

## EXACT DOUBLE DOMINATION IN GRAPHS

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### Abstract

In a graph a vertex is said to dominate itself and all its neighbours. A doubly dominating set of a graph  $G$  is a subset of vertices that dominates every vertex of  $G$  at least twice. A doubly dominating set is exact if every vertex of  $G$  is dominated exactly twice. We prove that the existence of an exact doubly dominating set is an NP-complete problem. We show that if an exact double dominating set exists then all such sets have the same size, and we establish bounds on this size. We give a constructive characterization of those trees that admit a doubly dominating set, and we establish a necessary and sufficient condition for the existence of an exact doubly dominating set in a connected cubic graph.

**Keywords:** double domination, exact double domination.

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## 1. Introduction

In a graph  $G = (V, E)$ , a subset  $S \subseteq V$  is a *dominating set* of  $G$  if every vertex  $v$  of  $V - S$  has a neighbour in  $S$ . The *domination number*  $\gamma(G)$  is the minimum cardinality of a dominating set of  $G$ . For a comprehensive treatment of domination in graphs and its variations, see [8, 9].

Harary and Haynes [7] defined and studied the concept of double domination, which generalizes domination in graphs. In a graph  $G = (V, E)$ , a subset  $S$  of  $V$  is a *doubly dominating set* of  $G$  if, for every vertex  $v \in V$ , either  $v$  is in  $S$  and has at least one neighbour in  $S$  or  $v$  is in  $V - S$  and has at least two neighbours in  $S$ . The *double domination number*  $\gamma_{\times 2}(G)$  is the minimum cardinality of a doubly dominating set of  $G$ . Double domination was also studied in [2, 3, 4]. Analogously to exact (or perfect) domination introduced by Bange, Barkauskas and Slater [1], Harary and Haynes [7] defined an *efficient doubly dominating set* as a subset  $S$  of  $V$  such that each vertex of  $V$  is dominated by exactly two vertices of  $S$ . We will prefer here to use the phrase *exact doubly dominating set*.

Every graph  $G = (V, E)$  with no isolated vertex has a doubly dominating set; for example  $V$  is such a set. In contrast, not all graphs with no isolated vertex admit an exact doubly dominating set; for example, the star  $K_{1,p}$  ( $p \geq 2$ ) does not. In Section 2 we prove that the existence of an exact doubly dominating set is an NP-complete problem. We then show in Section 3 that if a graph  $G$  admits an exact doubly dominating set then all such sets have the same size, and we give some bounds on this number. Finally, we give in Section 4 a constructive characterization of those trees that admit an exact doubly dominating set, and we establish a necessary and sufficient condition for the existence of an exact doubly dominating set in a connected cubic graph.

Let us give some definitions and notation. In a graph  $G = (V, E)$ , the *open neighbourhood* of a vertex  $v \in V$  is the set  $N(v) = \{u \in V \mid uv \in E\}$ , the *closed neighbourhood* is the set  $N[v] = N(v) \cup \{v\}$ , and the *degree* of  $v$  is the size of its open neighbourhood, denoted by  $\deg_G(v)$ . We denote respectively by  $n$ ,  $\delta$  and  $\Delta$  the *order* (number of vertices), *minimum degree* and *maximum degree* of a graph  $G$ .

## 2. NP-Completeness

In this section we consider the complexity of the problem of deciding whether

a graph admits an exact doubly dominating set.

EXACT DOUBLY DOMINATING SET (X2D)

Instance: A graph  $G$ ;

Question: Does  $G$  admit an exact doubly dominating set?

We show that this problem is NP-complete by reducing the following EXACT 3-COVER (X3C) problem to our problem.

EXACT 3-COVER (X3C)

Instance: A finite set  $X$  with  $|X| = 3q$  and a collection  $C$  of 3-element subsets of  $X$ ;

Question: Is there a subcollection  $C'$  of  $C$  such that every element of  $X$  appears in exactly one element of  $C'$ ?

EXACT 3-COVER is a well-known NP-complete problem [6].

