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ON SHORT CYCLES THROUGH PRESCRIBED VERTICES OF A POLYHEDRAL GRAPH

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Abstract

Guaranteed upper bounds on the length of a shortest cycle through $k \leq 5$ prescribed vertices of a polyhedral graph or plane triangulation are proved.

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1. Introduction and Results

G.A. Dirac [2] proved that for a given integer $c \geq 2$ any k $(1 \leq k \leq c)$ prescribed vertices of a c-connected graph belong to a common cycle. However, the complete bipartite graph $K_{c,c+1}$ shows that this is not true for $c+1$ prescribed vertices. In [3] we investigated the length of short cycles through k prescribed vertices with $1 \leq k \leq \min\{c, 3\}$ in a c-connected graph G. From A.K. Kelmans and M.V. Lomonosov [6] we know that any five vertices of a polyhedral graph (that is a planar and 3-connected graph) belong to a common cycle which is best possible.

For given integers k, l with $1 \leq k \leq 5$, $3 \leq l$ and $k \leq l$ let $n_k(l)$ denote the minimum number n such that there exists a polyhedral graph G of order n having a subset of k vertices with the property that the length of every cycle containing those k vertices is at least l. In [3] we proved

(i) $n_1(l) = 3l - 5$ for $l \geq 3$,

(ii) $n_2(l) = \lfloor \frac{3l-1}{2} \rfloor$ for $l \geq 3$,
(iii) $n_3(l) = \lfloor \frac{3l-1}{2} \rfloor$ for $l \geq 5$,

and the following results which will be proven here is a continuation of the investigation [3] of short cycles through prescribed vertices for a polyhedral graph.

Theorem 1.

$$n_4(l) = \begin{cases} l & if \quad l \in \{4, 8\}, \\ l+1 & if \quad l \in \{5, 6, 7, 9, 10\}, \\ l+2 & if \quad l \in \{11, 12\}, \\ \lceil \frac{4l-5}{3} \rceil & if \quad l \geq 13\,. \end{cases}$$

Theorem 2.

$$n_5(l) = \begin{cases} l & if \quad l = 5 \quad or \quad l \geq 8, \\ l+1 & if \quad l = 6 \quad or \quad 7. \end{cases}$$

For integers k, l with $2 \leq k \leq 5$, $3 \leq l$ and $k \leq l$ denote by $t_k(l)$ the minimum number n such that there exists a plane triangulation T of order n with certain k vertices such that the length of every cycle containing them is at least l. Then we have $n_k(l) \leq t_k(l)$ since every plane triangulation is 3-connected and thus a polyhedral graph. Notice that even $n_k(l) = t_k(l)$ holds in every considered case. If, namely, G is any one of the here or in [3], respectively, constructed graphs to prove an upper bound for $n_k(l)$ with certain k and l, then we were able to construct a plane triangulation T from G by adding edges only such that the length of a shortest cycle containing the prescribed k vertices is at least l.

2. Proofs

For terminology and notation not defined here we refer to [5]. Let G be a graph and $A, B \subseteq V(G)$. A path P of G with one end-vertex in A and B, respectively, and with $|V(P) \cap A| = |V(P) \cap B| = 1$ is called an *A-B-path*. If A or B consists of a single vertex x we write x instead of $\{x\}$. We use $[x, y]$ to denote an x-y-path and, moreover, $[x, y)$ or (x, y) to denote the segments obtained from $[x, y]$ by removing y or both end-vertices, respectively. A *path* *system* is a set of internally disjoint paths. For a path system $\mathcal{P}$ let $[\mathcal{P}]$ and $EV(\mathcal{P})$ denote the union of all paths and the set of all end-vertices of paths of $\mathcal{P}$, respectively. For some $a \in V(G)$ and $B \subseteq V(G) \setminus \{a\}$ a path system $\mathcal{P}$ of a-B-paths is called an a-B-fan if $P \cap Q = \{a\}$ for different $P, Q \in \mathcal{P}$.

We need the following lemma which is proved in [3] in more general form.

