$  will receive its dedicated color, whilst all the remaining children will get the color of  $v$ .

The number of used colors remains the same when we color the children of a vertex  $v \notin S$ , whilst it increases by precisely one when  $v \in S$ . Moreover, there is no leaf belonging to  $S$ . Taking also into account the color of the root, this means exactly  $n_0(T) + 1$  colors.

The obtained vertex coloring of  $T$  is a proper 3CC-coloring. Indeed, for every vertex  $v$ , in the neighborhood  $N[v]$  there occur at most two different colors because if  $u$  gets its dedicated color then  $u \notin S$  is assumed, hence its parent  $v \in S$  has monochromatic  $N[v] \setminus \{u\}$ .

This coloring algorithm proves that for a tree  $T$  the inequality  $\bar{\chi}_{3CC}(T) \geq n_0(T) + 1$  holds. We have already proved that also  $\bar{\chi}_{3CC}(T) \leq n_0(T) + 1$  is valid, hence the second statement follows. ■

We remark that the algorithm described in the proof actually yields a 3-consecutive coloring. Moreover, it is known that  $n_0(G)$  does not exceed the vertex covering number  $\alpha_0(G)$ , moreover  $n_0(G) = \alpha_0(G)$  for every triangle-free graph (see [9]; characterizations for other graph classes and complexity results on  $n_0(G)$  can be found in [6] and [4]). As a consequence, we have

**Corollary 1.**

- (i) For a connected graph  $G$ ,  $\bar{\chi}_{3CC}(G) \leq \alpha_0(G) + 1$ .
- (ii) For a tree  $T$ ,  $\chi_{3c}(T) = \bar{\chi}_{3CC}(T) = \alpha_0(T) + 1 = \beta_1(T) + 1$  where  $\beta_1(T)$  is the edge independence number.
- (iii) For trees, both  $\chi_{3c}$  and  $\bar{\chi}_{3CC}$  can be determined and an optimal coloring can be found in linear time.

A set  $S \subset V$  of vertices in a connected graph  $G = (V, E)$  is called a *connected dominating set* if (i) every vertex  $v \in V \setminus S$  is adjacent to at least one vertex in  $S$ , and (ii) the subgraph  $G[S]$  induced by  $S$  is connected (see [10]). The minimum cardinality of a connected dominating set  $S$  is called the *connected domination number*, and is denoted by  $\gamma_c(G)$ ; such a set  $S$  is called a  $\gamma_c$ -set. Condition (i) alone defines the notion of *dominating set*, the minimum cardinality of which is called *domination number* and is denoted by  $\gamma(G)$ .

Bounds on  $\bar{\chi}_{3CC}(G)$  involving  $\gamma_c$  and  $\gamma$  are as follows.

**Theorem 3.** For any connected graph  $G$ ,  $\bar{\chi}_{3CC}(G) \leq \gamma_c(G) + 1$  holds, moreover  $\bar{\chi}_{3CC}(G) \leq 2\gamma(G)$ .

**Proof.** If  $S$  is a connected dominating set of  $G$ , its vertices have an ordering  $v_1, \dots, v_{|S|}$  such that  $(N[v_i] \cap \bigcup_{1 \leq j < i} N[v_j]) \neq \emptyset$  for all  $2 \leq i \leq |S|$ . Similarly to the proof of the first part of Theorem 2, it can be proved that  $\bar{\chi}_{3CC}(G) \leq |S| + 1$ . Hence, choosing  $S$  to be a  $\gamma_c$ -set, the first inequality follows.

Since the closed neighborhood of each vertex contains at most two colors in any 3CC-coloring, and the closed neighborhoods of the vertices in a dominating set cover the entire vertex set, the second upper bound also holds. ■

A relation between  $\bar{\chi}_{3CC}(G)$  and the chromatic number  $\chi(G)$  is as follows: It is known that for a connected graph  $G$  of order  $p$ ,  $\gamma_c(G) \leq p - \Delta(G)$ , where  $\Delta(G)$  is the maximum degree of a vertex in  $G$  (cf. [5]) and  $\chi(G) \leq \Delta(G) + 1$ . Therefore we have the following:

**Corollary 2.** For any connected graph  $G$  of order  $p \geq 3$ ,  $\bar{\chi}_{3CC}(G) \leq p - \chi(G) + 2$ .

**Theorem 4.** For a unicyclic graph of order  $p \geq 3$ ,

$$\alpha_0(G) - 1 \leq \bar{\chi}_{3CC}(G) \leq \alpha_0(G) + 1.$$

**Proof.** In view of Corollary 1 we need to establish only the lower bound. Let  $C$  be the cycle in  $G$  and  $e$  be an edge of  $C$ . Then  $G - e$  is a tree and, again by Corollary 1,  $\bar{\chi}_{3CC}(G - e) = \alpha_0(G - e) + 1$ . Since  $G$  is unicyclic, either  $\alpha_0(G - e) = \alpha_0(G) - 1$ , or  $\alpha_0(G - e) = \alpha_0(G)$ . Therefore,  $\alpha_0(G - e) \geq \alpha_0(G) - 1$ . Also,  $\bar{\chi}_{3CC}(G) = \bar{\chi}_{3CC}(G - e)$  or  $\bar{\chi}_{3CC}(G) = \bar{\chi}_{3CC}(G - e) - 1$  and hence  $\bar{\chi}_{3CC}(G) \geq \bar{\chi}_{3CC}(G - e) - 1$ . Thus,  $\bar{\chi}_{3CC}(G) \geq \alpha_0(G) - 1$ . ■

The upper bound is attained e.g. if  $G$  is a cycle with exactly one pendant edge at each of its vertices. The lower bound is attained e.g. if  $G$  is an odd cycle of length at least 5.

### 3. CHARACTERIZATIONS

#### 3.1. Three-colorability

**Theorem 5.** *A connected graph  $G = (V, E)$  has a 3-consecutive C-coloring with exactly three colors — that is,  $\bar{\chi}_{3CC}(G) \geq 3$  — if and only if its diameter is at least 3.*

**Proof.** To prove necessity by a contradiction, assume a connected graph  $G$  with diameter at most 2 and its proper 3CC-coloring  $\varphi$  using exactly 3 colors.

Since  $G$  is connected, there exist two adjacent vertices  $x$  and  $y$  having different colors, say  $\varphi(x) = 1$  and  $\varphi(y) = 2$ . Moreover, consider a vertex  $z$  colored differently from each of them:  $\varphi(z) = 3$ . There cannot occur multicolored  $P_3$  and hence, vertex  $z$  is adjacent neither with  $x$  nor with  $y$ . Since the diameter equals 2, there exist common neighbors  $x'$  and  $y'$  for the vertex pairs  $x, z$  and  $y, z$ , respectively. Now, consider the  $P_3$  subgraphs  $x'xy$  and  $xx'z$ . The former one forces that  $x'$  does not have color 3, whilst the latter forbids color 2. Therefore, the color of  $x'$  should be 1, and by a similar argument we obtain that  $y'$  has color 2. (This implies that  $x'$  and  $y'$  are different vertices.) But in this case the forbidden multicolored  $P_3$   $x'zy'$  would appear. Consequently, if  $\bar{\chi}_{3CC}(G) \geq 3$ , then the diameter is at least 3.

Proving the opposite direction, we consider a graph  $G$  containing two vertices  $x$  and  $y$  at distance 3 apart. Let the vertex coloring  $\varphi$  assign color 1 to  $x$ , color 3 to  $y$ , and all the remaining vertices receive color 2. Since  $x$  and  $y$  cannot belong to a common  $P_3$ , every three consecutive vertices can

get at most two different colors. Therefore,  $\varphi$  is a proper 3CC-coloring and  $\bar{\chi}_{3CC}(G) \geq 3$  holds. ■

### 3.2. Four-colorability

Here we characterize the graphs admitting a 3-consecutive C-coloring with at least four colors. In the proof, the following notion will be used.

**Definition 1.** Let  $\varphi$  be a vertex coloring of a connected graph  $G$ . The *color-graph* of  $G$  with respect to coloring  $\varphi$ , denoted by  $C_\varphi(G)$ , has the colors occurring in  $\varphi$  as its vertices — called *color-vertices* — and two distinct color-vertices are adjacent in  $C_\varphi(G)$  if and only if there exist two adjacent vertices in  $G$  having the corresponding two colors.