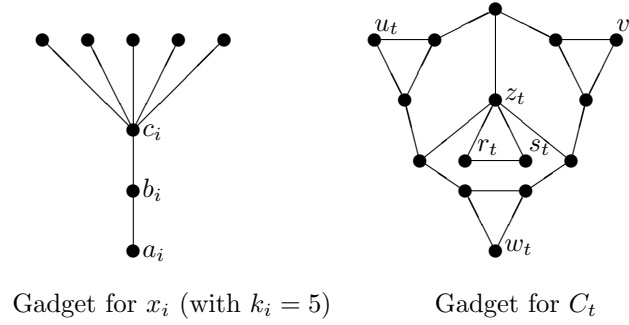
**Theorem 1.** *EXACT DOUBLY DOMINATING SET is NP-complete.*

**Proof.** Clearly, X2D is in NP. Let us now show how to transform any instance  $X, C$  of X3C into an instance  $G$  of X2D so that one of them has a solution if and only if the other has a solution.

For each  $x_i \in X$ , we build a “gadget” graph with vertices  $a_i, b_i, c_i$  and  $d_i^1, \dots, d_i^{k_i}$ , where  $k_i$  is the number of elements of  $C$  that contain  $x_i$ , and with edges  $a_i b_i, b_i c_i$  and  $c_i d_i^j$  ( $j = 1, \dots, k_i$ ). We view the  $d_i^j$ 's as points of this gadget, each of them being associated with an element of  $C$  that contains  $x_i$ . See Figure 1.

For each  $C_t \in C$ , we build a gadget graph with 15 vertices  $y_t^0, \dots, y_t^8, z_t, r_t, s_t, u_t, v_t, w_t$  and edges  $y_t^j y_t^{j+1}$  ( $j = 0, \dots, 8 \bmod 9$ ) (so that the  $y_t^j$ 's induce a  $C_9$ ) and  $z_t y_t^0, z_t y_t^3, z_t y_t^6, z_t r_t, z_t s_t, r_t s_t$  (so  $z_t, r_t, s_t$  induce a triangle), and  $u_t y_t^1, u_t y_t^2, v_t y_t^4, v_t y_t^5, w_t y_t^7, w_t y_t^8$ . We view  $u_t, v_t, w_t$  as the three points of this gadget, each of them being associated with an element of  $C_t$ . See Figure 1.

Now, for each  $C_t$ , if  $C_t = \{x_i, x_j, x_k\}$  say, we identify the first, second and third point of the gadget of  $C_t$  with the corresponding point in the gadget of  $x_i, x_j, x_k$  respectively. We call  $G$  the resulting graph. Clearly the size of  $G$  is polynomial in the size of  $X$  and  $C$ .

Figure 1: Gadgets for an element  $x_i$  and a triple  $C_t$ .

1. Suppose that the instance  $X, C$  of X3C has a solution  $C'$ . We build a set  $S$  of vertices of  $G$  as follows: for each  $C_t \in C'$ , we put in  $S$  the vertices  $u_t, y_t^1, v_t, y_t^4, w_t, y_t^7, z_t, r_t$ ; for each  $C_t \in C - C'$ , we put in  $S$  the vertices  $y_t^1, y_t^2, y_t^4, y_t^5, y_t^7, y_t^8, r_t, s_t$ ; for each  $x_i \in X$ , we put in  $S$  the vertices  $a_i, b_i$  (note that exactly one of the  $d_i^j$ 's has been put in  $S$ ). It is a routine matter to check that  $S$  is an exact doubly dominating set in  $G$ .

2. Conversely, suppose that  $G$  has an exact doubly dominating set  $S$ . Note the gadget of a given  $C_t$  is in exactly one of the following two possible states:

- (a)  $z_t \in S$ , and so exactly one of  $r_t, s_t$  is in  $S$ , none of  $y_t^0, y_t^3, y_t^6$  is in  $S$ , the other six  $y_t^j$ 's are in  $S$ , and none of  $u_t, v_t, w_t$  is in  $S$ ; or
- (b)  $z_t \notin S$ , both  $r_t, s_t$  are in  $S$ , none of  $y_t^0, y_t^3, y_t^6$  is in  $S$ , exactly one of  $\{y_t^1, y_t^4, y_t^7\}$ ,  $\{y_t^2, y_t^5, y_t^8\}$  is in  $S$  and the other is in  $V - S$ , and each of  $u_t, v_t, w_t$  is in  $S$ .

