Problems Column

Discussiones Mathematicae
Graph Theory 24 (2004) 545–549

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Abstract

The Erdős-Faber-Lovász conjecture states that if a graph G is the union of n cliques of size n no two of which share more than one vertex, then $\chi(G) = n$. We provide a formulation of this conjecture in terms of maximal partial clones of partial operations on a set.

Keywords: chromatic number, Erdős-Faber-Lovász conjecture, maximal partial clones.

2000 Mathematics Subject Classification: 05C15 (primary: graph colouring) 08A55 (secondary: Partial algebras).

1. Introduction

Suppose that there are n committees each with n members each, such that each pair of committees has at most one member in common. The committees hold their meetings in the committee room which has n chairs. Is it then possible for every person to select a chair which he/she will use in all the meetings of all the committees to which he/she belongs? The Erdős-Faber-Lovász conjecture states that the answer is yes; in graph theoretic terms, the conjecture can be restated as follows:

> If a graph G is the union of n cliques of size n no two of which share more than one vertex, then $\chi(G) = n$

(see [1, 8, 9]). The n constituent n-cliques of such an instance of the Erdős-Faber-Lovász conjecture can be viewed as the hyperedges in a hypergraph, hence the conjecture also admits a formulation in terms of strong colouring of hypergraphs:

> If an n-uniform hypergraph $\mathcal{H}$ has exactly n hyperedges no two of which share more than one point, then the strong chromatic number of $\mathcal{H}$ is n.

A hypergraph which satisfies the hypotheses of the Erdős-Faber-Lovász conjecture will be called an *instance* of the conjecture. Given such an instance $\mathcal{H}$, we can define the relational structure $(V_{\mathcal{H}}, R_{\mathcal{H}})$ whose base set $V_{\mathcal{H}}$ is the same as that of $\mathcal{H}$, where $R_{\mathcal{H}}$ is the n-ary relational structure consisting of all the n-tuples $(x_1, x_2, \ldots, x_n)$ such that $\{x_1, x_2, \ldots, x_n\}$ is an hyperedge of $\mathcal{H}$. Thus if $\mathcal{H}$ has n hyperedges, then $R_{\mathcal{H}}$ consists of the $n \cdot n!$ n-tuples obtained by linearly ordering each hyperedge of $\mathcal{H}$. The reason for considering $(V_{\mathcal{H}}, R_{\mathcal{H}})$ is that it allows to define products, homorphisms and partial homorphisms, and reinterpret the Erdős-Faber-Lovász conjecture from the point of view of partial clone theory. We prove the following.

Theorem 1. *The Erdős-Faber-Lovász conjecture is true if and only if for every instance $\mathcal{H}$ of the conjecture, the relational structure $(V_{\mathcal{H}}, R_{\mathcal{H}})$ determines a maximal partial clone.*

In the next section we introduce the necessary definitions and results needed to prove Theorem 1.

2. Partial Operations and Partial Clones

An *n-ary relational structure* is a couple (V, R) where V is a set and $R \subseteq V^n$. It is called *areflexive* if for every $(x_1, \ldots, x_n) \in R$, the elements $x_1, \ldots, x_n$ are all distinct, and *totally symmetric* if for every $(x_1, \ldots, x_n) \in R$ and every permutation π of $\{1, \ldots, n\}$, we have $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in R$. A *homomorphism* between two n-ary relational structures (V, R) and (V', R') is a map ϕ from V to V' such that $(\phi(x_1), \ldots, \phi(x_n)) \in R'$ for all $(x_1, \ldots, x_n) \in R$. A map ϕ from a subset dom_ϕ of V to V' is called a *partial homomorphism* if $(\phi(x_1), \ldots, \phi(x_n)) \in R'$ for all $x_1, \ldots, x_n \in \mathrm{dom}_\phi$ such that $(x_1, \ldots, x_n) \in R$.

For example, instances $\mathcal{H}$, $\mathcal{H}'$ of the Erdős-Faber-Lovász conjecture give rise to the areflexive, totally symmetric n-ary relational structures

