

## ON THE SIMPLEX GRAPH OPERATOR

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### Abstract

A simplex of a graph  $G$  is a subgraph of  $G$  which is a complete graph. The simplex graph  $\text{Simp}(G)$  of  $G$  is the graph whose vertex set is the set of all simplices of  $G$  and in which two vertices are adjacent if and only if they have a non-empty intersection. The simplex graph operator is the operator which to every graph  $G$  assigns its simplex graph  $\text{Simp}(G)$ . The paper studies graphs which are fixed in this operator and gives a partial answer to a problem suggested by E. Prisner.

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In [1], page 131, E. Prisner posed the problem whether there are infinite Simp-periodic graphs other than those consisting of isolated vertices. This paper is a contribution to that problem. We consider undirected graphs without loops and multiple edges.

A simplex in a graph  $G$  is a subgraph of  $G$  which is a complete graph. (It need not be maximal, hence this concept is broader than that of a clique). If a simplex has  $k$  vertices, it is called a  $k$ -simplex. Also a 1-simplex is considered; it consists of one vertex. The simplex graph  $\text{Simp}(G)$  of  $G$  is the graph whose vertex set is the set of all simplices of  $G$  and in which two vertices are adjacent if and only if they have a non-empty intersection (as simplices). The simplex graph operator is the operator which assigns to every graph  $G$  its simplex graph  $\text{Simp}(G)$ . A graph  $G$  is said to be Simp-fixed, if it is a fixpoint of the simplex graph operator, i.e. if  $\text{Simp}(G) \cong G$ .

The graph  $G$  is said to be Simp-periodic, if it is a fixpoint of some iteration of the simplex graph operator.

The mentioned problem from [1] concerns Simp-periodic graphs, but we shall treat only Simp-fixed graphs. Obviously every graph consisting of isolated vertices (regular graph of degree 0) is Simp-fixed and also the empty graph (in which both the vertex set and the edge set are empty) is Simp-fixed. No other finite graph is Simp-fixed. Namely, the set of simplices of  $G$  includes all 1-simplices and their number is equal to the number of vertices of  $G$ . If  $G$  has at least one edge, it has, moreover,  $k$ -simplices for  $k \geq 2$  and thus the vertex set of  $\text{Simp}(G)$  has more elements than the vertex set of  $G$  and  $\text{Simp}(G)$  cannot be isomorphic to  $G$ . Therefore there is a question whether an infinite graph exists which is Simp-fixed and has at least one edge.

The first theorem will have a preparatory character.

**Theorem 1.** *Let  $G$  be a Simp-fixed graph. Then no vertex of  $G$  has a finite degree greater than one.*

**Proof.** Suppose the contrary. Let  $r$  be the least integer greater than one such that  $G$  contains a vertex of degree  $r$ . Let a vertex  $v_0$  have the degree  $r$ . By  $v_1, \dots, v_r$  we denote the vertices adjacent to  $v_0$ . As  $G$  is Simp-fixed, there exist simplices  $S_0, S_1, \dots, S_r$  in  $G$  to which the vertices  $v_0, v_1, \dots, v_r$  correspond; the simplex  $S_0$  has non-empty intersections with all the simplices  $S_1, \dots, S_r$ . Suppose that  $S_0$  is a  $k$ -simplex for  $k \geq 2$ . Then it contains two distinct vertices  $w_1, w_2$ . If both  $w_1, w_2$  have degree 1, then  $S_0$  is a 2-simplex forming a connected component of  $G$ . Then  $G$  must contain a connected component whose image in the operator  $\text{Simp}$  is  $S_0$ ; but a 2-simplex is not a simplex graph for any graph. Therefore at least one of the vertices  $w_1, w_2$ , say  $w_1$ , has degree greater than one. As  $r$  is the minimum of such degrees, the degree of  $w_1$  is at least  $r$ . The vertex  $w_1$  is incident to at least  $r - 1$  edges distinct from  $w_1 w_2$ ; these edges with their end vertices form 2-simplices having a non-empty intersection with  $S_0$ . Further such simplices are 1-simplices consisting of  $w_1$  and consisting of  $w_2$ . There are at least  $r + 1$  simplices having non-empty intersections with  $S_0$  and thus the degree of  $v_0$  is at least  $r + 1$ , which is a contradiction. We have proved that  $S_0$  is a 1-simplex; let it consist of a vertex  $S_0$ . The vertex  $S_0$  cannot have degree 0 or 1, because so would have also  $v_0$ . Therefore the degree of  $S_0$  is at least  $r$ . Each edge incident with  $S_0$  forms a 2-simplex. As the degree of  $v_0$  is  $r$ , no  $k$ -simplices for  $k \geq 3$  containing  $S_0$  exist; the neighbours of  $S_0$  form an independent set and their number, i.e. the degree of  $S_0$ , is exactly  $r$ . Any two of the

