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COLOURING GAME AND GENERALIZED COLOURING GAME ON GRAPHS WITH CUT-VERTICES

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Abstract

For $k \geq 2$ we define a class of graphs $\mathcal{H}_k = \{G$: every block of G has at most k vertices$\}$. The class $\mathcal{H}_k$ contains among other graphs forests, Husimi trees, line graphs of forests, cactus graphs. We consider the colouring game and the generalized colouring game on graphs from $\mathcal{H}_k$.

Keywords: colouring game, generalized colouring game.

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1. Introduction

Every graph $G = (V, E)$ considered in this paper is finite and simple, i.e., undirected, loopless and without multiple edges. For $S \subseteq V(G)$, let $N(S) = \bigcup_{v \in V} N(v)$, where $N(v)$ denotes the neighbourhood of v and $G - S$ be the subgraph of G induced by $V(G) \setminus S$.

For undefined concepts we refer the reader to [6].

Let $\mathcal{C} = \{1, 2, \ldots, k\}$. Let G be a graph and $S \subseteq V(G)$. A function $c : S \to \mathcal{C}$ is a *k-colouring* (*colouring*) of G. We say that a colour i is *admissible* for an uncoloured vertex v, if $i \notin \{c(u) : u \in N(v)\}$. The k-colouring c is called *proper* if $c(v) \neq c(u)$ whenever $vu \in E(G)$ and for every uncoloured vertex v there is an admissible colour. We say that a graph G is *partially properly k-coloured* if G is properly k-coloured and it has at least one uncoloured vertex.

We consider the two-players *k-colouring game* defined as follows. Alice and Bob are the two players which play alternatively. Furthermore Alice has the first move. Given a graph G and a set of k colours, the players take turns colouring G in such a way that two adjacent vertices are not coloured with the same colour. If after $|V(G)|$ moves the graph is properly k-coloured then Alice wins. Bob wins whenever there is uncoloured vertex which has no admissible colour.

The following lemma is an immediate consequence of definitions.

Lemma 1. *If Alice has the strategy for k-colouring game such that the graph G is properly k-coloured after the move of each player, then Alice wins the k-colouring game.*

A *game chromatic number* of *a graph* G, denoted by $\chi_g(G)$, is defined as the smallest cardinality of $\mathcal{C}$ for which Alice has a winning strategy. For a family of graphs $\mathcal{P}$, let $\chi_g(\mathcal{P}) = \max\{\chi_g(G) : G \in \mathcal{P}\}$.

In [2] Bodlaender studied computational complexity of the game chromatic number. He also showed that the chromatic number of a forest is at most 5 and presented a tree T with $\chi_g(T) = 4$. Faigle *et al.* [11] improved this bound and showed that the chromatic number of the class of forest is 4. Since then, the game chromatic number of various classes of graphs have been studied. Namely, the game chromatic number of interval graphs [11, 15], outerplanar graphs [12], planar graphs [8, 14, 17, 22] and partial k-trees [17, 15, 21] have been studied.

The edge-colouring version of the colouring game was introduced by Cai and Zhu [5]. In this version in each move an uncoloured edge of G is being coloured with a colour from $\mathcal{C}$ in such a way that the adjacent edges are not coloured with the same colour. Alice wins if every edge is coloured at the end of the game, otherwise wins Bob. The *game chromatic index* $\chi'_g(G)$ of the graph G is the smallest cardinality of $\mathcal{C}$ for which Alice has a winning strategy. The game chromatic index of forests with bounded degree was studied by Erdös *et al.* [10] and Andres [1].

In [4] the concept of generalized colouring game was introduced. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ be additive hereditary properties of graphs. Let G be a k-coloured graph and $v \in V(G)$. We say that the colour i is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*admissible* for v if $i \in \{1, 2, \ldots, k\}$ and after colouring v with i the subgraph induced by the i-coloured vertices has the property $\mathcal{P}_i$. If properties

$\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ are clear from the context, we may omit them. A k-colouring of a graph G is called a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*colouring* (*generalized colouring*) of G if for every colour i, $1 \leq i \leq k$, every monochromatic i-coloured subgraph of G has the property $\mathcal{P}_i$ and for every uncoloured vertex there is an admissible colour. If $\mathcal{P}_i$ is a set of totally disconnected graphs for every i, then $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-colouring is the proper colouring. We say that a graph G is *partially* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*coloured* if G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured and has at least one uncoloured vertex.

More details about hereditary properties and generalized colouring of graphs can be found in [3].

The generalized colouring game is defined as follows. Let be given a graph G and an ordered set of additive hereditary properties $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$. The two players Alice and Bob play alternatively with Alice having the first move. The players take turns colouring vertices of G with $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colours. If after $|V(G)|$ moves the graph G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured, then Alice wins. Bob wins whenever a vertex for which there is no $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour appears. The above defined game is called a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*game* (*generalized game*).

From definitions it immediately follows.

Lemma 2. *If Alice has on G the strategy for $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game such that after the move of each player the graph G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured, then Alice wins the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game.*

The (k, d)-*relaxed colouring game*, introduced in [7], is the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game such that $\mathcal{P}_i$ $(i = 1, \ldots, k)$ is a set of graphs with maximum degree at most d. A d-*relaxed game chromatic number*, denoted by $\chi_g^{(d)}(G)$, is the smallest k for which Alice has a winning strategy for the (k, d)-relaxed colouring game. The d-relaxed game chromatic number for trees and outerplanar graphs was studied in [7, 9, 13, 18].

In this paper, we are interested in generalized colouring games which refer to two additive hereditary properties:
$\mathcal{O} = \{G : G \text{ is totally disconnected}\}$,
$\mathcal{O}_k = \{G : \text{each component of } G \text{ has at most } k+1 \text{ vertices}\}$.
We consider the colouring game and the generalized game on a special class of graphs $\mathcal{H}_k$ which includes line graphs of forests, cactus graphs and Husimi trees.

The *block* B of a graph G is the maximal subgraph of G which does not contain any cut-vertex. For $k \geq 2$, let

$$\mathcal{H}_k = \{G : \text{every block of } G \text{ has at most } k \text{ vertices}\}.$$

Then $\mathcal{H}_2$ is a set of forests. If $G \in \mathcal{H}_k$ and every block of G is a complete graph then G is a Husimi tree. If $G \in \mathcal{H}_k$ and every block is a cycle or K_2 then G is a cactus graph.

Proposition 3. *A graph G is a line graph of a forest if and only if $G \in \mathcal{H}_k$, every block of G is a complete graph and every vertex of G is in at most two blocks.*

In [19] Yang and Kierstead studied the game on line graphs. The edge-colouring game on a graph G one can see as the colouring game on a line graph of the graph G. Erdös *et al.* [10] proved that there is a tree with maximum degree Δ for which the game chromatic index is equal to $\Delta + 1$. Since a line graph of a tree with maximum degree Δ belongs to $\mathcal{H}_\Delta$, we obtain the lower bound for the game chromatic number of family $\mathcal{H}_k$.

Proposition 4 [10]**.** *Let $k \geq 2$. Then $\chi_g(\mathcal{H}_k) \geq k + 1$.*

In Section 2 we find an upper bound and we show that $\chi_g(\mathcal{H}_k) \leq k + 2$. In Section 3 we generalize the result of Erdös *et al.* [10] which proved that the game chromatic number of forest with maximum degree at least Δ ($\Delta \geq 6$) is at most $\Delta + 1$. We prove that $\chi_g(\mathcal{H}_k) = k + 1$ for $k \geq 6$.

From the result of Bodlaender [2] and Faigle et al. [11] we have $\chi_g(\mathcal{H}_2) = 4$. In [16] was presented a cactus G such that $G \in \mathcal{H}_3$ and $\chi_g(G) = 5$, thus $\chi_g(\mathcal{H}_3) = 5$. For $\mathcal{H}_4$ and $\mathcal{H}_5$, the question whether $\chi_g(\mathcal{H}_4) = 5$ or $\chi_g(\mathcal{H}_4) = 6$ and $\chi_g(\mathcal{H}_5) = 6$ or $\chi_g(\mathcal{H}_5) = 7$ remains open.

In Section 4 we investigate a generalized colouring game. We show that if players colour vertices using $k + 1$ colours in such a way that every vertex may have one neighbour coloured with its colour, then Alice can win a game on every graph from $\mathcal{H}_k$. Namely, we prove that Alice has a winning strategy for an $(\mathcal{O}_1, \mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game on every graph from $\mathcal{H}_3$ and for a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1})$-game on every graph from $\mathcal{H}_k$ ($k \geq 4$) where $\mathcal{P}_1 = \mathcal{O}_1$, $\mathcal{P}_i = \mathcal{O}$ ($i = 2, \ldots, k + 1$).

In general, the generalized colouring game is not monotone, i.e., from the fact that Alice has a wining strategy for a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game does not follows that Alice has a wining strategy for a $(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k)$-game, where $\mathcal{P}_i \subseteq \mathcal{P}'_i$ ($i = 1, 2, \ldots, k$). In Section 5 we discuss the monotonicity of games considered in Sections 2, 3 and 4.

2. Upper Bound for the Game Chromatic Number of $\mathcal{H}_k$

One can see a partially properly coloured graph as a graph G obtained during the colouring game after $|C|$ moves of players, where C is a set of coloured vertices of G. Thus, the players can continue the game and the first move of this part of the game can be made by Alice or Bob. If Alice starts, we say that the players play the *colouring game on a partially properly coloured graph*. If Bob starts, we say that the players play the *colouring game with the first move of Bob on a partially properly coloured graph*. Alice wins the colouring game on a partially properly coloured graph when all vertices are properly coloured.

