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COMBINATORIAL LEMMAS FOR POLYHEDRONS

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Abstract

We formulate general boundary conditions for a labelling to assure the existence of a balanced n-simplex in a triangulated polyhedron. Furthermore we prove a Knaster-Kuratowski-Mazurkiewicz type theorem for polyhedrons and generalize some theorems of Ichiishi and Idzik. We also formulate a necessary condition for a continuous function defined on a polyhedron to be an onto function.

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1. Preliminaries

By N and R we denote the set of natural numbers and reals, respectively. Let $n \in N$ and V be a finite set of cardinality at least $n+1$. $\mathbf{P}(V)$ is the family of all subsets of V and $\mathbf{P}_n(V)$ is the family of all subsets of V of cardinality $n+1$. For $A \subset R^n$ $\operatorname{co} A$ is the *convex hull* of A and $\operatorname{af} A$ is the *affine hull* of A (the minimal affine subspace containing A). Let $\operatorname{ri} Z$ and $\operatorname{bd} Z$ be the *relative interior* and the *boundary* of a set $Z \subset R^n$, respectively. The relative interior of the set Z is considered with respect to the affine hull of Z. *Dimension* of a set $A \subset R^n$ is the dimension of $\operatorname{af} A$. If for some $A \subset R^n$ the dimension of $\operatorname{af} A$ is $n-1$, then $\operatorname{af} A$ is called a *hyperplane*. And if for a finite set $A = \{a_0, \cdots, a_m\} \subset R^n$ $(m \in \{0, \cdots, n\})$ the dimension of $\operatorname{af} A$ is equal to m, then $\operatorname{co} A$ is called a *simplex* (precisely an m-*simplex*).

2. Polyhedrons

By a polyhedron we understand the convex hull of a finite set of R^n. Let $P \subset R^n$ be a polyhedron of dimension n. A *face* of the polyhedron P is the intersection of P with some of its supporting hyperplane. Denote the set of all k-dimensional faces of the polyhedron P by $\mathbf{F}_k(P)$ $(k \leq n)$ and the set of all vertices of the polyhedron P by $V(P)$ $(V(P) = \mathbf{F}_0(P))$. The maximal dimension proper faces of the polyhedron P are called *facets*. Let Tr_n be a family of n-simplexes such that $P = \bigcup_{\delta \in Tr_n} \delta$ and for any $\delta_1, \delta_2 \in Tr_n$, $\delta_1 \cap \delta_2$ is the empty set or their common face. A *triangulation* of the polyhedron P (we denote it by Tr) is a family of simplexes containing Tr_n and fulfilling the following condition: any face of any simplex of Tr also belongs to Tr. Let Tr_m $(m \in \{0, \cdots, n\})$ denote the family of m-simplexes belonging to a triangulation Tr. Hence $Tr = \bigcup_{i=0}^n Tr_i$. Let $V = Tr_0$ be the set of vertices of all simplexes of Tr. Notice, that $Tr_0 = \bigcup_{\delta \in Tr_n} V(\delta)$. An $(n-1)$-simplex of Tr_{n-1} is a *boundary* $(n-1)$-*simplex* if it is a facet of exactly one n-simplex of Tr_n.

Let U be a finite set. An n-*primoid* $\mathbf{L}_n^U$ over U is a nonempty family of subsets of U of cardinality $n+1$ fulfilling the following condition: for every set $T \in \mathbf{L}_n^U$ and for any $u \in U$ there exists exactly one $u' \in T$ such that a set $T \setminus \{u'\} \cup \{u\} \in \mathbf{L}_n^U$.

Each function $l : V \to U$ is called a *labelling*. An n-simplex $\delta \in Tr_n$ is *completely labelled* if $l(V(\delta)) \in \mathbf{L}_n^U$ and an $(n-1)$-simplex $\delta \in Tr_{n-1}$ is x-*labelled* $(x \in U)$ if $l(V(\delta)) \cup \{x\} \in \mathbf{L}_n^U$.