mentioned 2-simplices have a common vertex  $S_0$  and thus the neighbours of  $v_0$  form an  $r$ -simplex. We have proved that a vertex of  $G$  with degree  $r$  has the property that its neighbours form an  $r$ -simplex. But then this must hold for  $S_0$ , too, which is a contradiction. This proves the assertion. ■

At considerations concerning infinite cardinal numbers we shall suppose the validity of Axiom of Choice and the existence of well-ordering of cardinal numbers which follows from it. As usual, by  $\aleph_0$  we denote the cardinality of the set of positive integers, by  $\aleph_{\alpha+1}$  for a positive integer  $\alpha$  we denote the cardinal number immediately following after  $\aleph_\alpha$ . By  $\aleph_\omega$  we denote the least cardinal number which is greater than  $\aleph_\alpha$  for every non-negative integer  $\alpha$ . It is well-known  $\aleph_\omega = \sum_{\alpha < \omega} \aleph_\alpha$ .

**Theorem 2.** *Any graph  $G$  which contains at least one edge and whose vertex set has cardinality less than  $\aleph_\omega$  is not Simp-fixed.*

**Proof.** Suppose that there exists a Simp-fixed graph  $G$  having at least one edge. Then  $G$  contains vertices of non-zero degrees. If the maximum degree of a vertex of  $G$  is 1, then  $G$  contains at least one connected component which is a 2-simplex. The existence of such connected component was excluded in the proof of Theorem 1. Therefore  $G$  contains at least one vertex of degree greater than 1. According to Theorem 1 such a degree cannot be finite. Thus  $G$  contains a vertex  $v_0$  of infinite degree  $r$ . If  $r \geq \aleph_\omega$ , then also  $|V(G)| \geq \aleph_\omega$  and the assertion is true. Thus suppose  $r = \aleph_\alpha$  for some non-negative integer  $\alpha$ . The edges incident with  $v_0$  together with their end vertices form 2-simplices. Any two of these 2-simplices have a common vertex  $v_0$  and, as  $G$  is Simp-fixed, vertices of an  $\aleph_\alpha$ -simplex  $S_1$  in  $G$  correspond to them. Choose a vertex  $v_1$  in  $S_1$  and consider all simplices which are subgraphs of  $S$  and contain  $v_1$ . Their number is  $\exp \aleph_\alpha$  and any two of them have a common vertex  $v_1$ ; therefore vertices of an  $(\exp \aleph_\alpha)$ -simplex  $S_2$  in  $G$  correspond to them. We can proceed further, constructing always  $S_{n+1}$  from  $S_n$ . We have  $|V(S_2)| = \exp \aleph_\alpha \geq \aleph_{\alpha+1}$ ,  $|V(S_3)| = \exp \aleph_\alpha \geq \exp \aleph_{\alpha+1} \geq \aleph_{\alpha+2}$  etc., in general  $|V(S_n)| \geq \aleph_{\alpha+n+1}$ . Therefore the vertex set of  $G$  contains subsets of all cardinalities which are less than  $\aleph_\omega$  and hence its cardinality is at least  $\aleph_\omega$ . ■

A further theorem concerns a more general question.