Let G be a partially properly k-coloured graph and C be the set of coloured vertices. Let $G_1, G_2, \ldots, G_p$ be components of $G - C$. A *game component* of G is the subgraph induced by $V(G_i) \cup N(V(G_i))$ $(1 \leq i \leq p)$. If $p = 1$ and $G_1 = G$, then we say that G is a game component.

During the game players can see a graph G as a disconnected graph whose components are game components of G.

Definition (k-game closed family)**.** Let α be a family of partially properly k-coloured graphs. We say that the family α is *k-game closed* if for every $G \in \alpha$ the following conditions hold:

(i) Alice can colour a vertex of G with an admissible colour in such a way that all game components of G are in α or all vertices of G are properly coloured.

(ii) If Bob makes the first move (i.e., Bob colours a vertex x of G with an admissible colour) and if after his move G has an uncoloured vertex, then Alice can colour a vertex y with an admissible colour in such a way that all game components of the graph G (G with coloured x and y) are in α or all vertices of G are properly coloured.

Lemma 5. *Let α be a k-game closed family. If $G \in \alpha$, then Alice has a winning strategy on G for the k-colouring game and for the k-colouring game with the first move of Bob.*

Proof. Let $G \in \alpha$. The winning strategy of Alice is the following: she colours vertices in such a way that after her move every game component of G is in α or all vertices are coloured. We claim that she achieves this goal. Suppose that for a certain time of the game every game component of G is in α and it is Alice's turn. Let G_i be an arbitrary game component of

G, by Condition (i) of Definition 2 Alice can colour a vertex of G_i in such a way that every game component of G_i will be in α. Hence, after Alice's move every game component of G is in α. Suppose that it is Bob's turn. Let $G_1, G_2, \ldots, G_p$ be game components of G before Bob's move. Assume that Bob colours a vertex of G_i. If there is uncoloured vertex in G_i, then Condition (ii) of Definition 2 implies that after Bob's move Alice can colour a vertex of G_i in such a way that every game component of G_i will be in α. If all vertices of G_i are coloured, then Alice colours an uncoloured vertex of any other game component G_j in such a way that every game component of G_j will be in α. If all vertices of G are coloured, then Alice wins. Since α contains only properly k-coloured graphs, after every move of players the graph G is properly k-coloured. Thus, by Lemma 1 Alice wins the game. ■

Before we prove our results let us introduce some definitions. Let $\mathcal{H}_k^r$ be a family of partially r-coloured graphs from $\mathcal{H}_k$:

$\mathcal{H}_k^r = \{G : G$ is partially r-coloured and every block of G has at most k vertices$\}$.

Note that $\mathcal{H}_k \subseteq \mathcal{H}_k^r$, since every uncoloured graph is partially r-coloured.

Let $G \in \mathcal{H}_k^r$. A *pseudo-block* P is a subgraph of G such that if we add all edges to G which join non-adjacent vertices of P, then we obtain a graph which is still in $\mathcal{H}_k^r$. Let B be a block or a pseudo-block of G and $v \in V(B)$. A *v-branch* (*branch*) H of B is a connected subgraph of G such that $V(B) \cap V(H) = \{v\}$, and H is maximal with this property. The vertex v is a *root* of the v-branch. Let G be partially coloured graph. A *center* S of G is a block or a pseudo-block such that for every vertex $v \in V(S)$ the v-branch has at most one coloured vertex.

Note that if G has at most one coloured vertex, then every block of G is a center of G. If G has two coloured vertices, then each block which contains a coloured vertex is a center.

Let $S_1, S_2 \subseteq V(G)$, $S_1 \cap S_2 = \emptyset$. A *separator* of S_1 and S_2 is a vertex x such that in $G - x$ vertices of S_1 and S_2 are in distinct components. Let $X = \{x : x$ is a separator of v and $\{u, w\}\} \cup \{v\}$. If in X there is a vertex x such that u and w are in distinct components of $G - x$ then we call x a *v-cut-vertex* for the triple (v, u, w). Otherwise, a *v-cut-vertex* for the triple (v, u, w) is a vertex x such that the component of $G - x$ which contains u and w is minimal. The cut-vertex for the triple will be used extensively in this paper, since after colouring a cut-vertex for the triple with three coloured vertices the graph is split into the game components which have less coloured vertices.

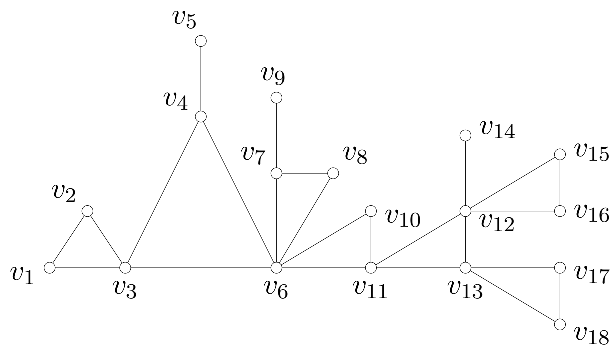

Figure 1

Example 6. Let G be a graph on Figure 1 and B be a block of G induced by vertices $\{v_3, v_4, v_6\}$. The v_6-branch of B is a subgraph induced by the vertices $\{v_6, v_7, \ldots, v_{18}\}$. The vertex v_{11} is the v_6-cut-vertex for the triple (v_6, v_{16}, v_{18}). The vertex v_{12} is the v_6-cut-vertex for the triple (v_6, v_{14}, v_{16}). The vertex v_6 is the v_6-cut-vertex for the triple (v_6, v_9, v_{11}).

Now we define a family α_1 which contains some special graphs of $\mathcal{H}_k^{k+2}$.

Definition (Family α_1)**.** Let $k \geq 2$.
$\alpha_1 = \{G : G \in \mathcal{H}_k^{k+2}$, and G is a partially properly $(k+2)$-coloured game component, and G has a center$\}$.

Note that every graph from $\mathcal{H}_k$ is partially properly $(k+2)$-coloured game component and has a center, therefore $\mathcal{H}_k \subseteq \alpha_1$. Moreover every graph from α_1 has at least one uncoloured vertex and if $G \in \alpha_1$ and v is a coloured vertex of the center S of G, then v-branch of S has exactly one vertex v, since G is a game component.

Lemma 7. *Family α_1 is $(k+2)$-game closed.*

Proof. Let $G \in \alpha_1$. We show that Conditions (i) and (ii) of Definition 2 hold. Since G has a center and it is a partially coloured game component, G has at least one uncoloured vertex in the center. If Alice starts to colour vertices, she colours a vertex of a center. If after her move G has an uncoloured vertex, then G might be split into smaller game components. All game components will be in α_1, hence the Condition (i) holds. Now we consider the case when Bob starts. If after his move all game components of G are in α_1, then obviously Alice can colour a vertex in such a way that also after her move all game components of G will be in α_1. Thus, assume that after Bob's move there is a game component which is not in α_1. It can happen only if G had two or more coloured vertices. Since before Bob's move G had a center S, it follows that after Bob's move there is a v-branch of S which has two coloured vertices, say u, w. Alice colours a v-cut-vertex x for the triple (v, u, w). Note that Alice always can find an admissible colour for the vertex x, in the worst case when w and u are adjacent to x and $x = v$, the vertex x has $k + 1$ coloured neighbours. After Alice's move G will have the following game components: First we consider the case when $x \neq v$. Then G will have the game component which contains vertices of S (the v-branch of S has one coloured vertex x), in this game component S will be still a center. If the vertices u and w are in distinct components of $G - x$, then G will also have two game components which have two coloured vertices u, x and w, x. If the vertices u and w are in one component of $G - x$, then G will have a game component which has three coloured vertices u, w, x and the block which contains x will be a center of this game component. The graph G might also have game components which have exactly one coloured vertex. If $x = v$, then the game component which contains the vertices of S will have a center or the vertices of S will be in distinct game components, but in each of the game component they will be in a center. The other game components are the same as above. Hence Condition (ii) of Definition 2 holds. ■

From Lemma 5 and Lemma 7 we have immediately:

Theorem 8. *Let $G \in \mathcal{H}_k$ $(k \geq 2)$. Then $\chi_g(G) \leq k + 2$.*

For $k = 2$ we obtain known result for forests.

Corollary 9 [11]. *For every forest F we have $\chi_g(F) \leq 4$.*

3. Game Chromatic Number of $\mathcal{H}_k$ for $k \geq 6$

From Proposition 4 and Theorem 8 we have that $k + 1 \leq \chi_g(\mathcal{H}_k) \leq k + 2$. In this Section we prove that $\chi_g(\mathcal{H}_k) = k + 1$ for $k \geq 6$. Let S be a center of a connected graph G, $G \in \mathcal{H}_k^r$. We denote by T a graph $T = G - S$. Let $s = |V(S) \cap C|$ and $t = |V(T) \cap C|$, where C is the set of coloured vertices of G. Let us denote $c(G) = \bigcup_{v \in C} c(v)$, recall that $c(v)$ is a colour of v. For $B \subseteq G$ and $v \in V(G) \setminus V(B)$, we say that v is adjacent to B if $v \in N(B)$.

Similarly as in the previous section we will define a family of graphs which contains $\mathcal{H}_k$ ($k \geq 6$) and next we will prove that this family is $(k+1)$-game closed.