**Theorem 6.** *A connected graph  $G = (V, E)$  admits a 3-consecutive C-coloring with exactly four colors — that is,  $\bar{\chi}_{3CC}(G) \geq 4$  — if and only if it satisfies at least one of the following properties:*

- (i) *There exist three vertices  $x, y, z \in V$  such that any two of them are at distance at least 3 apart.*
- (ii)  *$G$  has diameter at least 5.*
- (iii) *There exists a cycle  $C$  of length eight in  $G$  such that, for each vertex  $v \in V$ , there exists a vertex  $u$  in  $C$  for which  $d_G(u, v) = 4$  holds.*

**Proof.** ( $\Rightarrow$ ) To prove necessity, we assume a 3CC-coloring  $\varphi$  of  $G$  with exactly four colors. In this case,  $C_\varphi(G)$  has vertex set  $\{1, 2, 3, 4\}$ . Since  $G$  is connected and all the four colors are used in  $\varphi$ , the graph  $C_\varphi(G)$  is connected, too. We distinguish three cases on the basis of vertex degrees occurring in  $C_\varphi(G)$ .

1. First, we assume that there exists a color-vertex in  $C_\varphi(G)$  whose degree is 3. We suppose without loss of generality that this color-vertex is 1.

That is, there exist three vertices  $x, y$  and  $z$  in  $G$  with colors 2, 3 and 4, respectively, such that each of them is adjacent to a vertex from color class 1. We will prove that any two of the vertices  $x, y, z$  have distance at least 3. In a 3CC-coloring the closed neighborhood of any vertex can contain at most two different colors. This implies that  $x$  and  $y$  cannot have a common neighbor with color 1 or 4. On the other hand, if their common neighbor had color 2 (or 3) then in  $N[y]$  (or in  $N[x]$ ) there would occur three colors 1, 2, and 3. Therefore, the distance between  $x$  and  $y$  cannot



be smaller than 3. The analogous statement is true for the pairs  $(x, z)$  and  $(y, z)$  as well. Hence, condition (i) is fulfilled. In the sequel, we refer to three vertices with this property as *three distant vertices*.

2. Second, we suppose that the color-graph  $C_\varphi(G)$  is a path, where we assume the order 1, 2, 3, 4.

Choose a vertex  $x \in V$  with color 1 and a vertex  $y \in V$  with color 4. Due to the assumed structure of  $C_\varphi(G)$ , every path connecting  $x$  and  $y$  contains vertices with colors 2 and 3 as well. Moreover, there occur at least two vertices with color 2 in it. Indeed, assuming only one vertex from color-class 2 in this path, this would have neighbors colored with 1 and also with 3, yielding a forbidden multicolored  $P_3$ . Hence, every  $x$ - $y$  path has at least two internal vertices from color class 2 and, similarly, there exist at least two internal vertices with color 3. Consequently,  $d_G(x, y) \geq 5$ , complying with condition (ii).

3. In the remaining cases, every vertex of  $C_\varphi(G)$  has degree two; that is,  $C_\varphi(G)$  is a cycle. We assume the cyclic order 1-2-3-4-1 of colors, and all additions concerning them will be taken modulo 4. Also in this case, (i) and/or (ii) may be satisfied. But we assume throughout that none of the first two properties is valid for  $G$ , and then prove that (iii) necessarily holds under this assumption.

We can partition each color class  $\alpha$  ( $1 \leq \alpha \leq 4$ ) into two parts:

- $V_{\alpha, \alpha+1}$  contains the vertices colored with  $\alpha$  and having a neighbor of color  $\alpha + 1$ .
- $V_{\alpha, \alpha-1}$  contains the vertices colored with  $\alpha$  and having a neighbor of color  $\alpha - 1$ .

Both  $V_{\alpha, \alpha-1}$  and  $V_{\alpha, \alpha+1}$  are nonempty, but  $V_{\alpha, \alpha-1} \cap V_{\alpha, \alpha+1} = \emptyset$ . Now, suppose for a contradiction that there exists a vertex  $v$  of color  $\alpha$  which is not contained in  $V_{\alpha, \alpha-1} \cup V_{\alpha, \alpha+1}$ . Choose a vertex  $u \in V_{\alpha+1, \alpha+2}$ . All the neighbors of  $v$  have color  $\alpha$ , therefore any  $v$ - $u$  path contains an internal vertex with color  $\alpha$ , while  $u$  has no neighbor of color  $\alpha$ , forcing one more internal vertex. Consequently,  $d_G(v, u) \geq 3$ . Similarly, any vertex  $w \in V_{\alpha-1, \alpha-2}$  has distance at least 3 from both vertices  $v$  and  $u$ . This would mean three distant vertices complying with property (i), but this contradicts our present assumption.