Lemma 1. *Let G be a c-connected graph with $a \in V(G)$, $B \subseteq V(G) \setminus \{a\}$ and a path system $\mathcal{P}$ of $c-1$ a-B-paths. Let $B' = B \setminus EV(\mathcal{P})$ if this is not empty, and B' be an arbitrary nonempty subset of B otherwise. Then there is a vertex $b \in B'$ and a path system $\mathcal{Q}$ of c a-B-paths such that $EV(\mathcal{Q}) = EV(\mathcal{P}) \cup \{b\}$, all vertices of $B \setminus \{b\}$ are end-vertices of as many paths of $\mathcal{P}$ as of $\mathcal{Q}$, and $\mathcal{Q}$ has one more path with end-vertex b than does $\mathcal{P}$.*

We define five polyhedral graphs containing the vertices of a prescribed 4-element set X as follows. Let F_1 be the complete graph K_4 on X. Let F_2 denote the graph which is obtained from a 4-cycle C on X by connecting an additional vertex $a \notin X$ with all vertices of C. Let F_3 denote the graph which results from C and two adjacent vertices $a, b \notin X$ by connecting two adjacent vertices of C with a and the remaining two vertices of C with b. The graph F_4 is obtained if two non-adjacent vertices $a, b \notin X$ are connected with three vertices of a 4-path P on X, respectively, such that every vertex of X becomes degree 3. Eventually, let F_5 denote the cube graph containing the vertices of X such that no two vertices of X are adjacent.

Lemma 2. *Every polyhedral graph G with $X = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ has a subgraph H which is a subdivision of some F_i with $1 \leq i \leq 5$.*

Proof of Lemma 2. Lemma 1 implies that G has an x_1-x_2-path system $\{P_1, P_2, P_3\}$ which contains x_3 by planarity of G, i.e., we may assume that $x_3 \in V(P_1)$. Moreover, Lemma 1 yields an x_3-$V(P_2 \cup P_3)$-fan $\mathcal{Q} = \{[x_1, x_3], [x_2, x_3], [a, x_3]\}$, where we may assume that $a \in V(P_2)$. Thus, G has a path system $\mathcal{P} = \{[x_1, x_2], [x_1, x_3], [x_2, x_3], [a, x_1], [a, x_2], [a, x_3]\}$.

Suppose first, that x_4 is contained in $[\mathcal{P}]$. Considering symmetries we have to examine three different cases.

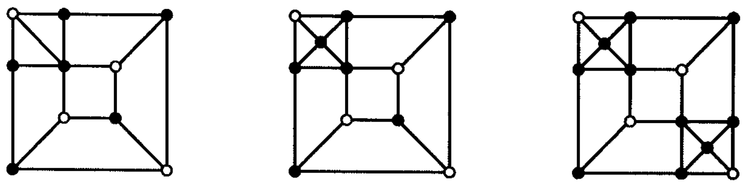
Case 1. $x_4 = a$.
Then $[\mathcal{P}]$ is a subdivision of F_1.

Case 2. $x_4 \in (x_1, x_2)$.
By Lemma 1 there is an x_4-$V([\mathcal{P}]\setminus(x_1,x_2))$-fan $\mathcal{Q} = \{[x_1,x_4],[x_2,x_4],[b,x_4]\}$ where $b \in V([\mathcal{P}]\setminus(x_1,x_2))$. Let H denote the subgraph $[\mathcal{P}\cup\mathcal{Q}]\setminus(x_1,x_2)$ of G, then by symmetries there are following subcases. If $b = x_3$ or $b = a$ then H is a subdivision of F_1 or F_2, respectively. If $b \in (x_1,x_3)$ or $b \in (a,x_1)$ then H is a subdivision of F_4 or F_3, respectively.

Case 3. $x_4 \in (a, x_1)$.
Applying Lemma 1 again there is an x_4-$V([\mathcal{P}]\setminus(a,x_1))$-fan $\mathcal{Q} = \{[x_1,x_4],[a,x_4],[b,x_4]\}$ where $b \in V([\mathcal{P}]\setminus(a,x_1))$. Let H denote the subgraph $[\mathcal{P}\cup\mathcal{Q}]\setminus(a,x_1)$ of G. Considering symmetries we have: If $b \in (x_1,x_2)$ or $b \in [x_2,a)$ then H is a subdivision of F_4 or F_1, respectively.

Suppose now, that x_4 is not contained in $[\mathcal{P}]$ and in any other such path system of G. Applying Lemma 1 we obtain an x_4-$V([\mathcal{P}])$-fan $\mathcal{Q} = \{[b,x_4],[c,x_4],[d,x_4]\}$ such that each path of $\mathcal{P}$ contains at most one vertex of $EV(\mathcal{Q})$ and that at most one path of $\mathcal{P}$ with end vertex a contains a vertex of $EV(\mathcal{Q})$. Thereby and since G is planar we may assume that $b \in (x_1,x_2)$, $c \in (x_2,x_3)$ and $d \in (x_1,x_3)$ which implies that $[\mathcal{P}\cup\mathcal{Q}]$ is a subdivision of F_5. ■

Figure 1 contains further three polyhedral graphs which contain the vertices of $X = \{x_1,x_2,x_3,x_4\}$ and which are needed to prove Theorem 1.