Definition (Family α_2)**.** $G \in \alpha_2$ if $G \in \mathcal{H}_k^{k+1}$, $k \geq 6$ and G is a partially properly $(k+1)$-coloured game component which has at least one of the following properties:

1. G has a center S, such that

- **1.1.** all coloured vertices of G are in S or
- **1.2.** $|V(S)| \leq k - 1$ or
- **1.3.** $|V(S)| = k$, and
  - **1.3.1.** $t = 3$, $s \neq 0$, and all vertices of T are adjacent to S, and they are coloured with colours from $c(S)$ or
  - **1.3.2.** $t = 2$ and $s = 1$ or
  - **1.3.3.** $t = 2$, $s = 2$ and if both vertices of T are adjacent to S, then at least one of them has a colour from $c(S)$ or
  - **1.3.4.** $t = 2$, $s \geq 3$ and if there is a coloured vertex of T which is adjacent to S, then it has a colour from $c(S)$ or
  - **1.3.5.** $t = 1$ and if $s \geq 4$, and if the coloured vertex of T is adjacent to S, then it has a colour from $c(S)$.

2. There are three coloured vertices and there is an uncoloured vertex u such that all coloured vertices are in blocks containing u.

Let $G \in \mathcal{H}_k^{k+1}$ ($k \geq 6$). Note that if G has at most two coloured vertices, then it always belongs to α_2. If all vertices of G are uncoloured or there is exactly one coloured vertex, then it has the property 1.1. If G has two coloured vertices and they are in one block or pseudo-block, then G also has the property 1.1. Otherwise, the block which has one coloured vertex may be a center of G, so G has the property 1.2 or 1.3.5. Moreover, observe that if $G \in \alpha_1$ and v is a coloured vertex of the center S (i.e., G has the property 1), then v-branch of S has exactly one vertex, since G is a game component.

Lemma 10. *Family α_2 is $(k+1)$-game closed.*

Proof. Let $G \in \alpha_2$. First we will show that Alice can colour a vertex of G in such a way that every game component will be in α_2. If G has a center S, then G has at least one uncoloured vertex in S. If G has the property 1.1 or 1.2, then Alice colours a vertex v of S. If after such a move G has uncoloured vertices, then it will have game components which are in α_2. Indeed, G will have a game component which contains all vertices of S (this game component will have property 1.1 or 1.2) or the vertices of S will be in distinct game components (all these game components will have the property 1.2) and G might also have game components which have exactly one coloured vertex v. If G has the property 1.3, then Alice colours a vertex $v \in V(S)$ such that the v-branch has a coloured vertex. After her move, G will have the following game components: the game component which has the center S (we will call it the main game component), the game component which has two coloured vertices and G also may have the game components which have exactly one coloured vertex v. If G has the property 1.3.1, then after Alice's move the main component of G will have the property 1.3.3 or 1.3.4. If G has one of the properties 1.3.2 – 1.3.4, then after Alice's move the main component of G will have the property 1.3.5. If G has the property 1.3.5, then the main component of G will have the property 1.1. It might also happen that after Alice's move vertices of S will be in distinct game components which will have the property 1.2. If G has the property 2, Alice colours the vertex u. After such a move every game component will have a center and all coloured vertices will be in the center of the game component, i.e, every game component will have the property 1.1. Hence Condition (i) of Definition 2 holds. If after Bob's move all game components of G are in α_2, then obviously Alice can colour a vertex in such a way that all game components of G are in α_2. Thus, assume that after Bob's move there is a game component which is not in α_2. The proof falls naturally into two cases:

Case 1. Before Bob's move G had the property 1.

Subcase 1.1. Suppose that G had the property 1.1.
Since after Bob's move $G \notin \alpha_2$, it follows that $|V(S)| = k,\ t = 1,\ s \geq 4$ and the coloured vertex $u \in T$ is adjacent to S, and $c(u) \notin c(S)$. Hence, Alice colours a root u' of a branch of S which contains u. After such a move, G will have the following game components: a game component which has the center S (or vertices of S are in distinct game components), a game

component which has two coloured vertices u and u' and G may also have game components which have one coloured vertex u'. The game component which contains S will have the property 1.1. If vertices of S are in distinct game components, then these game components will have the property 1.2.

Subcase 1.2. Assume that G had the property 1.2.
Thus, after Bob's move S is not a center. Therefore, there is a v-branch of S which has two coloured vertices u, w. Then Alice colours a v-cut-vertex x for the triple (v, u, w). After Alice's move G will have the following game components: a game component which contains vertices of S (the v-branch of S has one coloured vertex x), in this game component S will be still a center or the vertices of S will be in distinct game components which will have the property 1.2. If the vertices u and w are in distinct components of $G-x$, then G will also have two game components which have two coloured vertices u, x and w, x. If the vertices u and w are in one component of $G-x$, then G will have the game component which has three coloured vertices u, w, x and the block S' which contains x will be a center of this game component. If $u, w \in S'$, then this component will have the property 1.1. If $|V(S')| \leq k-1$, then it will have the property 1.2. Otherwise, it will have the property 1.3.2 or 1.3.5. The graph G might also have game components which have exactly one coloured vertex x.

Subcase 1.3. Now assume that G had the property 1.3.

Subcase 1.3.1. Suppose that G had the property 1.3.1.
Since after Bob's move $G \notin \alpha_2$, Bob has coloured a vertex of T and now we have $t = 4$. Let w be a vertex coloured by Bob and w' be a root of a branch of S which contains w. Then Alice colours w'. After such a move a game component which has the vertices of S will have the property 1.3.1 or 1.2 (if vertices of S are in one game component, then it will have the property 1.3.1, otherwise game components will have the property 1.2).

Subcase 1.3.2. Suppose that G had the property 1.3.2.
First suppose that Bob has coloured a vertex of S. Thus, $t = 2$ and $s = 2$. Since G is not in α_2, there are two coloured vertices u, w in T which are adjacent to S and they are coloured with colours which are not in $c(S)$. If $c(u) = c(w) = i$, then Alice colours a vertex of the center S with colour i. A game component which contains the vertices of S will have the property 1.3.3 or 1.2. If $c(u) \neq c(w)$, then Alice colours w' with colour $c(u)$, where w' is a root of a branch of S which contains w. A game component which contains the vertices of S will have the property 1.3.5 or 1.2.

Now suppose that Bob has coloured a vertex of T. Thus, $t = 3$ and $s = 1$. Let u, v, w be coloured vertices of T. Suppose that vertices u, v, w are in three distinct branches. If at least one vertex, say u, is not adjacent to S, then Alice colours a root u' of a branch of S which contains u. Suppose that the vertices u, v, w are adjacent to S. If $c(u) = c(w) = c(v) = i$ then $c \notin c(S)$, since G does not have the property 1.3.1, thus Alice colours a vertex of S with i. If $c(u) \neq c(w)$ or $c(u) \neq c(v)$, then Alice colours the vertex u' (a root of a branch containing u) in such a way that a game component of G which contains S will have the property 1.3.3, i.e., she colours u' with colours $c(w)$ or $c(v)$ or with an arbitrary colour if $c(w) = c(v) = i$ and $i \in c(S)$. Now suppose that u, v, w are not in three distinct branches. Let x be a vertex such that a x-branch contains u and v. Then Alice colours a x-cut-vertex y for the triple (x, v, u). After her move G will have a game component which contains the vertices of S (or several game component which contains the vertices of S), a game component which has three coloured vertices u, v, y or two game components which have two coloured vertices u, y or v, y, and also it might have components which have one coloured vertex. Observe that similarly as in Subcase 1.2 every game component is in α_2.

Subcase 1.3.3. Now, consider that case when G had the property 1.3.3. First suppose that Bob has coloured a vertex of S. Thus, $t = 2$ and $s = 3$. Since G is not in α_2, at least one coloured vertex of T is adjacent to S and it is coloured with a colour which is not in $c(S)$. Assume that u is adjacent to S and $c(u) \notin S$ and w is the second coloured vertex of T. Let u' be a root of a branch of S which contains u. Then Alice colours u' with colour $c(w)$ or with an arbitrary colour when $c(w) \in c(S)$. If she cannot make such a move, i.e., $c(u) = c(w) = i$, then Alice colours a vertex of S with colour i. Note that there is uncoloured vertex in S, since $|V(S)| \geq 6$.

Now suppose that Bob has coloured a vertex of T. Thus, $t = 3$ and $s = 2$. Let u, v, w be coloured vertices of T. Suppose that vertices u, v, w are in three distinct branches. Let u', v', w' be vertices such that u, v, w are in a u'-branch, a v'-branch, a w'-branch, respectively. Alice colours a vertex u' or v' or w' in such a way that after her move the game component which contains all vertices of S will have the property 1.3.4 (with $s = 3$) or will be several game components which have the vertices of S and they will have the property 1.2. Finally, suppose that u, v, w are not in three distinct branches. Let x be a root of a branch which contains u and v. Alice try to colour a x-cut-vertex y for the triple (x, v, u). If y is adjacent to S, she colour y with a colour from $c(S)$. If she cannot colour y, i.e., u, v are adjacent to y and

$c(S) = \{c(u), c(v)\}$, then she colours x. After her move a game component which contains the vertices of S will have the property 1.3.5 or 1.2 and the game component which contains vertices u, v, x will have the property 2.

Subcase 1.3.4. Assume that G had the property 1.3.4.
Since after Bob's move G is not in α_2, it follows that Bob has coloured a vertex $w \in V(T)$. Let w' be a root of a branch of S which contains w. If w is the only coloured vertex of the w'-branch, then Alice colours w'. After her move a game component containing the vertices of S will have the property 1.3.4 (but with $s \geq 4$) or 1.2. If w'-branch has two coloured vertices w and u, then Alice colours a w'-cut-vertex y for the triple (w', w, u). If y is adjacent to S, she colours y with a colour from $c(S)$. If in $c(S)$ there is no admissible colour for y, then similarly as in Subcase 1.3.3 Alice colours w'.

Subcase 1.3.5. Finally assume that G had the property 1.3.5.
First suppose that Bob has coloured a vertex of S. Since G is not in α_2, we have $s = 4$, $t = 1$ and the coloured vertex $u \in T$ is adjacent to S and $c(u) \notin c(S)$. Alice colours a root of a branch containing u.