Hence,  $\bigcup_{\alpha=1}^4 (V_{\alpha, \alpha-1} \cup V_{\alpha, \alpha+1}) = V$  holds. In other words, the vertex set of  $G$  is partitioned into eight nonempty disjoint sets admitting a cyclic order.

For the sake of simpler discussion, let us introduce the notation  $Q_1 = V_{1,2}$ ,  $Q_2 = V_{2,1}$ ,  $Q_3 = V_{2,3}$ ,  $Q_4 = V_{3,2}$ ,  $Q_5 = V_{3,4}$ ,  $Q_6 = V_{4,3}$ ,  $Q_7 = V_{4,1}$ ,  $Q_8 = V_{1,4}$ . Subscripts of the sets  $Q_i$  will be considered modulo 8.

Every edge of  $G$  must have its endpoints either in the same class  $Q_i$  or in two cyclically consecutive classes. Thus, any two vertices  $x \in Q_i$  and  $y \in Q_j$  ( $i < j$ ) are at distance at least  $\min\{j - i, 8 - (j - i)\}$  apart. On the other hand, we claim that any two vertices from the same or from two consecutive classes have distance at most 2. Indeed, if  $d_G(x, y) \geq 3$  for some  $x \in Q_i$  and  $y \in Q_j$ , where  $j = i$  or  $j = i + 1$ , then  $x$  and  $y$  together with any vertex from  $Q_{j+3}$  would be three distant vertices, what does not meet the present requirements.

Now, choose one vertex  $q_i$  from each class  $Q_i$ . Due to the previous observation, any two vertices  $q_i$  and  $q_{i+1}$  are adjacent or have a common neighbor which received the color of  $q_i$  or  $q_{i+1}$ . Hence, joining every two consecutive vertices by a shortest path, we obtain a cycle (or a closed walk), where all the four colors occur and, for each  $1 \leq \alpha \leq 4$ , the vertices having color  $\alpha$  form a connected arc (subpath). Hence, we can consider a shortest cycle with this property. This cycle  $C$  contains some vertex from each class  $Q_i$ ; thus, its length is at least 8. By minimality and the structure of  $C_\varphi(G)$ , for any two  $x, y \in C$ , the equality  $d_C(x, y) = d_G(x, y)$  holds. Consequently, if cycle  $C$  had nine or more vertices, we could choose three distant vertices, and they would have distances at least 3 not only in  $C$  but also in  $G$ . Hence, under the assumed conditions,  $C \cong C_8$  and it involves exactly one vertex  $r_i$  from each class  $Q_i$ .

Any vertex  $v \in V$  is contained in a uniquely determined class  $Q_j$ . Since property (ii) is not valid for  $G$ , we have  $d_G(v, r_{j+4}) \leq 4$ . On the other hand, every path from  $Q_j$  to  $Q_{j+4}$  has to involve vertices either from  $Q_{j+1}, Q_{j+2}$  and  $Q_{j+3}$ , or from  $Q_{j-1}, Q_{j-2}$  and  $Q_{j-3}$ , hence we obtain  $d_G(v, r_{j+4}) = 4$ , what completes the proof of necessity.

( $\Leftarrow$ ) To prove sufficiency, we will construct appropriate 3CC-colorings for graphs  $G$  having property (i) or (ii), and also for graphs  $G$  satisfying only (iii) from the three constraints.

(I) Assume that there exist three vertices  $x, y, z$  having mutual distances at least 3. By this property, there is no  $P_3$  involving at least two of them. Therefore, we can assign colors 1, 2 and 3 to the vertices  $x, y$  and  $z$ , respectively. If all the remaining vertices receive color 4, the assignment obtained is a proper 3CC-coloring with four colors, hence  $\bar{\chi}_{3CC}(G) \geq 4$ .