F_6 F_7 F_8

Figure 1

Proof of Theorem **1.** For $l = 6, 7, 11$ and $l \geq 13$ connect a vertex a with each vertex of a 4-cycle $C = x_1x_2x_3x_4x_1$. Put $\alpha = \lfloor \frac{l-5}{3} \rfloor$ and suppose $l \equiv r(\text{mod } 3)$ where $r \in \{0,1,2\}$. Subdivide every edge e of C with respect to r by the number of new vertices given in Table 1. Connect every new vertex with a and denote the so constructed polyhedral graph by G.

Table 1

$r\backslash e$	x_1x_2	x_2x_3	x_3x_4	x_4x_1
0	$\alpha+1$	$\alpha+1$	α	α
1	$\alpha+1$	$\alpha+1$	$\alpha+1$	α
2	α	α	α	α

A simple calculation shows that the length of a shortest cycle in G containing $X = \{x_1, x_2, x_3, x_4\}$ is l and that the order of G is $\lceil \frac{4l-5}{3} \rceil$, in every case.

For $l = 4, 5, 8, 9, 10, 12$ let G be $F_1, F_4, F_5, F_6, F_7, F_8$, respectively, with $X \subseteq V(G)$. In these special cases it is not hard to see that the length of a shortest cycle of G containing X is l. That together with $n_4(l) \le |G|$ completes the proof of the upper bound.

Suppose, now, that G is a polyhedral graph of order n with a 4-element subset $X = \{x_1, x_2, x_3, x_4\}$ of $V(G)$ such that the length of a shortest cycle containing X is at least l. Because of Lemma 2 it is sufficient to estimate for $i = 1, \ldots, 5$ the order of a subgraph H of G which is a subdivision of F_i with $X \subseteq V(F_i)$ and to deduce a lower bound for $n_4(l)$.

$i = 1$: H has three different cycles C_1, C_2, C_3 passing each vertex of F_1. Every vertex of $V(H) \setminus V(F_1)$ occurs in precisely two of these three cycles. Thus, $2|H| + 4 \ge |C_1| + |C_2| + |C_3| \ge 3l$ and, consequently, $|H| \ge \lceil \frac{3l-4}{2} \rceil$.

$i = 2$: H has four cycles $C_1, \ldots, C_4$ containing all vertices of F_2 and one cycle C_5 containing X but no other vertex of F_2. Every vertex of $V(H) \setminus V(F_2) \setminus V(C_5)$ occurs in precisely two and every vertex of $V(C_5) \setminus V(F_2)$ in precisely three of the cycles $C_1, \ldots, C_4$. Thus, $2|H| + |C_5| + 4 \cdot 1 + 2 \ge |C_1| + \ldots + |C_4| \ge 4l$ and, thereby, $2|H| + |C_5| + 6 \ge 4l$. From $|C_5| \le |H| - 1$ we further obtain $|H| \ge \lceil \frac{4l-5}{3} \rceil$.

$i = 3, 4$: H has three different cycles C_1, C_2, C_3 passing each vertex of F_i. Every vertex of $V(H) \setminus V(F_i)$ occurs in precisely two of these three cycles. Thus, $2|H| + 6 \ge |C_1| + |C_2| + |C_3| \ge 3l$ and, consequently, $|H| \ge \lceil \frac{3l-6}{2} \rceil$.

$i = 5$: H has six different cycles $C_1, \ldots, C_6$ passing each vertex of F_5. Every vertex of $V(H) \setminus V(F_5)$ occurs in precisely four of these six cycles. Thus, $4|H| + 2 \cdot 8 \ge |C_1| + \ldots + |C_6| \ge 6l$ and, consequently, $|H| \ge \lceil \frac{3l-8}{2} \rceil$. Because of $|G| \ge \min\{|H_i| : 1 \le i \le 5\}$ and $|G| \ge l$ we obtain

$$n_4(l) \geq \begin{cases} l & \text{if } l \in \{4,5,6,8\}, \\ l+1 & \text{if } l \in \{7,9,10\}, \\ l+2 & \text{if } l \in \{11,12\}, \\ \lceil \frac{4l-5}{3} \rceil & \text{if } l \geq 13\,. \end{cases}$$

In the special cases $l = 5, 6$ one can observe that since G has a subgraph H which is a subdivision of F_i for some $i \in \{1, \ldots, 5\}$ the order of G can not be smaller than 6 or 7, respectively. That proves the lower bound. ■

***Proof of Theorem* 2.** For $l = 5, 6, 7, 8, 9$ let G_l be the polyhedral graphs with $X = \{x_1, \ldots, x_5\} \subseteq V(G_l)$ given in Figure 2.