The following theorem is a special case of the theorem of Idzik and Junosza-Szaniawski formulated for geometric complexes. This theorem generalizes the well known Sperner lemma [9].

Theorem 2.1 (Theorem 6.1 in [3]). *Let Tr be a triangulation of an n-dimensional polyhedron $P \subset R^n$, $V = Tr_0$, $\mathbf{L}_n^U$ be an n-primoid over a set U and $x \in U$ be a fixed element. Let $l : V \to U$ be a labelling. Then the number of completely labelled n-simplexes in Tr is congruent to the number of boundary x-labelled $(n-1)$-simplexes in Tr modulo* 2.

Let $U \subset R^n$ be a finite set containing $V(P)$ and let $b \in \operatorname{ri} P$ be a point, which is not a convex combination of fewer than $n+1$ points of the set U. The family $\mathbf{L}_n^b = \{T \subset U : |T| = n+1,\ b \in \operatorname{co} T\}$ is a primoid over the set U (see Example 3.6 in [3]). We say a *b-balanced n-simplex* instead of a completely labelled n-simplex if $\mathbf{L}_n^U = \mathbf{L}_n^b$. In the case $b = 0$ a b-balanced n-simplex is called a *balanced n-simplex.*

3. Main Theorem

Theorem 3.1. *Let $P \subset R^n$ be a polyhedron of dimension n, Tr be a triangulation of the polyhedron P, $V = Tr_0$. Let $U \subset R^n$ be a finite set containing $V(P)$, let $b \in \operatorname{ri} P$ be a point which is not a convex combination of fewer than $n+1$ points of U and let $l : V \to U$ be a labelling. If for every facet F_{n-1} of the polyhedron P we have $l(V \cap F_{n-1}) \subset F_{n-1}$, then the number of b-balanced n-simplexes in the triangulation Tr is odd.*

Remark 3.2. Notice that the condition $l(V \cap F_{n-1}) \subset F_{n-1}$ implies that for each lower dimensional face F we have $l(V \cap F) \subset F$, because: $l(V \cap F) \subset \bigcap_{F \subset F_{n-1} \in \mathbf{F}_{n-1}(P)} F_{n-1} = F$.

***Proof of Theorem* 3.1.** We apply the induction with respect to dimension of the polyhedron P. If dimension of P is equal to 1, then the theorem is obvious. Assume that the theorem is true for all polyhedrons of dimension k ($k \in N$). Consider a polyhedron P of dimension $k+1$. Choose a vertex of P and denote it by x. Let b' be a point different from x, lying on the boundary of P and on the straight line passing through points b and x. Let $F_{b'}$ be a face of P containing b'. Observe that dimension of $F_{b'}$ is equal to k, because otherwise the point b would be a convex combination of fewer than $(k+1)$+1 points of $V(P)$.

Let us count x-labeled k-simplexes on $\operatorname{bd} P$. For any facet F different from $F_{b'}$ there is no x-labeled k-simplex contained in F since for all $\delta \in Tr^k \cap F$ $\operatorname{co} l(V(\delta)) \subset F$ and $b \notin \operatorname{co}(\{x\} \cup V(F))$. Hence all x-labeled k-simplexes are contained in $F_{b'}$. Notice that a k-simplex $\delta \in Tr^k \cap F_{b'}$ is the x-labelled k-simplex if and only if δ is a b'-balanced k-simplex. Because of Remark 3.2 we may apply the induction assumption for $F_{b'}$ ($F_{b'}$ is considered as a subset of $\operatorname{af} F_{b'}$) and the point b'. Therefore the number of b'-balanced k-simplexes on $F_{b'}$ is odd. Thus the number of boundary x-labeled k-simplexes in Tr is odd and by Theorem the number of the b-balanced $(k+1)$-simplexes in Tr is odd. ■

Observe that for any polyhedron Q, triangulation Tr' of $\operatorname{bd} Q$ and a point $c \in \operatorname{ri} Q$ the family $Tr = \{\operatorname{co}(\{c\} \cup V(\delta)) : \delta \in Tr'\} \cup Tr' \cup \{c\}$ is a triangulation of the polyhedron Q.