(II) Assuming property (ii), there exist two vertices  $x$  and  $y$  at distance 5 apart. Consider the following coloring  $\varphi$ :

- $\varphi(x) = 1$ ;  $\varphi(y) = 4$ ;
- $\varphi(v) = 2$  if  $1 \leq d_G(x, v) \leq 2$ ;
- $\varphi(v) = 3$  if  $d_G(x, v) > 2$  and  $v \neq y$ .

This yields a proper 3CC-coloring, since color 1 can occur together only with color 2 in a  $P_3$ , and similarly, color 4 appears only with color 3. Consequently, no multicolored  $P_3$  can arise.

(III) Assume that the graph  $G$  satisfies (iii) but none of the conditions (i) and (ii). Let the cycle corresponding to (iii) be  $C_8 = r_1r_2r_3r_4r_5r_6r_7r_8$ .

Consider a vertex  $v \in V$ . By condition (iii), there exists a vertex  $r_i$  in the cycle whose distance from  $v$  equals 4. Since  $4 = d_G(v, r_i) \leq d_G(v, r_{i+1}) + 1$ , we obtain  $d_G(v, r_{i+1}) \geq 3$  and, similarly,  $d_G(v, r_{i-1}) \geq 3$  must hold for the other neighbor of  $r_i$ , too. Let us assume that there exists a further vertex  $r_j$  in the cycle whose distance from  $v$  is at least 3. In this case  $r_{i-1}, r_j, v$  or  $r_{i+1}, r_j, v$  would be three distant vertices. Since (i) is supposed to be not valid, this is a contradiction. Therefore, for all vertices  $r_k$  distinct from  $r_{i-1}, r_i$  and  $r_{i+1}$ , the inequality  $d_G(v, r_k) \leq 2$  holds. Taking into account that the relations  $d_G(v, r_i) \geq 4$  and  $d_G(v, r_{i\pm 2}) \leq 2$  imply  $d_G(v, r_{i\pm 1}) = 3$ , the above argument also yields that for every  $v$  there exists precisely one  $r_i \in C_8$  at distance 4.

This uniqueness makes it possible to define the partition of the vertex set into eight disjoint classes (subscript addition taken modulo 8):

$$v \in Q_j \iff d_G(v, r_{j+4}) = 4, \quad \text{for all } 1 \leq j \leq 8.$$

Since  $r_j \in Q_j$  holds for every  $j$ , none of the partition classes is empty. Summarizing the previous observations:

If  $v \in Q_j$ ,

- $d_G(v, r_{j+4}) = 4$ ;
- $d_G(v, r_{j+3}) = d_G(v, r_{j+5}) = 3$ ;
- $d_G(v, r_k) \leq 2$  otherwise.

Next, we prove that any two adjacent vertices  $x$  and  $y$  belong either to the same  $Q_i$  or to two consecutive partition classes.

Assume  $x \in Q_i$ ,  $y \in Q_j$ , and  $d_G(x, y) = 1$ . By the properties of the distance function we obtain

$$\begin{aligned}d_G(x, y) + d_G(y, r_{i+4}) &\geq d_G(x, r_{i+4}) = 4, \\d_G(y, r_{i+4}) &\geq 3.\end{aligned}$$

The inequalities can be fulfilled only if  $i + 4$  equals either  $j + 4$  or  $j + 3$  or  $j + 5$ . These correspond to the cases where  $i = j$ ,  $i = j - 1$  or  $i = j + 1$ ; that is, when  $x$  and  $y$  belong either to the same or two consecutive partition classes.

Now, we can define an appropriate 3CC-coloring  $\varphi$  with four colors:

- $\varphi(v) = k$  if  $v \in Q_{2k-1} \cup Q_{2k}$ , for all  $1 \leq k \leq 4$ .

As we have shown, there occur edges only between consecutive partition classes and inside one class, hence no multicolored  $P_3$  can arise. This proves the assertion for the last case. ■

**Remark 1.** As it can be read out from the proof, in Theorem 6 the prescribed property (iii) can be replaced by other statements without changing validity:

- $C_8$  can be assumed to be an induced subgraph of  $G$ .
- We can prescribe that for every vertex  $v \in V$  there exists precisely one vertex  $u \in C_8$  at distance 4.

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