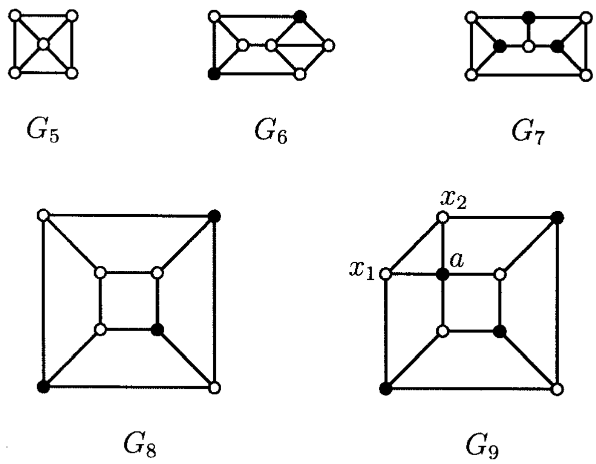


Figure 2

For $l > 9$ let G_l be the polyhedral graph which results from G_9 by subdividing x_1x_2 by $l-9$ new vertices and connecting each of them with $a \notin X$. Notice that $|G_l| = l$ if $l = 5$ or $l \geq 8$ and $|G_l| = l+1$ if $l = 6$ or 7. It is not hard to see that for every $l \geq 5$ the length of any cycle of G_l passing all the vertices of X is at least l.

So, it remains to prove $n_5(l) > l$ for $l = 6, 7$. Let $l = 6$ and suppose that there exists a polyhedral graph G of order 6 with $V(G) = X \cup \{a\}$ such that every cycle which contains the vertices of X is a hamiltonian one. Let $\mathcal{C}(G)$ denote the set of all cycles of G. Then we may suppose that $x_1x_2x_3x_4x_5ax_1 \in \mathcal{C}(G)$. Clearly, $x_1x_5 \notin E(G)$ which implies that x_1x_3

or $x_1x_4 \in E(G)$. If $x_1x_3 \in E(G)$ then $x_2x_5 \notin E(G)$ because otherwise $x_1x_2x_5x_4x_3x_1 \in \mathcal{C}(G)$. Thus, $x_3x_5 \in E(G)$ and also $x_1x_4, x_2x_4 \notin E(G)$ because otherwise $x_1x_2x_3x_5x_4x_1$ or $x_1x_2x_4x_5x_3x_1 \in \mathcal{C}(G)$, respectively. Thereby, x_2 and x_4 are connected with a which yields that $\{x_3, a\}$ is a cutset, a contradiction. So, we have that $x_1x_3 \notin E(G)$ and $x_1x_4 \in E(G)$ which implies that $x_3x_5 \notin E(G)$ because otherwise $x_1x_2x_3x_5x_4x_1 \in \mathcal{C}(G)$. That implies $x_2x_5 \in E(G)$ and thereby $d_G(x_3) = 2$, a contradiction.

Now, let $l = 7$ and suppose that there exists a polyhedral graph G of order 7 with $V(G) = X \cup \{a, b\}$ such that every cycle which contains the vertices of X is a hamiltonian one. We may assume that $\mathcal{C}(G)$ contains one of the cycles $C_1 = x_1x_2x_3x_4x_5abx_1$, $C_2 = x_1x_2x_3x_4ax_5bx_1$, $C_3 = x_1x_2x_3ax_4x_5bx_1$.

Case 1. $C_1 \in \mathcal{C}(G)$.
Clearly, $x_1x_5, x_1a, x_5b \notin E(G)$. If $x_1x_3 \in E(G)$ then x_2x_5, $x_2a \notin E(G)$ because otherwise $x_1x_2x_5x_4x_3x_1$ or $x_1x_2ax_5x_4x_3x_1 \in \mathcal{C}(G)$, respectively. Thus, $x_3x_5 \in E(G)$ which yields $x_1x_4, x_2x_4, x_4b \notin E(G)$ because otherwise $x_1x_2x_3x_5x_4x_1$ or $x_1x_2x_4x_5x_3x_1$ or $x_1x_2x_3x_5x_4bx_1 \in \mathcal{C}(G)$, respectively. That implies $x_2b, x_4a \in E(G)$ which means that $\{x_3, a\}$ or $\{x_3, b\}$ would be a cutset of G, a contradiction. If $x_1x_3 \notin E(G)$ we have $x_1x_4 \in E(G)$ and $x_3x_5, x_3a \notin E(G)$ because otherwise $x_1x_2x_3x_5x_4x_1$ or $x_1x_2x_3ax_5x_4x_1 \in \mathcal{C}(G)$, respectively. That implies $x_2x_5 \in E(G)$ which means by planarity that $x_3b \notin E(G)$. Thus, $d_G(x_3) = 2$, a contradiction.