Now suppose that Bob has coloured a vertex $w \in T$. Let u, w be coloured vertices of T. First assume that u and w are in one branch of S with a root x. Alice colours a x-cut-vertex y for the triple (x, w, u). If y is adjacent to S and $s \geq 4$, then she must colour y with a colour from $c(S)$, otherwise she colours it with an arbitrary admissible colour. Suppose that she cannot make such a move, i.e., y is adjacent to S and in $c(S)$ there is no admissible colour for y. Then Alice colours x, after her move the game component which contains the vertices of S will have the property 1.1 and the game component which contains the vertices u, w, x will have the property 2. Now suppose that u and w are in distinct branches of S. Let w'-branch be a branch of S which contains w. If $s \geq 4$ or $0 \leq s \leq 2$, then Alice colours w'. Clearly after her move a game component which contains the vertices of S will have the property 1.3.5 or 1.2. Assume that $s = 3$. Since G does not have the property 1.3.4, at least one of vertices u, w, say u, is adjacent to S and $c(u) \notin c(S)$. Alice colours w' with colour $c(u)$. Hence after her move a game component which contains the vertices of S will have the property 1.3.5 or 1.2. If she cannot make such a move, i.e., w is adjacent to S and $c(w) = c(u) = i$, then Alice colours a vertex of S with colour i. After her move the game component which contains S will have the property 1.3.4.

Case 2. Before Bob's move G had the property 2.
Since $G \notin \alpha_2$, Bob has coloured a vertex $w \neq u$ which is not adjacent to u.

Alice colours the vertex u. After her move G will have the game components which will have the property 1.1 and at most one game component which will have the property 1.3.5. ■

From Lemma 5 and Lemma 10 and from the fact that $\mathcal{H}_k \subseteq \alpha_2$ we have immediately:

Theorem 11. *If $k \geq 6$, then $\chi_g(\mathcal{H}_k) = k + 1$.*

The result obtained by Erdös et al. [10] follows from Proposition 3 and Theorem 11.

Corollary 12 [10]**.** *If F is a forest with maximum degree Δ and $\Delta \geq 6$, then $\chi'_g(F) \leq \Delta + 1$.*

4. Generalized Colouring Game

In [4] it was proven that Alice has a winning strategy for an $(\mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game on every forest. Thus, Alice can win a generalized game on every graph from $\mathcal{H}_2$ if players use 3 colours. In this section we show that if players colour vertices using $k+1$ colours in such a way that every vertex can have one neighbour coloured with its colour then Alice can win a game on every graph from $\mathcal{H}_k$ ($k \geq 3$).

Let C be a set of coloured vertices of G, $C \neq V(G)$. One can see the partially $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured graph as a graph obtained during the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game after $|C|$ moves of players. Thus, the players can continue the game and the first move of this part of the game can be made by Alice or Bob. If Alice starts, we say that the players play the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*game*. If Bob starts, we say that the players play the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*game with the first move of Bob*. Alice wins the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game on a partially $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured graph G when all vertices of G are $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured.

Let G_1 and G_2 be vertex disjoint subgraphs of G. We say that the graph G_1 *is joined to* G_2 if there is a vertex $v_1 \in V(G_1)$ and a vertex $v_2 \in V(G_2)$ such that $v_1v_2 \in E(G)$.

Let G be a partially k-coloured graph and
$M = \{uv \in E(G) : (uv \text{ is bichromatic edge) or } (u \text{ is coloured, and } v \text{ is uncoloured, and } c(u) \text{ is not an admissible colour for } v)\}$.

Let $G_1, G_2, \ldots, G_p$ be components of $G - M$ such that each of them contains at least one uncoloured vertex. A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*game component* of G is the subgraph induced by $V(G_i)$ and vertices of every connected monochromatic subgraph which is joined to $G_i - C$ $(1 \le i \le p)$. If $p = 1$ and $G_1 = G$, then we say that G is a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game component. If properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ are clear from the context, the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game component will be called the game component, for short.

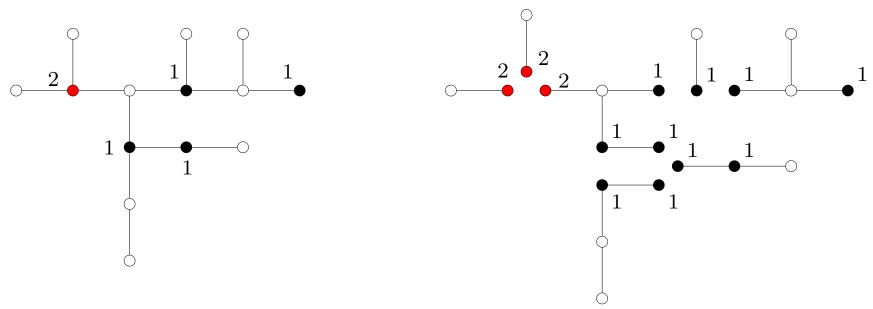

Figure 2. The graph G and its $(\mathcal{O}_1, \mathcal{O})$-game components. The labeled vertices are coloured with colour 1 or 2.

Observe that a graph G can be split into its $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components even if G is not $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured, i.e., there is a monochromatic subgraph coloured with i which does not have the property $\mathcal{P}_i$ and a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game component need not be $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured. We will use this property in Section 5. In other sections we will determine $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components of graphs which are partially $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured.

Lemma 13. *Every uncoloured vertex is in exactly one* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$*-game component of* G.

During the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game players can see a graph G as a disconnected graph whose components are $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components of G. For an uncoloured vertex v the sets of admissible colours in G and in the game component are the same. The graph G contains a forbidden monochromatic subgraph if and only if one of its game components contains such a subgraph.

Definition ($(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game closed family)**.** Let β be a family of partially $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured graphs. We say that the family β is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*game closed* if for every $G \in \beta$ the following conditions hold:

(i) Alice can colour a vertex of G with a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour in such a way that all $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components of G are in β or all vertices of G are $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured.

(ii) If Bob colours a vertex with a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour and if after his move G has an uncoloured vertex, then Alice can colour a vertex with a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour in such a way that all $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components of the graph G are in β or all vertices of G are $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured.

The next lemma follows from Lemma 2 (similarly as Lemma 5).

Lemma 14. *Let β be a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game closed family. If $G \in \beta$, then Alice has a winning strategy on G for the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game and for the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game with the first move of Bob.*

We use Lemma 14 to prove, that Alice has a winning strategy for $(\mathcal{O}_1, \mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game on every graph from $\mathcal{H}_3$. Firstly we construct an $(\mathcal{O}_1, \mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game closed family, which contains all uncoloured graphs from $\mathcal{H}_3$.

Let $G \in \mathcal{H}_k^r$ and $v \in V(G)$. Every component of $G - v$ is called a *steam* of v. Recall that a center of G is a block or a pseudo-block S such that for every vertex $v \in V(S)$ the v-branch has at most one coloured vertex. Similarly as in the previous section, let us denote $s = |V(S) \cap C|$, $T = G - S$, $t = |V(T) \cap C|$, where C is the set of the coloured vertices of G. We say that two blocks (or block and pseudo-block) are *adjacent* if they have a common vertex.

Example 15. Let G be a graph on Figure 1. The vertex v_6 has three steams: the subgraph induced by the vertices $\{v_1, v_2, \ldots, v_5\}$, the subgraph induced by the vertices $\{v_7, v_8, v_9\}$ and the subgraph induced by the vertices $\{v_{10}, v_{11}, \ldots, v_{18}\}$.

Observe that for a center S and $v \in V(S)$ there is one v-branch of S and v belongs to the v-branch. A graph G can have several steams of v and v does not belong to any steam of v.

Definition (Family β_3)**.** $G \in \beta_3$ if $G \in \mathcal{H}_3^4$, and G is an $(\mathcal{O}_1, \mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game component, and G is partially $(\mathcal{O}_1, \mathcal{O}_1, \mathcal{O}, \mathcal{O})$-coloured, and has at least one of the following properties:

1. G has a center S and if G has at least one coloured vertex, then the center S has a coloured vertex.
2. There is a vertex v which is coloured with colour 1 (colour 2), such that every steam of v has at most two coloured vertices. If a steam has two coloured vertices, then one of them is adjacent to v. Moreover, if a vertex u of a steam which has two coloured vertices is adjacent to v and is coloured with 2 (with 1), then the second coloured vertex of this steam is adjacent to u and is also coloured with 2 (with 1).
3. There is an uncoloured vertex u such that every coloured vertex is adjacent to u and is coloured with one of the colours $\{1, 3, 4\}$ ($\{2, 3, 4\}$).

Note that if $G \in \mathcal{H}_3^4$ and G has at most two coloured vertices, then it always belongs to β_3 (it has the property 1). Moreover, from the fact that every graph of β_3 is a game component it follows: If $G \in \beta_3$ and v is a cut-vertex of G, then v can be coloured neither with colour 3 nor with colour 4. If there is a block B such that $|V(B)| = 3$ and there are two 1-coloured (2-coloured) vertices u, w in B, then the third vertex of B is uncoloured and the u-branch and the w-branch of B have exactly one vertex. If there is a block B such that $|V(B)| = 2$ and both vertices of B are 1-coloured (2-coloured), then at most one branch of B has more than one vertex.