Clearly, for each  $x_i \in X$ , we have  $a_i, b_i \in S$  (else  $a_i$  would not be doubly dominated), then  $c_i \notin S$  (else  $b_i$  would be dominated three times), and it follows that exactly one of the  $d_i^j$ 's is in  $S$ . For each  $i = 1, \dots, 3q$ , let  $t(i)$  be the integer such that this special  $d_i^j$  is equal to one point of  $C_{t(i)} \in C$ , and let us say that  $C_{t(i)}$  is selected by  $x_i$ . Thus the gadget of  $C_{t(i)}$  is in state (b), which means that  $C_{t(i)}$  is selected by each of its 3 elements. Therefore, the collection  $C'$  of all selected elements of  $C$  (i.e., those whose three points are in  $S$ ) is an exact 3-cover. ■

### 3. Exact Doubly Dominating Sets

We begin by the following observation which is a straightforward property

of exact doubly dominating sets in graphs. A *matching* in a graph  $G$  is a set of pairwise non-incident edges of  $E$ .

**Observation 2.** The vertex set of every exact doubly dominating set induces a matching.

Next, we show that all exact doubly dominating sets (if any) have the same size.

**Proposition 3.** *If  $G$  has an exact doubly dominating set then all such sets have the same size.*

**Proof.** Let  $D_1, D_2$  be two exact doubly dominating sets of  $G$ . Let us write  $I = D_1 \cap D_2$ , and let  $X_0$  and  $X_1$  be the subsets of  $D_1 - I$  such that every vertex of  $X_0$  has zero neighbours in  $I$  and every vertex of  $X_1$  has one neighbour in  $I$ . Clearly  $D_1 - I = X_0 \cup X_1$ . We define similarly subsets  $Y_0$  and  $Y_1$  of  $D_2 - I$ . We claim that  $|X_1| = |Y_1|$ . Indeed, let  $x$  be any vertex of  $X_1$ , adjacent to a vertex  $z \in I$ . Since  $D_2$  is an exact doubly dominating set,  $z$  has a unique neighbour  $y$  in  $D_2$ . We have  $y \in D_2 - I$ , for otherwise  $z$  has two neighbours  $x, y$  in  $D_2$ , a contradiction. Thus  $y \in Y_1$ . The symmetric argument holds for every vertex of  $Y_1$ , and so  $|X_1| = |Y_1|$ . Since  $D_2$  is an exact doubly dominating set, every vertex of  $X_1$  has exactly one neighbour in  $Y_0 \cup Y_1$  and every vertex of  $X_0$  has exactly two neighbours in  $Y_0 \cup Y_1$ . The same holds about the vertices of  $Y_1$  and  $Y_0$ . This implies  $|X_0| = |Y_0|$ , and thus  $|D_1| = |D_2|$ . ■

The next result relates the size of an exact doubly dominating set with the order and minimum degree  $\delta$  of a graph  $G$ .

**Proposition 4.** *If  $S$  is an exact doubly dominating set of a graph  $G$ , then  $|S| \leq 2n/(\delta + 1)$ .*

**Proof.** Let  $S$  be an exact doubly dominating set of a graph  $G$  and let  $t$  denote the number of edges joining the vertices of  $S$  to the vertices of  $V - S$ . Then  $t = 2|V - S|$  since  $S$  is an exact doubly dominating set. By Observation 2,  $S$  induces a matching of  $G$ , and so every vertex  $v$  of  $S$  has exactly  $\deg_G(v) - 1$  neighbours in  $V - S$ . Thus  $t = \sum_{v \in S} (\deg_G(v) - 1)$ . So  $|S|(\delta - 1) \leq t = 2|V - S|$ . Hence  $|S| \leq 2n/(\delta + 1)$ . ■