For any $(n-1)$-dimensional hyperplane h_b^F containing the point b and disjoint with a facet F of the polyhedron P let H_b^F denote the open halfspace containing F and such that h_b^F is its boundary.

Theorem 3.3. *Let $P \subset R^n$ be a polyhedron of dimension n, Tr be a triangulation of the polyhedron P, $V = Tr_0$. Let $U \subset R^n$ be a finite set containing $V(P)$, let $b \in \operatorname{ri} P$ be a point which is not a convex combination of fewer than $n+1$ points of U and let $l : V \to U$ be a labelling. If for every facet F_{n-1} of the polyhedron P there exists an $(n-1)$-dimensional hyperplane $h_b^{F_{n-1}}$ containing the point b and disjoint with F_{n-1} such that $l(V \cap F_{n-1}) \subset H_b^{F_{n-1}}$, then the number of b-balanced n-simplexes in the triangulation Tr is odd.*

Proof. For $n = 1$ the theorem is obvious, so we consider $n > 1$. Let $V(P) = \{a_0, \cdots, a_k\}$ $(k \geq n)$. Let $a'_i = 2a_i - b$ for $i \in \{0, \cdots, k\}$ and let $P' = \operatorname{co}\{a'_0, \cdots, a'_k\}$. Notice that $P \subset P'$.

Now we define a triangulation of P', which is an extension of the triangulation Tr on P. We will define a triangulation of $P' \setminus \operatorname{ri} P$.

For every face $F = \operatorname{co}\{a_{i(0)}, \cdots, a_{i(l)}\}$ $(\{a_{i(0)}, \cdots, a_{i(l)}\} \subset V(P))$ of the polyhedron P we denote $F' = \operatorname{co}\{a'_{i(0)}, \cdots, a'_{i(l)}\}$. Every face F of P has one-to-one correspondence to the face F' of P'.

Let us denote $FF' = \operatorname{co}\{F \cup F'\}$. Thus $P' \setminus \operatorname{ri} P = \bigcup_{F \in \mathbf{F}_{n-1}(P)} FF'$.

For $n = 1$ the triangulation of P' is trivial, so we may assume $n > 1$.

For any face $F_1 \in \mathbf{F}_1(P)$ we choose a point $v_{F'_1} \in \operatorname{ri} F'_1$ in such a way that the point b is not a convex hull of less than $n+1$ points of $U \cup \{v_{F'_1} :$

$F_1 \in \mathbf{F}_1(P)\}$. We join $v_{F_1'}$ with every vertex of the face F_1'. Thus we receive triangulation of F_1'. We choose a point $v_{F_1F_1'} \in \operatorname{ri} F_1F_1'$ in such a way that the point b is not a convex hull of less than $n+1$ points of $U \cup \{v_{F_1'}, v_{F_1F_1'} : F_1 \in \mathbf{F}_1(P)\}$. We join $v_{F_1F_1'}$ with every vertex of the face F_1', with the point $v_{F_1'}$ and with every vertex of $V \cap F_1$. Thus we receive triangulation of F_1F_1'.

Now we apply the induction for $k \in \{2, \cdots, n-1\}$: For any face $F_k \in \mathbf{F}_k(P)$ we choose a point $v_{F_k'} \in \operatorname{ri} F_k'$ in such a way that the point b is not a convex hull of less than $n+1$ points of $U \cup \bigcup_{i=1}^{k}\{v_{F'} : F \in \mathbf{F}_i(P)\} \cup \bigcup_{i=1}^{k-1}\{v_{FF'} : F \in \mathbf{F}_i(P)\}$. We join $v_{F_k'}$ with every vertex of F_k' and every point of the set $\bigcup_{F' \subset F_k'}\{v_{F'}\}$. Thus we get a triangulation of the face F_k'. We choose a point $v_{F_kF_k'} \in \operatorname{ri} F_kF_k'$ in such a way that the point b is not a convex hull of less than $n+1$ points of $U \cup \bigcup_{i=1}^{k}\{v_{F'}, v_{FF'} : F \in \mathbf{F}_i(P)\}$. For each $F_k \in \mathbf{F}_k(P)$ we join the vertex $v_{F_kF_k'}$ with the vertex $v_{F'}$, with all the vertices of $V \cap F_k$, vertices of F_k' and with the vertices of the set $\bigcup_{F \subset F_k}\{v_{F'}, v_{FF'}\}$.