Lemma 16. *Family β_3 is $(\mathcal{O}_1, \mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game closed.*

Proof. Let $G \in \beta_3$. We will show that Condition (i) and (ii) of Definition 4 hold. Suppose that Alice colours first. First assume that G has the property 1 of Definition 4. Thus, G has a center S. Note that G has at most three coloured vertices. If there is an uncoloured vertex $v \in S$ which does not have a 3-coloured neighbour or a 4-coloured neighbour, then Alice colours v with colour 3 or 4. If after such a move all vertices of S are coloured, then every game component of G will have at most two coloured vertices. If after Alice's move there is an uncoloured vertex in S, then the game component which has the vertices of S will have the property 1 and all other game component will have at most two coloured vertices. If there is an uncoloured vertex $v \in S$ but it has a 3-coloured neighbour and a 4-coloured neighbour, then Alice colours v with colour 1 or 2. After such a move, G will have the property 2 or if all vertices of S are coloured, then G will have game components which have at most two coloured vertices or G might also have the game component which has three coloured vertices such that it will have the property 2. If before Alice's move all vertices of S are coloured, then from the fact that G is a game component it follows that S has exactly two 1-coloured (2-coloured) vertices and these vertices are the only coloured vertices of G. Then Alice colours a vertex adjacent to S. After such a move G will have the game component containing vertices of S (it will have the property 2) and also it might have game components which have exactly one coloured vertex.

Suppose that G has the property 2. If v has an uncoloured neighbours, then Alice colours it. If after such a move v still have any uncoloured neighbour, then the game component which contains v will have the property 2 the other game components of G have at most one coloured vertex. If after Alice's move all neighbours of v are coloured, then every game component of G will have at most two coloured vertices. Now suppose that v has no uncoloured neighbours. Since G is a game component, v has only one neighbour w and $c(v) = c(w)$. Hence w must have an uncoloured neighbour. Then Alice colours the uncoloured neighbour of w, in such a way that after her move the game component of G will have the property 2. If G has the property 3, Alice colours the vertex u in such a way that G will have the property 2. Thus, Condition (i) holds. Now we will show that Condition (ii) also holds. Suppose that Bob has coloured a vertex of G and after his move there is a game component of G which is not in β_3.

Case 1. Before Bob's move G had the property 1.
Suppose that Bob has coloured a vertex $w \in S$. Since $G \notin \beta_3$, the w-branch of S has two coloured vertices and $c(w) = 1$ or 2, say $c(w) = 1$. Observe that in S there is exactly one 1-coloured vertex, otherwise every game component of G is in β_3. Moreover S has exactly one uncoloured vertex. Then Alice colours the vertex of S in such a way that after her move every game component of G will have at most two coloured vertices.

Suppose that Bob has coloured a vertex of T. Thus, S has a branch which contains two coloured vertices. Let w' be a root of this branch. Assume that w' is not coloured and w, u are the two coloured vertices of the w'-branch. Alice try to colour a w'-cut-vertex x for the triple (w', w, u). If $x = w'$ and w' is not adjacent to any 3-coloured vertex or to any 4-coloured, then Alice colours w' with 3 or 4, respectively. After such a move the game component which contains all vertices S, will have the property 1 or the vertices of S are in distinct game components and every this game component will have the property 1. If the vertices u and w are in distinct components of $G-x$, then G will also have two game components which have two coloured vertices u,x and w,x. If the vertices u and w are in one component of $G-x$, then G will have a game component which has three coloured vertices u,w,x. The block S' which contains x will be a center of this game component. G might also have game components which have one coloured vertex. If w' is adjacent to a 3-coloured vertex and a 4-coloured vertex, then Alice colours w' with colour 1 (or with 2 whenever w' has also 2-coloured neighbour). After such a move G will have the property 2. Now assume that $x \neq w'$. If x is not adjacent to any 3-coloured vertex or to any 4-coloured vertex, then Alice colours x with 3 or 4. The game component which contains the vertices of S will have the property 1. If the vertices u and w are in distinct components of $G-x$, then G will also have two game components which have two coloured vertices u,x and w,x. If the vertices u and w are in one component of $G-x$, then G will have a game component which has three coloured vertices u,w,x. The block S' which contains x will be a center of this game component. If x is adjacent to a 3-coloured vertex and a 4-coloured vertex, then Alice colours a neighbour y of x which is a separator of x and w'. She colours y with colour 3 or 4, after such a move a game component containing vertices y,w,u will have the property 3. If she cannot make such a move, i.e., $w'=y$ and $c(S)=\{3,4\}$, then Alice colours w' with 1. After that a move the game component containing vertices w',w,u will have the property 3.

Finally, assume that w' is coloured, say $c(w')=1$, and w is the second coloured vertex of the w'-branch. Since $G \notin \beta_3$, in S there is exactly one 1-coloured vertex. Since G does not have the property 3, w' has a 2-coloured neighbour u. Thus, Alice colours a common neighbour of w' and u. After Alice's move every game component of G will have at most two coloured vertices.

Case 2. Before Bob's move G had the property 2.
Let $c(v)=1$. Since $G \notin \beta_3$, Bob has coloured a vertex in such a way that:

(a) there is a steam which has one 2-coloured vertex u which is adjacent to v and a coloured vertex w such that $c(w) \neq 2$ or w is not adjacent to u, or

(b) there is a steam which has two coloured vertices which are not adjacent to v, or

(c) there is a steam which has three coloured vertices.

First suppose that (a) holds. Note that w is not adjacent to v, otherwise v, u, w are in three distinct game components which are in β_3. Then Alice colours the common neighbour of v and u with colour 3 or 4. After her move the game component which contains v will have the property 2, the other game components will have at most two coloured vertices. Next assume that (b) holds. Let u, w be two coloured vertices of this steam. Then Alice try to colour a v-cut-vertex x for a triple (v, u, w) with colour 3 or 4. After such a move the game component which contains v will have the property 2. The game component which contains x, u, w will have the property 1 (or these vertices are in distinct game components which have at most two coloured vertices). If neither colour 3 nor 4 is admissible for x, then Alice colours a neighbours y of x which is a separator of x and v with colour 3 or 4. After such a move the game component which contains x, u, w will have the property 3. If she cannot make such a move, i.e., the vertex y is coloured ($y = v$), then Alice colours x with 1 or 2. After her move G will have property 2, but now the central vertex is x or G may be split into game components which will have the property 2. Now suppose that (c) holds. Let u be the coloured vertex adjacent to v and w, z be the other coloured vertices of this steam. If $c(u) = 2$ then Alice colours a common neighbour of v and u. Since z is adjacent to u and $c(z) = 2$, every game component will be in β_3. Suppose that $c(u) \neq 2$. Then Alice colours a v-cut-vertex x for a triple (v, z, w). If x is adjacent to v, she colours it with an arbitrary admissible colour. Otherwise, she try to colour x with colour 3 or 4. If neither colour 3 nor 4 is admissible for x, then Alice colours a neighbour y of x which is a separator of x and v with colour 3 or 4.

Case 3. Before Bob's move G had the property 3.
If Bob has coloured a vertex which is adjacent to v, then Alice colours v with colour 1 or 2. After her move G will have property 2 or can be split into game components which will have the property 2. Suppose that Bob has coloured a vertex w which is in distance at least two to v. Alice try to colour a neighbour x of w which is a separator of w and v with colour 3 or 4. If she cannot make such a move because x has a 3-coloured neighbour and a 4-coloured neighbour, then Alice colours v with colour 1. After her move G will have the property 2. If she cannot make such a move because x is coloured, say with colour 1, then Alice colours a common neighbour of x and v with 1. If there is no common uncoloured neighbour of x and v, then

Alice colours v with colour 2. ■

From Lemma 14 and Lemma 16 and the fact that $\mathcal{H}_3 \subseteq \beta_3$, we obtain the following.

Theorem 17. *If $G \in \mathcal{H}_3$, then Alice has a winning strategy for the $(\mathcal{O}_1, \mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game on G.*

Next we show that Alice has a winning strategy for $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1})$-game on every graph from $\mathcal{H}_k,\ k \geq 4$, where $\mathcal{P}_1 = \mathcal{O}_1, \mathcal{P}_i = \mathcal{O}$ $(2 \leq i \leq k+1)$.

Let $G \in \mathcal{H}_k^r$. A *weak-center of* G is a block or a pseudo-block W such that for every uncoloured vertex $v \in W$ a v-branch has at most one coloured vertex and for every 1-coloured vertex $w \in W$, a w-branch has at most two coloured vertices (the vertex w and the other vertex). A *weak-branch* is a branch of a weak-center W which has two coloured vertices (the root and the other vertex). By Q we denote the subgraph induced by the vertices which are neither in the weak-center W nor in weak-branches. Let $w = |V(W) \cap C|,\ q = |V(Q) \cap C|$, where C is the set of coloured vertices of G.

Definition (Family β_4)**.** $G \in \beta_4$ if $G \in \mathcal{H}_k^{k+1},\ k \geq 4$, and G is a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1})$-game component, and G is partially $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1})$-coloured, where $\mathcal{P}_1 = \mathcal{O}_1, \mathcal{P}_i = \mathcal{O}$ $(2 \leq i \leq k+1)$, and G has at least one of the following properties:

1. G has a center S such that

- **1.1.** $|V(S)| \leq k-1$ or
- **1.2.** $|V(S)| = k$, and
  - **1.2.1.** $t \leq 1$ or
  - **1.2.2.** $t = 2$ and $s = 1$ or
  - **1.2.3.** $t = 2,\ s \geq 2$ and if in S there is exactly one 1-coloured vertex x and both vertices of T are in distance two to x, then at least one of them has a colour from $c(S)$.

2. G has a weak-center W such that

- **2.1.** $|V(W)| \leq k-1$ or
- **2.2.** $|V(W)| = k$, and
  - **2.2.1.** $q = 0$ or

2.2.2. $q = 1$ and $w = 1$ or

2.2.3. $q = 1$, $w \geq 2$ and if in W there is exactly one 1-coloured vertex x and a vertex of Q is in distance two to x, then it has a colour from $c(W)$ or

2.2.4. $q = 2$, $w \geq 1$ and two coloured vertices of Q are adjacent to W and they are coloured with colours from $c(W)$.

