

A NOTE ON MAXIMAL COMMON SUBGRAPHS
OF THE DIRAC'S FAMILY OF GRAPHS

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Abstract

Let \mathcal{F}^n be a given set of unlabeled simple graphs of order n . A *maximal common subgraph* of the graphs of the set \mathcal{F}^n is a common subgraph F of order n of each member of \mathcal{F}^n , that is not properly contained in any larger common subgraph of each member of \mathcal{F}^n . By well-known Dirac's Theorem, the Dirac's family \mathcal{DF}^n of the graphs of order n and minimum degree $\delta \geq \frac{n}{2}$ has a maximal common subgraph

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containing C_n . In this note we study the problem of determining all maximal common subgraphs of the Dirac's family \mathcal{DF}^{2n} for $n \geq 2$.

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We follow the definitions and terminology of [1]. Let \mathcal{F}^n be a given set of unlabeled simple graphs of order n . A *maximal common subgraph* of the graphs of the set \mathcal{F}^n is a common subgraph F of order n of each member of \mathcal{F}^n , that is not properly contained in any larger common subgraph of each member of \mathcal{F}^n . By well-known Dirac's Theorem, the Dirac's family \mathcal{DF}^n of the graphs of order n and minimum degree $\delta \geq \frac{n}{2}$ has a maximal common subgraph containing C_n (see [2, 3, 4]). The cycles C_4 and C_6 are maximal common subgraphs of \mathcal{DF}^4 and \mathcal{DF}^6 , respectively. While C_4 is the unique maximal common subgraph of \mathcal{DF}^4 , for \mathcal{DF}^6 it is easy to check that there are exactly two maximal common subgraphs: C_6 and the graph F_6 (see Figure 1).

Figure 1. Maximal common subgraphs of \mathcal{DF}^6 .

In this note we study the problem of determining maximal common subgraphs of the Dirac's family \mathcal{DF}^{2n} for $n \geq 2$. It is easy to see that to determine all maximal common subgraphs of the Dirac's family \mathcal{DF}^n , it is enough to consider the maximal common subgraphs of the family of the minimal elements of the set \mathcal{DF}^n partially ordered by the relation \subseteq - *to be a subgraph*. The minimal Dirac's graphs of order 8 are presented in Figure 2.

Because the complete bipartite graph $K_{4,4}$ is a member of the set of minimal elements of \mathcal{DF}^8 , each maximal common subgraph of the set \mathcal{DF}^8 must be a bipartite graph with a balanced regular two-colouring (i.e., four vertices in each colour class). Using this fact we determined all maximal

common subgraphs of the set \mathcal{DF}^8 . They are presented in the Figure 3. Since they could also be found by a computer search, we omit a detailed proof here.



Figure 2. Minimal Dirac's graphs of order 8.

H_1 H_2 H_3

Figure 3. The maximal common subgraphs of \mathcal{DF}^8 .

The problem of determining the maximal common subgraphs for the Dirac's family \mathcal{DF}^n is much more complicated for odd n and we can mention only that the wheel $W_5 = K_1 + C_4$ is the unique maximal common subgraph of \mathcal{DF}^5 , however for \mathcal{DF}^7 there are at least 5 different maximal common subgraphs.

As the main result of this note we will show that the Hamiltonian cycle C_n is not a maximal common subgraph of the Dirac's family \mathcal{DF}^n for $n \geq 7$. The proof is based on the following lemma.

Lemma 1. *Let G be a graph of order $n \geq 7$ satisfying Dirac's condition $\delta(G) \geq n/2$. Let $H = abcd$ be a 4-cycle in G having a tail $T = [x_0 \cdots x_k]$ of maximum length k . Then $k = n - 4$.*

Proof. Without loss of generality, we may assume that $x_k = a$. Assume, to the contrary, that $V \setminus (H \cup T)$ is nonempty and let y be a vertex in this set. We will produce a contradiction by finding in G a 4-cycle with a longer tail.

Denote by $x_{i_1} = x_1, \dots, x_{i_p}$ the neighbours of x_0 belonging to T .

Case 1. If there is an $i \in \{1, \dots, p\}$ such that $yx_{i-1} \in E$, then $[yx_{i-1} \cdots x_0x_i \cdots x_k]$ is a tail of length $k + 1$ for H . Assume henceforth the contrary. Let now q be the number of neighbours of x_0 in the set $\{b, c, d\}$. We have by hypothesis $p + q \geq n/2$. Let q_1 be the number of neighbours of y in the set $\{a, b, c, d\}$. Note that y has at most $k - p$ neighbours in the set $T \setminus \{a\}$ and at most $n - k - 5$ neighbours outside the set $H \cup T$.