In [7], Harary and Haynes gave a lower bound for the doubly domination number:

**Theorem 5** ([7]). *If  $G$  has no isolated vertices, then  $\gamma_{\times 2}(G) \geq 2n/(\Delta + 1)$ .*

From Proposition 4 and Theorem 5, we have:

**Corollary 6.** *If  $S$  is an exact doubly dominating set of a regular graph  $G$ , then  $|S| = 2n/(\Delta + 1)$ .*

Next, we establish a bound on the double domination number based on the neighbourhood packing number for any graph with no isolated vertices. Recall that a set  $R \subseteq V(G)$  is a *neighbourhood packing set* of  $G$  if  $N[x] \cap N[y] = \emptyset$  holds for any two distinct vertices  $x, y \in R$ . The *neighbourhood packing number*  $\rho(G)$  is the maximum cardinality of a neighbourhood packing in  $G$ . It is easy to see (see [8]) that every graph  $G$  satisfies  $\rho(G) \leq \gamma(G)$ .

**Theorem 7.** *If  $G$  is a graph without isolated vertices, then  $\gamma_{\times 2}(G) \geq 2\rho(G)$ .*

**Proof.** Let  $R$  be a maximum neighbourhood packing set of  $G$ . Then for every  $v \in R$ , every doubly dominating set of  $G$  contains at least 2 vertices of  $N[v]$  to doubly dominate  $v$ . Since  $N[v] \cap N[u] = \emptyset$  holds for each pair of vertices  $v, u$  of  $R$ , we have  $\gamma_{\times 2}(G) \geq 2|R|$ . ■

**Corollary 8.** *If  $S$  is an exact doubly dominating set of  $G$ , then  $|S| \geq 2\rho(G)$ .*

Farber [5] proved that the domination number and neighbourhood packing number are equal for any strongly chordal graph. Thus we have the following corollary to Theorem 7 which extends the result of Blidia *et al.* [3] for trees.

**Corollary 9.** *If  $G$  is a strongly chordal graph without isolated vertices, then  $\gamma_{\times 2}(G) \geq 2\gamma(G)$ .*

## 4. Graphs with Exact Doubly Dominating Sets

We first consider paths and cycles. The double domination number for cycles  $C_n$  and nontrivial paths  $P_n$  were given in [7] and [3] respectively:

$$[7] \quad \gamma_{\times 2}(C_n) = \lceil \frac{2n}{3} \rceil.$$

$$[3] \quad \gamma_{\times 2}(P_n) = 2\lceil \frac{n}{3} \rceil + 1 \text{ if } n \equiv 0 \pmod{3} \text{ and } \gamma_{\times 2}(P_n) = 2\lceil \frac{n}{3} \rceil \text{ otherwise.}$$

Now we establish similar results for the exact doubly dominating sets in cycles and paths.

**Proposition 10.** *A cycle  $C_n$  has an exact doubly dominating set if and only if  $n \equiv 0 \pmod{3}$ . If this holds the size of any such set is  $2n/3$ .*

**Proof.** Let  $S$  be an exact doubly dominating set of a cycle  $C_n$ . By Corollary 6, we have  $|S| = 2n/3$  and so  $n \equiv 0 \pmod{3}$ . Conversely, assume the vertices of  $C_n$  are labelled  $v_1, v_2, \dots, v_n, v_1$ . If  $n \equiv 0 \pmod{3}$ , then it is easy to check that the set  $\{v_i, v_{i+1} \mid i \equiv 1 \pmod{3}, 1 \leq i \leq n-1\}$  is an exact doubly dominating set of  $C_n$ . ■

**Proposition 11.** *A path  $P_n$  has an exact doubly dominating set if and only if  $n \equiv 2 \pmod{3}$ . If this holds the size of any such set is  $2(n+1)/3$ .*

**Proof.** If  $n = 2$  the fact is obvious, so let us assume  $n \geq 3$ . Let  $S$  be an exact doubly dominating set of a path  $P_n$ . Note that for every vertex  $v$  of degree 2, either  $v$  or its two neighbours are in  $S$ . So  $V - S$  is an independent set, and  $N(v) \cap N(w) = \emptyset$  for any two  $v, w \in V - S$ . By Observation 2, every vertex of  $S$  has exactly one neighbour in  $V - S$ . Thus  $|S| - 2 = 2|V - S|$  and so  $n = |S| + |V - S| = 3|V - S| + 2$ .