We get the triangulation of $P' \setminus \operatorname{ri} P$ and we denote it by Tr''. Hence $Tr' = Tr \cup Tr''$ is a triangulation of P', which is an extension of the triangulation Tr on P.

Let $U' = U \cup \bigcup_{i=1}^{n-1}\{v_{F'}, v_{FF'} : F \in \mathbf{F}_i(P)\}$. Let $V' = Tr_0'$. We define a labelling $l' : V' \to U'$. Let $l'(v) = l(v)$ for $v \in V$ and $l(v) = v$ for $v \in V' \setminus V$. Notice that the labelling l' satisfies conditions of Theorem 3.1. Thus there exists an odd number of b-balanced n-simplexes in Tr'. All b-balanced n-simplexes belong to Tr since for any facet F of P we have $l'(V' \cap FF') \subset H_b^F$, where H_b^F is an open halfspace such that the point b is on its boundary. ■

In the proof of Theorems 3.1, 3.3 the condition: $b \in \operatorname{ri} P$ is a point which is not a convex combination of fewer than $n+1$ elements of $l(V)$ is essential. If we omit this condition we may still prove that there exists at least one b-balanced n-simplex (not necessarily an odd number of such n-simplexes). Related results were obtained by van der Laan, Talman and Yang [6, 7].

Theorem 3.4. *Let $P \subset R^n$ be a polyhedron of dimension n, Tr be a triangulation of the polyhedron P, $V = Tr_0$. Let $U \subset R^n$ be a finite set, let $b \in \operatorname{ri} P$ and let $l : V \to U$ be a labelling. If for every facet F of the polyhedron P there exists an $(n-1)$-dimensional hyperplane h_b^F containing the point b and disjoint with F such that $l(V \cap F) \subset H_b^F$, then there exists a b-balanced n-simplex in the triangulation Tr.*

Proof. Take a sequence of points b_k, which converges to the point b and b_k is not a convex combination of fewer that $n+1$ elements of $l(V)$ for any $k \in N$. For sufficiently large k we may assume that $H_b^F \cap l(V \cap F) = H_{b_k}^F \cap l(V \cap F)$ for some $(n-1)$-dimensional hyperplane $h_{b_k}^F$ and every facet F of P and apply Theorem 3.3 to b_k. Thus there exists a b_k-balanced n-simplex in Tr_n. Since the points b_k converge to the point b and the set U is finite, then there exists at least one b-balanced n-simplex in Tr_n. ∎

Theorem 3.4 applied to the n-dimensional cube implies the Poincaré-Miranda theorem [5].

Theorem 3.5. *Let P be an n-dimensional polyhedron, $b \in \operatorname{ri} P$ and $U \subset R^n$ be a finite set containing $V(P)$. Let $\{C_u \subset R^n : u \in U\}$ be a family of closed sets such that $P \subset \bigcup_{u\in U} C_u$ and for every facet F_{n-1} of the polyhedron P there exists a hyperplane $h_b^{F_{n-1}}$ containing b and disjoint with F_{n-1} such that for every face F of P we have $F \subset \bigcup_{u\in U\cap H_b^F} C_u$, where $H_b^F = \bigcap_{F\subset F_{n-1}\in \mathbf{F}_{n-1}} H_b^{F_{n-1}}$. Then there exists $T \subset U$, $|T| = n+1$, such that $b \in \operatorname{co} T$ and $\bigcap_{u\in T} C_u \neq \emptyset$.*