3. There is an uncoloured vertex u such that all coloured vertices are in blocks which contain u and are coloured with colour distinct from 1.

4. There is a 1-coloured vertex v such that all coloured vertices are in blocks which contain v.

Let $G \in \mathcal{H}_k^{k+1}$ $(k \geq 4)$. Note that if G has at most two coloured vertices, then it always belongs to β_4 and it has the property 1.1 or 1.2.1. If G has the property 1.2.3 (2.2.3), then the vertices which are in distance two to x are adjacent to S (to W) and roots of their branches are adjacent to x. From the fact that every graph $G \in \beta_4$ is a game component we have the following properties: If v is a coloured with colour distinct from 1 and v is in the center or the weak-center S, then the v-branch of S has only one vertex (i.e., the vertex v). If all vertices of the center (the weak-center) S are coloured, then $|V(S)| = 2$ and $c(S) = 1$ and there is at most one branch (the weak-branch) of S which has uncoloured vertices. Moreover if $v \in S$ is the root of this branch, then v has two steams. If S is the center (the weak-center) and $|V(S)| \geq 3$ and S has two adjacent 1-coloured vertices u_1, u_2, then the u_i-branch (the weak u_i-branch) of S, $i \in \{1, 2\}$, has exactly one vertex. If v is an 1-coloured vertex and $B_1, B_2, \ldots, B_p$ are steams such that $v \in B_i$ and B_i has uncoloured vertices $(1 \leq i \leq p,\ p \geq 2)$, then in every steam B_i $(1 \leq i \leq p)$ there is a neighbour of v which can be coloured with 1.

Lemma 18. *The family β_4 is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1})$-game closed, where $\mathcal{P}_1 = \mathcal{O}_1, \mathcal{P}_i = \mathcal{O}$ $(2 \leq i \leq k+1)$.*

Proof. Let $G \in \beta_4$. First we will show that Alice can colour a vertex of G in such a way that every game component will be in β_4. Suppose that G has the property 1 of Definition 4, hence G has the center S. If all vertices of S are coloured, then Alice colours a vertex which is adjacent to the S with colour distinct from 1. If there is an uncoloured vertex $v \in S$

and G has the property 1.1 or 1.2.1, then Alice colours v with a colour distinct from 1. After such a move the game component which contains all vertices of S will have the property 1.1 or 1.2.1 or the vertices of S are in distinct game components which will have the property 1.1. The other game components will have at most one coloured vertex or at most two coloured vertices whenever the v-branch had a coloured vertex. Assume that there is any uncoloured vertex in S and G has the property 1.2.2 or 1.2.3. Let v be an uncoloured vertex of S such that the v-branch of S has one coloured vertex. Then Alice colours v with a colour distinct from 1. After Alice's move the game component which contains the vertices of S will have the property 1.2.1 or 1.1. Suppose that G has the property 2, hence G has the weak-center W. Suppose that all vertices of W are coloured, i.e, $|W| = 2$ and $c(W) = \{1\}$. Let B be the block adjacent to W. Since in G there is one coloured vertex which does not belong to W, B is a weak-center and has at least one uncoloured vertex. Hence we may assume that the weak-center has at least one uncoloured vertex. If G has the property 2.1 or 2.2.1, then Alice colours an uncoloured vertex $v \in W$ with a colour distinct from 1. After such a move the game component which contains the vertices of W, will have the property 2.1 or 2.2.1. The other game components will have at most one coloured vertex or at most two coloured vertices whenever the v-branch had a coloured vertex. If G has the property 2.2.2 or 2.2.3, or 2.2.4, then Alice colours an uncoloured vertex v such that the v-branch of W has one coloured vertex. After such a move the game component which contains the vertices of W will have the property 2.2.1, 2.2.2 or 2.2.4 (also it might have the property 2.1). Now assume that G has the property 3. Then Alice colours u with 1, after her move G will have the property 4. If G has the property 4, Alice colours an uncoloured neighbours of v. If she cannot make such a move, i.e., all neighbours of v are coloured, then v has a neighbour coloured with 1 and these are the only coloured vertices of G. But in this case G has also the property 1.1, thus Alice can make the same move as above. Thus, Condition (i) of Definition 4 holds. Next we will prove that Condition (ii) of Definition 4 also holds. Suppose that Bob has coloured a vertex of G and after his move there is a game component of G which is not in β_4. Let us consider the following cases.

Case 1. Before Bob's move G had the property 1.

Subcase 1.1. Suppose that G had the property 1.1.
Thus, after Bob's move S is not a center of G, hence there is a branch of

S which has two coloured vertices u, w ($u, w \in T$) (if Bob had coloured a vertex of S, then G would have had a center of a weak-center). Let v be a root of a branch which contains u, w. Alice colours a v-cut-vertex x for the triple (v, u, w) with a colour distinct from 1 or with 1 if it is the only admissible colour. Note that always there is an admissible colour for x, since x has at most k coloured neighbours. After Alice's move the game component which contains the vertices of S will have the property 1.1. If in $G-x$ the vertices u, w are in distinct components, then G will also have two game components which have two coloured vertices. If in $G-x$ the vertices u, w are in one component, then G will have the game component which has three coloured vertices and has a center. All other game components will have at most one coloured vertex.

Subcase 1.2. Now, assume that G had the property 1.2.

Subcase 1.2.1. First, let us consider the case when G had the property 1.2.1.
Since after Bob's move G is not in β_4, it follows that Bob has coloured a vertex $w \in T$. Let w' be a root of a branch of S which contains w. If w' is coloured (it must be coloured with 1), then Alice colours a root of a branch which contains the second coloured vertex of T. If w' is uncoloured and w is the only coloured vertex of the w'-branch, then Alice colours w' with a colour distinct from 1. After her move the game component which contains vertices of S has the property 1.2.1 or the property 1.1. Suppose that the w'-branch has two coloured vertices w and u ($u \neq w'$). Alice colours a w'-cut-vertex x for the triple (w', w, u) with a colour distinct from 1 or with colour 1 when it is the only admissible colour for w'. Note that always there is an admissible colour for x, since x has at most $k+1$ coloured neighbours. If $|c(N(x))| = k+1$, then exactly one vertex is coloured with colour 1, so Alice can colour x with colour 1.

Subcase 1.2.2. Suppose that G had the property 1.2.2.
First assume that Bob has coloured a vertex of S. Thus, $t = 2$ and $s = 2$. Let $v, u \in V(T) \cap C$ and u', v' be vertices such that u, v are in a u'-branch and a v'-branch, respectively. Since G is not in β_4, it follows that S is the center with exactly one 1-coloured vertex x and u, v are in distance two to x and $c(u) \notin c(S)$, $c(v) \notin c(S)$ or S is the weak-center (u' is a vertex which was coloured by Bob, $c(u') = 1$) and v is in distance two to x and $c(v) \notin c(S)$. Alice colours an uncoloured root of a branch which has a coloured vertex in such a way that the game component which contains vertices of S will

have the property 1.2.1 (with $t = 1$) or the property 2.2.1 (with $q = 0$) or the vertices of S will be in distinct game components which will have the property 1.1 or 2.1.

Now assume that Bob has coloured a vertex of T. Thus, $t = 3$ and $s = 1$. Let $v, u, w \in V(T) \cap C$. Suppose that coloured vertices of T are in three distinct branches. Let u', v', w' be such vertices that u, v, w are in a u'-branch, a v'-branch and a w'-branch, respectively. If u', v', w' are uncoloured, then Alice colours the vertex u' or v' or w', in such a way that the game component which contains vertices of S will have the property 1.2.3 (if $c(v) = c(u) = c(w) = c$ and S has exactly one 1-coloured vertex, then she colours a root with colour 1) or the vertices of S will be in distinct game components which have the property 1.1.

Suppose that one of the roots, say w', is coloured with 1 and w has just been coloured by Bob. Thus, Alice colours u' with $c(v')$ or with arbitrary colour whenever $c(v) \in c(S)$ or v is not in distance two to w'. After her move the game component which contains vertices of S has the property 2.2.3 or the property 1.1 or the property 2.1. If she cannot make such a move, i.e., $c(u) = c(v) = i$ and $c \notin c(S)$ and u, v are in distance two to w', then Alice colours uncoloured vertex of S with i. After that a game component which contains the vertices of S will have the property 2.2.4 or 1.2 or 2.1.

Finally, suppose that u, v, w are not in three distinct branches. Let x be a root of a branch which contains u and v. Alice colours a x-cut-vertex for the triple (x, v, u) with a colour distinct from 1.

Subcase 1.2.3. Suppose that G had the property 1.2.3.
First assume that Bob has coloured a vertex of S. Since $G \notin \beta_4$, Bob has coloured a vertex of S with colour 1. Alice colours an uncoloured root of a branch which has a coloured vertex in such a way that a game component which contains the vertices of S will have the property 1.2.1 or the property 2.2.1 (or the property 1.1, 2.1).

Now assume that Bob has coloured a vertex of T. Thus, $t = 3$ and $s \geq 2$. Let u, v, w be coloured vertices of T. Suppose that the vertices u, v, w are in three distinct branches. Let u', v', w' be such vertices that u, v, w are in a u'-branch and a v'-branch and a w'-branch, respectively. If the vertices u', v', w' are uncoloured, then Alice colours one of them in such a way that the game component which contains vertices of S will have the property 1.2.3 or 1.1. If one of vertices is coloured, say w', (w' is coloured with 1 and w has just been coloured by Bob), then Alice colours u' or v'. After her move the game component which contains vertices of S has the property

2.2.3 or the property 1.1 or 2.1.