Conversely, assume that the vertices of  $P_n$  are labelled  $v_1, v_2, \dots, v_n$ . If  $n \equiv 2 \pmod{3}$  then it is easy to check that the set  $\{v_i, v_{i+1} \mid i \equiv 1 \pmod{3}, 1 \leq i \leq n-1\}$  is an exact doubly dominating set of  $P_n$ . ■

Chellali and Haynes [4] established the following upper bound for the double domination number:

**Theorem 12** ([4]). *Every graph  $G$  without isolated vertices satisfies*

$$\gamma_{\times 2}(G) \leq n - \delta + 1.$$

**Theorem 13.** *Let  $G$  be a graph that admits an exact doubly dominating set  $S$ . Then  $|S| = n - \delta + 1$  if and only if either  $G = tK_2$  with  $t \geq 1$ , if  $\delta = 1$ , or  $G = K_n$  with  $n \geq 3$  otherwise.*

**Proof.** Let  $S$  be an exact doubly dominating set of  $G$  such that  $|S| = n - \delta + 1$ . If  $\delta = 1$ , then  $|S| = n$ . Since  $S$  induces a 1-regular subgraph,  $G$  itself is 1-regular, i.e.,  $G = tK_2$  with  $t \geq 1$ . Now assume that  $\delta \geq 2$ . Let  $v$  be a vertex of  $S$ . Then  $V - S$  contains all the neighbours of  $v$  except one, and so  $\deg_G(v) - 1 \leq |V - S| = n - (n - \delta + 1) = \delta - 1$ . Thus all the vertices of  $S$  have the same degree  $\delta$ , and  $|V - S| = \delta - 1$ . Let  $u$  be a vertex of  $N(v) \cap S$ . Then  $u$  is adjacent to all the vertices of  $V - S$  and

hence at this point every vertex of  $V - S$  is doubly dominated by  $u$  and  $v$ . Thus  $S = \{u, v\}$  and all the vertices of  $V - S$  are mutually adjacent. So  $G$  is a complete graph. ■

Next, we consider nontrivial trees. A vertex of degree 1 is called a *leaf*, and a *support vertex* is any vertex adjacent to a leaf. It is easy to see that a star with at least three vertices is an example of a tree that does not admit an exact doubly dominating set. The following observation generalizes this remark.

**Observation 14.**

- If a graph  $G$  has a leaf, then any doubly dominating set of  $G$  contains this leaf and its neighbour.
- If a graph  $G$  has an exact doubly dominating set, then every support vertex is adjacent to exactly one leaf, and no two support vertices are adjacent.

We now define recursively a collection  $\mathcal{T}$  of trees, where each tree  $T \in \mathcal{T}$  has two distinguished subsets  $A(T), B(T)$  of vertices. First,  $\mathcal{T}$  contains any tree  $T_1$  with two vertices  $x, y$ , and for such a tree we set  $A(T_1) = \{x, y\}$  and  $B(T_1) = \{y\}$ . Next, if  $T'$  is any tree in  $\mathcal{T}$ , then we put in  $\mathcal{T}$  any tree  $T$  that can be obtained from  $T'$  by any of the following two operations:

*Type-1 operation:* Attach a path  $P_3 = uvw$ , with  $u, v, w \notin V(T')$ , by adding an edge from  $w$  to one vertex of  $A(T')$ . Set  $A(T) = A(T') \cup \{u, v\}$  and  $B(T) = B(T') \cup \{u\}$ .