Case 2. $C_2 \in \mathcal{C}(G)$.
Clearly, $x_1x_5, x_4x_5 \notin E(G)$. Suppose, first, $x_1x_3 \in E(G)$ then $x_2x_5 \notin E(G)$ because otherwise $x_1x_2x_5ax_4x_3x_1 \in \mathcal{C}(G)$. Thereby, $x_3x_5 \in E(G)$ which implies that $x_1x_4, x_2x_4 \notin E(G)$ because otherwise $x_1x_2x_3x_5ax_4x_1$ or $x_1x_2x_4ax_5x_3x_1 \in \mathcal{C}(G)$, respectively. Thus, $x_4b \in E(G)$ which yields by planarity $x_1a, x_2a \notin E(G)$, i.e., $\{x_3, b\}$ would be a cutset of G, a contradiction. Suppose, now, $x_1x_3 \notin E(G)$ and $x_1x_4 \in E(G)$. Then $x_2x_5, x_3x_5 \notin E(G)$ because otherwise $x_1x_4x_3x_2x_5bx_1$ or $x_1x_2x_3x_5ax_4x_1 \in \mathcal{C}(G)$, respectively. That yields $d_G(x_5) = 2$, a contradiction. Suppose $x_1x_3, x_1x_4 \notin E(G)$ then $x_1a \in E(G)$. If, here, $x_2x_5 \in E(G)$ then $x_3x_5 \notin E(G)$ because otherwise $x_1x_2x_5x_3x_4ax_1 \in \mathcal{C}(G)$. By planarity, $x_3b, x_4b \notin E(G)$ which means that $\{x_2, a\}$ would be a cutset of G, a contradiction. If $x_2x_5 \notin E(G)$ then $x_3x_5 \in E(G)$ and, consequently, $x_2x_4 \notin E(G)$ because otherwise $x_1x_2x_4x_3x_5ax_1 \in \mathcal{C}(G)$. Planarity implies $x_4b \notin E(G)$ and, hence, $d_G(x_4) = 2$, a contradiction.

Case 3. $C_3 \in \mathcal{C}(G)$.
Clearly, $x_1x_5, x_3x_4 \notin E(G)$. Suppose, first, $x_1x_3 \in E(G)$ then $x_2x_4, x_2x_5 \notin E(G)$ because otherwise $x_1x_3x_2x_4x_5bx_1$ or $x_1x_3ax_4x_5x_2x_1 \in \mathcal{C}(G)$, respectively. That implies x_1x_4 or $x_4b \in E(G)$. If $x_1x_4 \in E(G)$ then $x_2b \notin E(G)$ because otherwise $x_1x_3x_2bx_5x_4x_1 \in \mathcal{C}(G)$. Thereby, $x_2a \in E(G)$ which implies $x_3x_5, x_3b \notin E(G)$ because otherwise $x_1x_2x_3x_5x_4x_1$ or $x_1x_2x_3bx_5x_4x_1 \in \mathcal{C}(G)$, respectively. That gives $d_G(x_3) = 2$, a contradiction. If $x_1x_4 \notin E(G)$ then $x_4b \in E(G)$ which yields $x_3x_5 \notin E(G)$ because otherwise $x_1x_2x_3x_5x_4bx_1 \in \mathcal{C}(G)$. Thus, $x_5a \in E(G)$ and $\{a, b\}$ would be a cutset of G, a contradiction.

Suppose, now, $x_1x_3 \notin E(G)$ and $x_1x_4 \in E(G)$. Then $x_3x_5, x_3b \notin E(G)$ because otherwise $x_1x_2x_3x_5x_4x_1$ or $x_1x_2x_3bx_5x_4x_1 \in \mathcal{C}(G)$, respectively. That implies $d_G(x_3) = 2$, a contradiction.

Suppose, eventually, $x_1x_3, x_1x_4 \notin E(G)$ then $x_1a \in E(G)$. That implies $x_3x_5 \notin E(G)$ because otherwise $x_1x_2x_3x_5x_4ax_1 \in \mathcal{C}(G)$. Thereby, $x_3b \in E(G)$ and by planarity $x_2x_4, x_2x_5 \notin E(G)$ which means that $\{a, b\}$ would be a cutset of G, a contradiction, and the proof is complete. ■

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