Finally, suppose that u, v, w are not in three distinct branches. Let x be a root of a branch which contains u and v. Then Alice colours a x-cut-vertex y for the triple (x, v, u). If S has exactly one 1-coloured vertex and y and w are in distance two to this vertex, then Alice colours y with colour from $c(S) \setminus \{1\}$. If in $c(S) \setminus \{1\}$ there is no admissible colour for y, then Alice colours x. Otherwise, she colours y with a colour distinct from 1 or with colour 1 when it is the only admissible colour. Note, that colour 1 is the only admissible colour if $x = y$ and $k - 2$ vertices of S are coloured with distinct colours.

Case 2. Before Bob's move G had the property 2.

Subcase 2.1. Assume that G had the property 2.1.
First suppose that $|W| = 2$ and two adjacent vertices of W are coloured with 1. Hence W has only one branch, say the v-branch (it is a weak-branch) and v has only one steam which has uncoloured vertices. Since after Bob's move $G \notin \beta_4$, the v-branch has three coloured vertices, v, u, w. Let W' be a block adjacent to W. If u and w are in two distinct branches of W', then Alice colours a root of the branch containing the coloured vertex in such a way that a game component which contains the vertices of W' will have the property 2.2.1 or 2.2.3, or 2.1. She can also colour a vertex of W' in such a way that a game component which contains W' will have the property 2.2.4. Now assume that u, w are in one branch of W'. Let x be a root of the branch of W' which contains u and w. Alice colours an x-cut-vertex for the triple (x, u, w) with a colour distinct from 1.

Now assume that $|W| \geq 2$ and W has an uncoloured vertex. Since after Bob's move W is not a weak-center of G, there is a branch of W which has two coloured vertices or there is a weak-branch which has three coloured vertices.

Suppose that there is a branch of W which has two coloured vertices. Let v be a root of a branch of W which contains coloured vertices u, w (v is uncoloured). Alice colours a v-cut-vertex for the triple (v, u, w) with a colour distinct from 1.

Finally, suppose that there is a weak-branch which has three coloured vertices. Let v be a coloured root of a weak-branch of W which contains coloured vertices u, w ($c(v) = 1$). Note that since G is a game component and there are at least two steams of v which have uncoloured vertices, in every steam of v there is a neighbour of v which can be coloured with 1, particularly in W there is a vertex which can be coloured with 1. If u and w are in two distinct steams of v, then Alice colours a vertex of W with 1. Assume that u and w are in one steam of v and let W' be a block adjacent to W which is in this steam. If u and w are in distinct branches of W', then Alice colours a vertex of $N(v) \cap V(W')$ with 1. Otherwise, let x be a root of a branch of W' containing u and w. Alice colours an x-cut-vertex for the triple (x, u, w) with a colour distinct from 1.

Subcase 2.2. Suppose that G had the property 2.2.

Subcase 2.2.1. Now assume that G had the property 2.2.1.
Since $G \notin \beta_4$, Bob has coloured a vertex $w \notin W$. If w is the only coloured vertex of a branch of W, then Alice colours a root w' of this branch. Let w and u be two coloured vertices of a weak-branch of W and u' be a root of this weak-branch and $c(u') = 1$. If u and w are in two distinct steams of u', then Alice colours a vertex of W with 1. Note that in $N(u') \cap V(W)$ there is a vertex which can be coloured with 1, since G is a game component and there is more than one steam of u' which has an uncoloured vertex. Assume that u and w are in one steam of u' and let W' be a block adjacent to W which is in this steam. If u and w are in distinct branches of W', then Alice colours a vertex of $N(v) \cap V(W')$ with 1. Otherwise, let x be a root of a branch of W' containing u and w. Then Alice colours an x-cut-vertex for the triple (x, u, w) with a colour distinct from 1.

Subcase 2.2.2. Suppose that G had the property 2.2.2.
Let u' be a coloured vertex of W, so $c(u') = 1$. First assume that Bob has coloured a vertex of W. Since $G \notin \beta_4$, Bob has coloured a vertex which is a root of an uncoloured branch of W. Thus, after Bob's move $q = 1$ and $w = 2$. Let v be a coloured vertex of Q. Thus, Alice colours a root of a branch of W which contains v.

Next assume that Bob has coloured a vertex of Q. Note that the root of a branch which contains the vertex which has just been coloured by Bob is uncoloured, otherwise $G \in \beta_4$. Thus, $q = 2$ and $w = 1$. Let v, w be coloured vertices of Q. Suppose that v and w are in two distinct branches of W. If one of vertices v, w, say v, is not in distance two to u', then Alice colours a root of a branch which contains v with an arbitrary admissible colour. If both vertices are in distance two to u', then Alice colours a root of a branch containing v with colour $c(w)$ or with an arbitrary colour when $c(w) \in c(W)$. If she cannot make such a move, i.e., $c(v) = c(w) = i$, then Alice colours a vertex of W with i. After such a move the game component which contains the vertices of W has the property 2.2.4 or 2.1, or 1.1. If v and w are in one branch, say an x-branch, then Alice colours a x-cut-vertex for the triple (x, w, v).

Finally, assume that Bob has coloured a vertex of the weak-branch. Let u' be a root of this weak-branch. Let u, w be the coloured vertices of the u'-branch. If u and w are in two distinct steams of u', then Alice colours a vertex of W with 1. Assume that u and w are in one steam of u' and let W' be a block adjacent to W which is in this steam. If u and w are in distinct branches of W' or one of them is in W', then Alice colours a vertex of $N(u') \cap V(W')$ with 1. Otherwise, let x be a root of a branch of W' containing u and w. Alice colours a x-cut-vertex for the triple (x, u, w) with a colour distinct from 1.

Subcase 2.2.3. Assume that G had the property 2.2.3.
Since $G \notin \beta_4$, Bob has coloured a vertex which is not in W. Assume first that Bob has coloured a vertex $w \in Q$. Note that a root w' of a branch containing w is uncoloured, otherwise $G \in \beta_4$. Thus, $q = 2$ and $w \geq 2$. If two coloured vertices of Q are in distinct branches, then Alice colours w' with a colour distinct from 1. If the w'-branch has two coloured vertices w and u, then Alice colours a w'-cut-vertex y for the triple (w', w, u). If Q has exactly one 1-coloured vertex and y is in distance two to this vertex, then Alice colours y with colour from $c(Q) \setminus \{1\}$. If in $c(Q) \setminus \{1\}$ there is no admissible colour for y, then Alice colours x. Otherwise, she colours y with a colour distinct from 1 or with colour 1 if it is the only admissible colour.

Now assume that Bob has coloured a vertex of the weak-branch. Let u' be a root of this weak-branch. Let u, w be the coloured vertices of the u'-branch. If u and w are in two distinct steams of u', then Alice colours a vertex of W with 1. Assume that u and w are in one steam of u' and let W' be a block adjacent to W which is in this steam. If u and w are in distinct branches of W' or one of them is in W', then Alice colours a vertex of $N(u') \cap V(W')$ with 1. Otherwise, let x be a root of a branch of W' containing u and w. Alice colours a x-cut-vertex for the triple (x, u, w).

Subcase 2.2.4. Finally assume that G had the property 2.2.4.
Since $G \notin \beta_4$, Bob has coloured a vertex which is not in W. First assume that Bob has coloured a vertex $w \in Q$. Then Alice colours a root of this branch with a colour distinct from 1.

Now assume that Bob has coloured a vertex of the weak-branch. Let u' be a root of this weak-branch and u, w be coloured vertices of u'-branch. If

u and w are in two distinct steams of u', then Alice colours a vertex of W with 1. Assume that u and w are in one steam of u' and let W' be a block adjacent to W which is in this steam. If u and w are in distinct branches of W' or one them is in W', then Alice colours a vertex of $N(u') \cap V(W')$ with 1. Otherwise, let x be a root of a branch of W' containing u and w. Alice colours an x-cut-vertex for the triple (x, u, w) with a colour distinct from 1 or with colour 1 when it is the only admissible colour.

Case 3. Before Bob's move G had the property 3.
If Bob colours a vertex which is in a block containing u, then Alice colours u with colour 1. If Bob colours a vertex w which is not in the block containing u, then Alice colours a separator of u and w which is in the block containing u. She colours this vertex a colour distinct from 1.

Case 4. Before Bob's move G had the property 4.
Since G does not have the property 4, Bob colours a vertex w which is not in the block containing v. Thus, Alice colours a separator of v and w which is in the block containing v with colour distinct from 1. ■

Next theorem follows from Lemma 14 and Lemma 16 and the fact that $\mathcal{H}_k \subseteq \beta_4$ for $k \geq 4$.

Theorem 19. *Let $G \in \mathcal{H}_k$ and $k \geq 4$. Then Alice has a winning strategy for a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1})$-game on G, where $\mathcal{P}_1 = \mathcal{O}_1,\ \mathcal{P}_i = \mathcal{O}\ (i = 2, \ldots, k+1)$.*

5. Monotonicity of Generalized Colouring Game

We begin this section with an example. Let $K_{n,n}$ $(n \geq 4)$ be a complete bipartite graph. It is easy to see that $\chi_g(K_{n,n}) = 3$. Hence Alice has a winning strategy for the $(\mathcal{O}, \mathcal{O}, \mathcal{O})$-game (the proper colouring) on $K_{n,n}$. However one can observe that Bob has a winning strategy for the $(\mathcal{O}_1, \mathcal{O}_1, \mathcal{O}_1)$-game on $K_{n,n}$. Thus, in general the generalized colouring game is not monotone. In this section we show that the generalized colouring games which were discussed in previous sections on graphs from $\mathcal{H}_k$ are monotone.