*Type-2 operation:* Attach a path  $P_5 = a_1a_2a_3a_4a_5$ , with  $a_1, a_2, a_3, a_4, a_5 \notin V(T')$ , by adding an edge from  $a_3$  to one vertex of  $V(T') - A(T')$ . Set  $A(T) = A(T') \cup \{a_1, a_2, a_4, a_5\}$  and  $B(T) = B(T') \cup \{a_1, a_5\}$ .

**Lemma 15.** *If  $T \in \mathcal{T}$ , then:*

- (a)  $A(T)$  is the unique exact doubly dominating set of  $T$ .
- (b)  $B(T)$  is a neighbourhood packing set of  $T$ .
- (c)  $|A(T)| = 2\gamma(T)$ .

**Proof.** Consider any  $T \in \mathcal{T}$ . So  $T$  can be obtained from a sequence  $T_1, T_2, \dots, T_k$  ( $k \geq 1$ ) of trees of  $\mathcal{T}$ , where  $T_1$  is the tree with two vertices,  $T = T_k$ , and, if  $1 \leq i \leq k - 1$ , the tree  $T_{i+1}$  is obtained from  $T_i$  by one of the two operations. We prove (a) by induction on  $k$ . If  $k = 1$ , then  $A(T)$  is



obviously the unique exact doubly dominating set of  $T$ . Assume now that  $k \geq 2$  holds for  $T$  and that the result holds for all trees in  $\mathcal{T}$  that can be constructed by a sequence of length at most  $k - 1$ . Let  $T' = T_{k-1}$ . We distinguish between two cases.

*Case 1.*  $T$  is obtained from  $T'$  by using the Type-1 operation. Note that  $A(T)$  is an exact doubly dominating set of  $T$  since, by the induction hypothesis,  $A(T')$  is an exact doubly dominating set of  $T'$  and  $u, v$  and the neighbour of  $w$  in  $T'$  are in  $A(T)$ . Now let  $S$  be any exact doubly dominating set of  $T$ . By Observation 14, we have  $\{u, v\} \subset S$ , and consequently  $w \notin S$  (for otherwise  $v$  would be dominated three times by  $S$ ). If  $x$  is any vertex in  $V(T')$ , then  $x$  is not dominated by any of  $u, v$ , so  $S - \{u, v\}$  is an exact doubly dominating set of  $T'$ . By the inductive hypothesis  $A(T')$  is the unique such set, so  $S - \{u, v\} = A(T')$ , and so  $S = A(T)$ , which shows the unicity announced in (a).

*Case 2.*  $T$  is obtained from  $T'$  by using the Type-2 operation. Note that  $A(T)$  is an exact doubly dominating set of  $T$  since, by the induction hypothesis,  $A(T')$  is an exact doubly dominating set of  $T'$  and the neighbour of  $a_3$  in  $T'$  is not in  $A(T')$  while  $a_1, a_2, a_4, a_5$  are in  $A(T)$ . Now let  $S$  be any exact doubly dominating set of  $T$ . By Observation 14, we have  $\{a_1, a_2, a_4, a_5\} \subseteq S$ , and consequently  $a_3 \notin S$ . If  $x$  is any vertex in  $V(T')$ , then  $x$  is not dominated by any of  $a_1, a_2, a_4, a_5$ , so  $S - \{a_1, a_2, a_4, a_5\}$  is an exact doubly dominating set of  $T'$ . By the inductive hypothesis we have  $S - \{a_1, a_2, a_4, a_5\} = A(T')$ , and so  $S = A(T)$ . So (a) is proved.

It is a routine matter to check item (b). Note that the tree  $T_1$  with two vertices has  $|A(T_1)| = 2$  and  $|B(T_1)| = 1$ ; moreover, each operation adds twice as many vertices to  $A(T)$  as to  $B(T)$ , so  $|A(T)| = 2|B(T)|$  holds for every tree  $T \in \mathcal{T}$ . It follows from this and from (a) and (b) that  $\gamma_{\times 2}(T) \leq |A(T)| = 2|B(T)| \leq 2\gamma(T)$ , and we have equality throughout by Corollary 9. This proves part (c) and concludes the proof of the lemma. ■

We now are ready to give a constructive characterization of trees with an exact doubly dominating sets.