$(V_{\mathcal{H}}, R_{\mathcal{H}})$, $(V_{\mathcal{H}'}, R_{\mathcal{H}'})$, and also to totally symmetric binary relational structures, namely the corresponding graphs G and G'. Since the elements of $R_{\mathcal{H}}$ and $R_{\mathcal{H}'}$ correspond to cliques in G and G' respectively, a homomorphism from $(V_{\mathcal{H}}, R_{\mathcal{H}})$ to $(V_{\mathcal{H}'}, R_{\mathcal{H}'})$ naturally induces a homomorphism form G to G'. However the converse does not necessarily holds. For instance G may be n-colourable (as the conjecture claims) and G' may contain a n-clique C which does not correspond to any hyperedge of $\mathcal{H}'$. Identifying the elements of C with the colours in a n-colouring of G defines a homomorphism form G to G' which does not correspond to any homomorphism from $(V_{\mathcal{H}}, R_{\mathcal{H}})$ to $(V_{\mathcal{H}'}, R_{\mathcal{H}'})$. Now the partial homomorphisms from $(V_{\mathcal{H}}, R_{\mathcal{H}})$ to $(V_{\mathcal{H}'}, R_{\mathcal{H}'})$ need not even induce partial homomorphisms from G to G'. For instance every hyperedge of $\mathcal{H}$ contains a vertex which does not belong to any other hyperedge. Removing such a vertex from every hyperedge yields a subset X of $V_{\mathcal{H}}$ which does not contain any hyperedge. Thus *any* map from X to $V_{\mathcal{H}'}$ is a partial homomorphism from $(V_{\mathcal{H}}, R_{\mathcal{H}})$ to $(V_{\mathcal{H}'}, R_{\mathcal{H}'})$, while the partial homomorphisms from G to G' with domain X need to preserve the edges of the subgraph induced by X. Thus from the point of view of partial homomorphisms, the n-ary relational structures induced by instances of the Erdős-Faber-Lovász conjecture do not behave like the corresponding graphs.

Given an integer m, the *m-th power* $(V, R)^m$ of a n-ary relational structure (V, R) is the n-ary relational structure (V^m, R'), where

$$R' = \{(X_1, \ldots, X_n) \in (V^m)^n : (\mathrm{pr}_i(X_1), \ldots, \mathrm{pr}_i(X_n)) \in R, i = 1, \ldots, m\}$$

(where pr_i is the projection given by $pr_i(a_1, \ldots, a_m) = a_i$). A partial function from a power of V to V is called a *partial operation* on V. Let $pPol(V, R)$ denote the set of partial operations on V which are partial homomorphisms from some power of (V, R) to (V, R). Note that $pPol(V, R)$ contains all the projections, and is closed under the following composition:

If $\phi_1, \dots, \phi_k$ are partial homomorphisms from $(V,R)^m$ to (V,R) and ψ a partial homomorphism from $(V,R)^k$ to (V,R), then $\psi(\phi_1, \dots, \phi_k)$ is the partial homomorphism ψ' given by

$$\psi'(X_1, \dots, X_m) = \psi(\phi_1(X_1, \dots, X_m), \dots, \phi_k(X_1, \dots, X_m)),$$

on the domain consisting of the m-tuples $(X_1, \dots, X_m)$ for which the above expression is well defined. A set of partial operations which contains all projections and is closed under composition is called a *partial clone.* The family of partial clones on a set V, ordered by inclusion, forms a dually atomic lattice. A partial clone is called *maximal* if it is not properly contained in any other partial clone, apart from the set of all partial operations on V. In other words, a partial clone $\mathcal{C}$ on a set V is maximal if for any two partial operations f, g on V, neither one in $\mathcal{C}$, g can be obtained by composition using only f and elements of $\mathcal{C}$. The characterization of maximal partial clones on a finite set resembles that of maximal clones and generating sets of boolean operations, though the list is far more complex in the partial case. (see [2, 3, 4, 5, 6, 7]). In particular, the following is known:

Theorem 2 ([5])**.** *Let (V,R) be an n-ary areflexive, totally symmetric relational structure. Then pPol(V,R) is a maximal partial clone if and only if (V,R) admits a strong n-colouring.*

Here, a *strong n-colouring* of (V,R) is just a strong n-colouring of the hypergraph whose hyperedges are the sets $\{x_1, \dots, x_n\}$ such that $(x_1, \dots, x_n) \in R$. Thus we have the following.

Corollary 3. *Let $\mathcal{H}$ be an instance of the Erdős-Faber-Lovász conjecture. Then $\mathcal{H}$ admits a strong n-colouring if and only if pPol$(V_{\mathcal{H}}, R_{\mathcal{H}})$ is a maximal partial clone.*

Theorem 1 follows directly from this corollary. Note that although this partial clone theoretic formulation uses the context of the Erdős-Faber-Lovász conjecture, the hypothesis that the hyperedges be almost disjoint is not necessary for the correspondence. It would be interesting to see whether this hypothesis can be incorporated to a partial clone-theoretic treatment of the subject.

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Received 12 July 2004