Let G be a partially k-coloured graph. A subgraph G' of G is a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*subgraph* if it contains all uncoloured vertices of G and the coloured vertices such that every uncoloured vertex v has the same $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colours in G' as it has in G. We say that the $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-subgraph G' is *minimal* if G' does not contain any proper $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-subgraph.

Let $G_1, G_2, \ldots, G_p$ be $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components of G. The graphs $G_1', G_2', \ldots, G_p'$ are *minimal* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$*-game components* of G if $G_1', G_2', \ldots, G_p'$ are $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-minimal subgraphs of $G_1, G_2, \ldots, G_p$, respectively.

Let G be a partially k-coloured graph. Note that if $\mathcal{P}_i = \mathcal{O}$ for $1 \leq i \leq k$, then the minimal $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components of G are $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloured graphs.

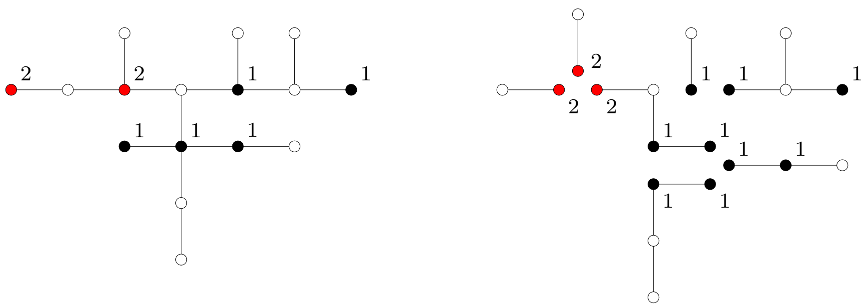

Figure 3. The graph G and its minimal $(\mathcal{O}_1, \mathcal{O})$-game components
The labeled vertices are coloured with colour 1 or 2.

Definition (Monotone $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game closed family)**.** Let γ be a family of partially k-coloured graphs. We say that the family γ is *monotone* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$*-game closed* if for every $G \in \gamma$ the following conditions hold:

(i) Alice can colour a vertex of G with a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour in such a way that all minimal $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components of G are in γ or all vertices of G are coloured.

(ii) If Bob colours a vertex with an arbitrary colour and if after his move G has an uncoloured vertex, then Alice can colour a vertex with a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour in such a way that all minimal $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components of the graph G are in γ or all vertices of G are coloured.

Lemma 20. *Let γ be a monotone $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game closed family, $G \in \gamma$ and $u \in V(G)$ be an uncoloured vertex. Then u has a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour.*

Proof. Assume that in γ there is a graph G such that G has an uncoloured vertex u which has no $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour and G has the minimum number of uncoloured vertices. If u is the only uncoloured vertex, then G does not satisfy Condition (i) of Definition 5, a contradiction. If G has more than one uncoloured vertex, then Bob can colour a vertex other than u. By Condition (ii) of Definition 5 Alice can colour a vertex in such a way that all minimal $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components of G are in γ. Thus after her move there is a minimal game component which has less uncoloured vertices than G and has a vertex which has no $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour, a contradiction. ■

Lemma 21. *Let γ be a monotone $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game closed family and $(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k)$ be additive hereditary properties such that $\mathcal{P}_i \subseteq \mathcal{P}'_i$ $(1 \leq i \leq k)$. If $G \in \gamma$ and G is partially $(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k)$-coloured, then Alice has a winning strategy on G for the $(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k)$-game and for the $(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k)$-game with the first move of Bob.*

Proof. Let $G \in \gamma$. First note that if players play the $(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k)$-game on G then Alice can see a graph G as a disconnected graph which components are minimal $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components of G. If colour i is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible for a vertex v in a minimal $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game component of G, then i is also $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible for the vertex v in G. If the player colours a vertex of a minimal game component G_i, then in all other minimal game components the set of $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colours for vertices do not change.

The winning strategy of Alice is the following: she colours the vertices in such a way that every minimal $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game component of G is in γ or all vertices are coloured. From the definition of the monotone $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game closed family easy follows that she achieves this goal.

Note that Alice's strategy implies that after every move of players every uncoloured vertex has a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour. Indeed, after Alice's move every minimal $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game component is in γ, hence every uncoloured vertex has a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour. After Bob's move there is a vertex which can be coloured by Alice in such a way that all minimal $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-game components will be in γ. Hence also after Bob's move every uncoloured vertex has a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible colour. If a colour is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-admissible, then it is also $(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k)$-admissible. Since during the game players never create a forbidden monochromatic subgraph, after every move of players the graph G is $(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k)$-coloured. Thus, Alice wins the game. ■

Now we construct a family of forests which is monotone $(\mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game closed, to show that Alice has a winning strategy for a $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$-game on every forest, where $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are arbitrary additive hereditary properties such that $\mathcal{O}_1 \subseteq \mathcal{P}_1$. This improve the result obtained in [4], which says that Alice has a winning strategy for the $(\mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game on every forest.

Definition (Family γ_2).
$\gamma_2 = \{G : G \in \mathcal{H}_2^3$ and G has at most two coloured vertices$\}$.

Lemma 22. *Family γ_2 is monotone $(\mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game closed.*

Proof. Let $G \in \gamma_2$. It is easy to check that Alice always can colour a vertex in such a way that all minimal $(\mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game components are in γ_2 or all vertices of G are $(\mathcal{O}_1, \mathcal{O}, \mathcal{O})$-coloured. Suppose that after Bob's move G has a minimal $(\mathcal{O}_1, \mathcal{O}, \mathcal{O})$-game component which is not in γ_2. Hence before Bob's move G had two coloured vertices u, v. Let w be a vertex which has just been coloured by Bob. If u and v are adjacent, $c(u) = c(v) = 1$ and w is adjacent to u or v, then every minimal game component is in γ_2 (even if Bob has coloured w with 1). If w is adjacent neither to u nor to v, then Alice colours with a colour distinct from 1 a neighbour of u or v which is a separator of $\{u, v\}$ and w. Now assume that u and v are not adjacent. If w is adjacent to u and $c(u) \neq 1$, then w and u are in distinct minimal game components which are in γ_2 (even if Bob coloured w with $c(u)$). So, we may assume that the vertices u, v, w form an independent set. Suppose that there is a path which contains u, v, w and v is the middle vertex of this path. If $c(v) \neq 1$, then every game component of G has two coloured vertices, hence it is in γ_2. So, $c(v) = 1$. Thus, Alice colours a neighbour of v which is on this path with colour 2 or 3. Assume that there is no path which contains u, v, w and let x be a vertex such that the vertices u, v, w are in distinct steams of x. Alice colours x with colour 2 or 3. If she cannot make such a move, i.e, the vertices u, v, w are adjacent to x and coloured with distinct colours, then Alice colour x with 1. After such a move the vertices u, v, w are in distinct minimal game components which are in γ_2. ■

Theorem 23. *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be additive hereditary properties such that $\mathcal{O}_1 \subseteq \mathcal{P}_1$. Then Alice has a winning strategy for a $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$-game on every forest.*

One can observe that game closed families discussed in previous sections can be easily extended to monotone game closed. In families α_1, α_2 it is enough to replace the condition which says that a graph is partially properly $(k+2)$-coloured ($(k+1)$-coloured) with the condition that a graph is partially $(k+2)$-coloured ($(k+1)$-coloured). In families β_3, β_4 it is enough to replace the condition which says that a graph is partially $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1})$-coloured with the condition that a graph is partially $(k+1)$-coloured. From proofs of these theorems it follows that if we allow Bob to colour a vertex with an arbitrary colour (no necessary admissible), then Alice can colour a vertex in the same way as in case when Bob colours a vertex with an admissible color. After Alice's move the game components are in the corresponding family. Thus, from Lemmas 7, 10, 16 and 18 we obtain the new results.

Theorem 24. *Let $k \geq 2$, $G \in \mathcal{H}_k$ and $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+2}$ be additive hereditary properties. Then Alice has a winning strategy for a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+2})$-game on G.*

Theorem 25. *Let $k \geq 6$, $G \in \mathcal{H}_k$ and $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1}$ be additive hereditary properties. Then Alice has a winning strategy for a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1})$-game on G.*

Theorem 26. *Let $G \in \mathcal{H}_3$ and $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be additive hereditary properties such that $\mathcal{O}_1 \subseteq \mathcal{P}_i$ ($i \in \{1, 2\}$). Then Alice has a winning strategy for a $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$-game on G.*

Theorem 27. *Let $k \in \{4, 5\}$, $G \in \mathcal{H}_k$ and $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1}$ be additive hereditary properties such that $\mathcal{O}_1 \subseteq \mathcal{P}_1$. Then Alice has a winning strategy for a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k+1})$-game on G.*

In [20] Zhu suggested that if $\chi_g(G) = k$, then Alice has a winning strategy for the t-colouring game on G for any $t \geq k$. Our results confirm this hypothesis. By Lemma 5 and Lemma 7, Alice has a winning strategy for the $(k+2)$-colouring game on $G \in \mathcal{H}_k$ ($k \geq 2$). If $G \in \mathcal{H}_k$, then also $G \in \mathcal{H}_p$ for any $p \geq k$. Hence Alice has a winning strategy for the t-colouring game on $G \in \mathcal{H}_k$ ($k \geq 2$) for any $t \geq k+2$. Similarly, from Lemma 5 and Lemma 10 it follows that Alice has a winning strategy for the t-colouring game on $G \in \mathcal{H}_k$ ($k \geq 6$) for any $t \geq k+1$. So, also in this sense the colouring game number for graphs from $\mathcal{H}_k$ is monotone.

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