**Theorem 16.** *Let  $T$  be a tree. Then  $T$  has an exact doubly dominating set if and only if  $T \in \mathcal{T}$ .*

**Proof.** First suppose that  $T \in \mathcal{T}$ . Then Lemma 15 implies that  $T$  has an exact doubly dominating set. Conversely, assume that  $T$  is a tree that has

an exact doubly dominating set  $S$ , and let  $n$  be the order of  $T$ . Clearly,  $n \geq 2$ . If  $n = 2$ , then  $T$  is in  $\mathcal{T}$ . Observation 14 implies that  $n \in \{3, 4\}$  is impossible and that  $n = 5$  implies that  $T$  is a path on 5 vertices, which is in  $\mathcal{T}$  since it can be obtained from  $T_1$  by the Type-1 operation.

Now assume that  $n \geq 6$  and that every tree  $T'$  of order  $n'$  with  $2 \leq n' < n$  such that  $T'$  has an exact doubly dominating set is in  $\mathcal{T}$ . Root  $T$  at a vertex  $r$ . Let  $u$  be a leaf at maximum distance from  $r$ , let  $v$  be the parent of  $u$  in the rooted tree, and let  $w$  be the parent of  $v$ . By Observation 14,  $u$  is the unique child of  $v$ ,  $\{u, v\} \subseteq S$ ,  $w \notin S$ , and  $w$  is neither a support vertex nor a leaf. This implies that every child of  $w$  is a support vertex. Furthermore  $w$  has at most two children, for otherwise  $w$  would be dominated at least 3 times by  $S$ , a contradiction. So  $w \neq r$ . Let  $z$  be the parent of  $w$  in the rooted tree.

Suppose that  $w$  has exactly one child in the rooted tree. Let  $T' = T - \{u, v, w\}$ . Since  $\{u, v\} \subseteq S$  and  $w \notin S$ , we have  $z \in S$  so that  $w$  is dominated twice by  $S$ . Moreover,  $S - \{u, v\}$  is an exact doubly dominating set of  $T'$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$  and, by Lemma 15,  $S - \{u, v\} = A(T')$  is the unique exact doubly dominating set of  $T'$ . Thus  $T$  can be obtained from  $T'$  by using Type-1 operation (with the path  $uvw$  and since  $z \in A(T')$ ), so  $T \in \mathcal{T}$ .

Now suppose that  $w$  has exactly two children  $v, v'$  in the rooted tree. Let  $T_w$  be the subtree of  $T$  induced by  $w$  and its descendants, rooted at  $w$ . By Observation 14, each child of  $w$  has exactly one child, and we call  $u'$  the child of  $v'$ , so  $T_w$  is a path on five vertices  $uvwv'u'$  with central vertex  $w$ . Moreover, by Observation 14, we have  $\{u, v, u', v'\} \subseteq S$ ,  $w \notin S$ , and  $z \notin S$  since  $w$  is dominated twice in  $S$  by  $v, v'$ . Thus  $z$  is doubly dominated by  $S \cap V(T')$  and consequently  $S \cap V(T')$  is an exact doubly dominating set of  $T'$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$  and, by Lemma 15,  $S \cap V(T') = A(T')$  is the unique exact doubly dominating set of  $T'$ . Thus  $T$  can be obtained from  $T'$  by using Type-2 operation (with the path  $uvwv'u'$  and since  $z \notin A(T')$ ), so  $T \in \mathcal{T}$ . This completes the proof of the theorem. ■