Case 2. If b and d are both neighbours of y , then $[x_0 \cdots x_kb]$ is a tail of length $k + 1$ for the C_4 $ybcd$ of G . So we may assume that we have

$q_1 \leq 3$. Now we obtain by hypothesis for the number of neighbours of y : $n/2 \leq \deg(y) \leq (k-p) + q_1 + n - k - 5 = n + q_1 - 5 - p \leq n/2 + q + q_1 - 5$, so $q + q_1 \geq 5$ implying $q \geq 2$. So x_0 must have a neighbour in the set $\{b, d\}$. By symmetry, we may suppose $bx_0 \in E$.

Case 3. If a and c are both neighbours of y , then $[x_{k-1} \cdots x_0ba]$ is a tail of length $k+1$ for the C_4 $yadc$ of G . This being not the case, we have $q_1 \leq 2$ therefore $q = 3$ and $q_1 = 2$. Now the three vertices b, c, d are neighbours of x_0 , and by symmetry we may suppose $yb \in E$. There remains only two cases, according to whether a or c is the other neighbour of y in H .

Case 4. If a is neighbour of y , then $[x_1 \cdots x_kyb]$ is a tail of length $k+1$ for the cycle x_0bcd .

Case 5. If c is neighbour of y , then $[da \cdots x_0]$ is a tail of length $k+1$ for the cycle yx_0c . ■

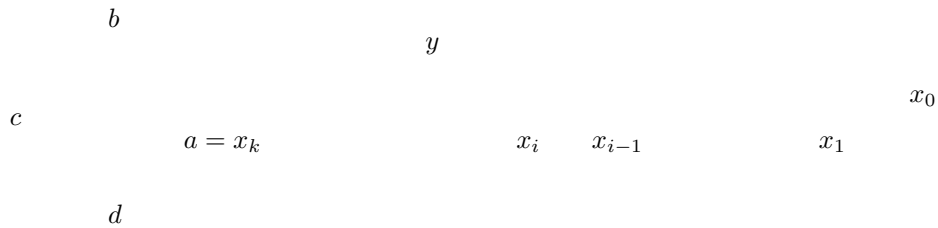


Figure 4. Case 1 in the proof of Lemma 1.

Theorem 2. *Let $G = (V, E)$ be a graph of order $|V| = n$, with $n \geq 7$. If G satisfies the Dirac's condition $\delta(G) \geq n/2$, then G contains as a subgraph, a Hamiltonian cycle with a chord that skips two vertices on this cycle.*

Proof. It is straightforward that G contains a cycle C_4 as a subgraph. For a subgraph H of G , a tail of H is any path $[x_0 \cdots x_k]$ in G sharing with H only the vertex x_k . We now complete the proof of the theorem, by examining a 4-cycle $H = abcd$ with a tail T of length $k = n - 4 \geq 3$. Such a cycle exists by the previous lemma. We assume, as before, that $x_k = a$ and keep the same notations as in the proof of the lemma. In the same way, we study and eliminate all possible cases.

Case 1. If b or d , say b by symmetry, is neighbour of x_{i-1} , with $i \in \{i_1, \dots, i_p\}$ then the Hamiltonian cycle $adcbx_{i-1} \cdots x_0x_i \cdots x_k$ has the chord ab .

If this is not the case, then $q \leq 1$ and we must have for the neighbours of b (or d) : $n/2 \leq \deg(b) \leq n-1-p \leq n/2+q-1$, so $q=1$ and we have $cx_0 \in E$, $p = n/2 - 1$ (hence n is even and $n \geq 8$ in this case). Moreover, $x_i, 1 \leq i \leq k-1$ is neighbour of b if and only if it is also neighbour of d and $i+1$ is not in the set $\{i_2, \dots, i_p\}$. Finally, we must have, for the above inequalities being equalities, $bd \in E$.

Case 2. If $bx_{k-1} \in E$, then the Hamiltonian cycle $cdabx_{k-1} \cdots x_0c$ has the chord cb . Assuming the contrary we must have $ax_0 \in E$, otherwise $bx_{k-1} \notin E$ and we must have $ax_0 \in E$.

Case 3. If $bx_{k-2} \in E$, it is a chord of the Hamiltonian cycle $cdba \cdots x_0c$. At last, we may assume $x_0x_{k-1} \in E$, otherwise $bx_{k-2} \notin E$. Since $n \geq 8$, b (as well as d) must have a neighbour x_i with $1 \leq i \leq k-3$, forming a 4-cycle bx_idc with the tail $[x_{i+1} \cdots ax_0 \cdots x_i]$ or the tail $[x_{i-1} \cdots x_0a \cdots x_i]$. In these configurations, x_{i+1} or x_{i-1} play the role of x_0 , and c keeps its own one. Therefore, after eliminating the first case, we obtain that both $x_{i-1}c$ and $x_{i+1}c$ are in E . Now the cycle $bcd a$ has the tail $x_{k-1} \cdots x_0c$, in which vertices a and c exchange their places. Therefore, it remains to consider only the case when both ax_{i+1} and ax_{i-1} are in E . In this case the Hamiltonian cycle $x_{i+1} \cdots x_{k-1}x_0 \cdots x_i b a d c x_{i+1}$ has the chord ax_{i+1} . ■

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