Proof. Let Tr^k $(k \in N)$ be a sequence of triangulations of P with the diameter of simplexes tending to zero, when k tends to infinity. Denote $V_k = Tr_0^k$. We define a labelling l_k on the vertices V_k $(k \in N)$ in the following way: for $v \in V_k$ let $l_k(v) = u$ for some u such, that $v \in C_u$ and furthermore if $v \in \operatorname{bd} P$, then $u \in \bigcap_{F_{n-1}\ni v, F_{n-1}\in \mathbf{F}_{n-1}(P)} H_b^{F_{n-1}}$.

Since $P \subset \bigcup_{u\in U} C_u$ and $F \subset \bigcup_{u\in H_b^F} C_u$, then the labelling l_k is well defined and it satisfies the conditions of Theorem 3.4. Thus there exists a b-balanced n-simplex $\delta_k \in Tr^k$. Let $V(\delta_k) = \{v_0^k, \cdots, v_n^k\}$. Hence for $i \in \{0, \cdots, n\}$ $v_i^k \in C_{l_k(v_i^k)}$. Because the diameter of simplexes of Tr^k tends to zero, there exists $z \in P$ and a subsequence of v_i^k which converges to z for each $i \in N$. Since C_u is a closed set for $u \in U$ and U is a finite set, then $z \in C_{t_i}$ for $i \in \{0, \cdots, n\}$ and $T = \{t_0, \cdots, t_n\}$, $|T| = n+1$, $b \in \operatorname{co} T$ and thus $\bigcap_{u\in T} C_u \neq \emptyset$. ∎

Theorem 3.5 is a generalization of an earlier result of Ichiishi and Idzik:

Theorem 3.6 (Theorem 3.1 in [1])**.** *Let P be an n-dimensional polyhedron, $b \in \operatorname{ri} P$ and $U \subset R^n$ be a finite set containing $V(P)$. Let $\{C_u \subset R^n : u \in U\}$ be a family of closed sets such that $P \subset \bigcup_{u\in U} C_u$ and $F \subset \bigcup_{u\in U\cap \operatorname{af} F} C_u$ for every face F of the polyhedron P. Then there exists $T \subset U$, $|T| = n+1$, such that $b \in \operatorname{co} T$ and $\bigcap_{u\in T} C_u \neq \emptyset$.*

Notice that the theorem of Ichiishi and Idzik is more general than the Knaster-Kuratowski-Mazurkiewicz covering lemma [4] and Shapley's covering lemma (Theorem 7.3 in [8]).

The theorem below is related to Corollary 4.2 in [2].

Theorem 3.7. *Let $P \subset R^n$ be an n-dimensional polyhedron and $f : P \to R^n$ be a continuous function. If for every facet F of the polyhedron P the set $f(F)$ is in the closed halfspace H^F, such that $\operatorname{bd} H^F = \operatorname{af} F$ and P is not contained in H^F, then $P \subset f(P)$.*

Proof. Let $b \in \operatorname{ri} P$ be a fixed point. Let Tr^k be a triangulation of the polyhedron P with the diameter of simplexes tending to zero and with a set of vertices denoted by V_k ($k \in N$). We define a labelling $l_k : V_k \to R^n$ by putting $l_k(v) = f(v)$ ($v \in V_k$, $k \in N$). Notice that the labelling l_k satisfies the conditions of Theorem 3.4 and there exists a b-balanced n-simplex in Tr^k. Denote this n-simplex by δ_k. Without loss of generality we may assume that there exists $x \in P$ such that $x = \lim_{k\to\infty} x_k$ for every $x_k \in \delta_k$. Because f is a continuous function and $b \in \operatorname{co} f(V(\delta_k))$ we have $f(x) = b$.

We have proved that $\operatorname{ri} P \subset f(P)$. Since the set $f(P)$ is closed, we have $P \subset f(P)$. ■

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