**Theorem 3.** *Let  $G$  be a connected graph such that the cardinalities of  $V(G)$  and of  $V(\text{Simp}(\text{Simp}(G)))$  are equal. Then the cardinality of  $V(G)$  is 0, 1 or a limit cardinal number.*

**Proof.** At the beginning of this paper we have written that for a finite graph  $G$  having at least one edge always  $|V(\text{Simp}(G))| > |V(G)|$ . Thus suppose that  $|V(G)|$  is equal to some isolated infinite cardinal number  $\aleph_{\beta+1}$ , where  $\beta$  is an ordinal number. If  $G$  contains a vertex of degree  $\aleph_{\beta+1}$ , then  $\text{Simp}(G)$  contains an  $\aleph_{\beta+1}$ -simplex and  $\text{Simp}(\text{Simp}(G))$  contains an  $(\exp \aleph_{\beta+1})$ -simplex and thus  $|V(\text{Simp}(\text{Simp}(G)))| \geq \exp \aleph_{\beta+1} > \aleph_{\beta+1}$ . Hence all vertices of  $G$  must have degrees less than  $\aleph_{\beta+1}$ , i.e. less than or equal to  $\aleph_\beta$ . Choose a vertex  $v$  of  $G$  and for each non-negative integer  $k$  by  $N_k(v)$  denote the set of all vertices whose distance from  $v$  in  $G$  is equal to  $k$ . As  $G$  is connected, the union of  $N_k(v)$  for all non-negative integers  $k$  is  $V(G)$ . By induction we prove that  $|N_k(v)| \leq \aleph_\beta$  for each  $k$ . For  $k = 0$  we have  $N_0(v) = \{v\}$  and  $|N_0(v)| = 1 < \aleph_\beta$ . Now suppose that the assertion is true for some  $k$ . Each vertex of  $N_{k+1}(v)$  is adjacent to a vertex of  $N_k(v)$  and the cardinality of  $N_{k+1}(v)$  cannot exceed the cardinality of the set of edges joining vertices of  $N_k(v)$  with vertices of  $N_{k+1}(v)$ . As  $|N_k(v)| \leq \aleph_\beta$  and each vertex of  $N_k(v)$  has degree at most  $\aleph_\beta$ , there are at most  $\aleph_\beta$  such edges and  $|N_{k+1}(v)| \leq \aleph_\beta$ . And then  $V(G)$  is the union of  $\aleph_0$  disjoint sets of cardinalities at most  $\aleph_\beta$ , hence also  $|V(G)| \leq \aleph_\beta < \aleph_{\beta+1}$ , which is a contradiction. This proves the assertion. ■

Note that in this case also the limit cardinal number  $\aleph_0$  may occur. This theorem has importance for Simp-periodic graphs. By  $\text{Simp}^k$  we denote the  $k$ -th iteration of Simp, where  $k$  is a positive integer. From the inequality  $|V(G)| \leq |V(\text{Simp}(G))|$  it is clear that if  $\text{Simp}^k(G) \cong G$ , then  $|V(G)| = |V(\text{Simp}(\text{Simp}(G)))|$ , the number  $k$  being an arbitrary positive integer, and the following corollary holds.

**Corollary.** *Let  $G$  be a connected Simp-periodic graph. Then the cardinality of  $V(G)$  is 0, 1 or a limit cardinal number.* ■

The last theorem will concern locally finite graphs. Remember the well-known fact that a connected infinite locally finite graph has always a countable vertex set.

**Theorem 4.** *Let  $G$  be an infinite locally finite graph. Then so is  $\text{Simp}(G)$ .*

**Proof.** Let  $S$  be a simplex in  $G$ ; as  $G$  is locally finite,  $S$  is finite. Each vertex  $v$  of  $S$  can be contained only in a finite number of simplices of  $G$ , because this number cannot exceed the number of all subsets of the set of neighbours of  $v$ . As also  $S$  is finite, the set of all simplices having non-empty intersections with  $S$  is finite: the vertex of  $\text{Simp}(G)$  corresponding to  $S$  has

a finite degree. As  $S$  was chosen arbitrarily, the graph  $\text{Simp}(G)$  is locally finite. ■

## REFERENCES

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