The proof of the theorem suggests a polynomial-time algorithm which, given a tree  $T$  with  $n$  vertices, decides whether  $T$  is in  $\mathcal{T}$  and, if it is, returns the set  $A(T)$ . Here is an outline of the algorithm. If  $T$  is a path on 2 or 5 vertices, answer  $T \in \mathcal{T}$ , return the obvious set  $A(T)$ , and stop. Else, if either  $n \leq 5$  or  $T$  is a star, answer  $T \notin \mathcal{T}$  and stop. Now suppose  $n \geq 6$ . Pick a vertex  $r$ , root the tree  $T$  at  $r$ , and pick a vertex  $u$  at maximum distance from  $r$ . Let  $v$  be the parent of  $u$  in the rooted tree and  $w$  be the

parent of  $v$ . If either  $v$  has at least two children, or  $w$  has at least three children, or  $w$  has exactly two children and its second child has either zero or at least two children, then return the answer  $T \notin \mathcal{T}$  and stop. Else, let  $z$  be the parent of  $w$ . If  $w$  has exactly one child, call the algorithm recursively on the tree  $T' = T - \{u, v, w\}$ ; if the answer to the recursive call is  $T' \in \mathcal{T}$  and  $z \in A(T')$ , then answer  $T \in \mathcal{T}$ , return  $A(T) = A(T') \cup \{u, v\}$ , and stop, else answer  $T \notin \mathcal{T}$  and stop. If  $w$  has exactly two children  $v, v'$ , call the algorithm recursively on the tree  $T' = T - \{u, v, w, v', u'\}$  (where  $u'$  is the child of  $v'$ ); if the answer to the recursive call is  $T' \in \mathcal{T}$  and  $z \notin A(T')$ , then answer  $T \in \mathcal{T}$ , return  $A(T) = A(T') \cup \{u, v, u', v'\}$  and stop, else answer  $T \notin \mathcal{T}$  and stop.

Next, we give a necessary and sufficient condition for the existence of an exact doubly dominating set in a connected cubic graph. Recall that a matching in a graph  $G = (V, E)$  is *perfect* if its size is  $|V|/2$ . With any perfect matching  $M = \{e_1, e_2, \dots, e_{n/2}\}$  of a graph  $G$  we associate a simple graph denoted by  $G_M = (V_M, E_M)$  where each edge  $e_i \in M$  is represented by a vertex in  $V_M$  and two vertices of  $V_M$  are adjacent if the corresponding edges in  $M$  are joined by an edge in  $G$ . A graph is an *equitable bipartite* graph if its vertex set can be partitioned into two independent sets  $S_1$  and  $S_2$  such that  $|S_1| = |S_2|$ , and in this case  $(S_1, S_2)$  is called an *equitable bipartition* of  $G$ .

**Theorem 17.** *Let  $G$  be a connected cubic graph. Then  $G$  has an exact doubly dominating set if and only if  $G$  has a perfect matching  $M$  such that the associated graph  $G_M$  is an equitable bipartite graph.*

**Proof.** Let  $G$  be a connected cubic graph with an exact doubly dominating set  $S$ . So  $S$  induces a 1-regular graph, whose edges form a matching  $M_1$ , and every vertex of  $S$  has two neighbours in  $V - S$ . Since every vertex of  $V - S$  has exactly two neighbours in  $S$ , the subgraph induced by  $V - S$  is 1-regular, and its edges form a matching  $M_2$ . Thus  $G$  admits a perfect matching  $M = M_1 \cup M_2$ . Each edge of  $E - M$  joins a vertex of  $S$  with a vertex of  $V - S$ , and the bipartite subgraph  $(S, V - S; E - M)$  is 2-regular, so  $|S| = |V - S|$ , and so  $|M_1| = |M_2|$ . It follows that the graph  $G_M$  associated with  $M$  is an equitable bipartite graph with equitable bipartition  $(M_1, M_2)$ .

Conversely, let  $M$  be a perfect matching of a connected cubic graph  $G$  such that the associated graph  $G_M$  is equitable bipartite, with equitable bipartition  $(A, B)$ . Let  $A_M$  (resp.  $B_M$ ) be the vertices of  $G$  that are contained in the edges corresponding to the vertices of  $A$  (resp.  $B$ ). Since  $A$  (resp.  $B$ )

is independent in  $G_M$ , the subgraph of  $G$  induced by  $A_M$  (resp. by  $B_M$ ) is 1-regular. This also implies that every vertex of  $A_M$  (resp. of  $B_M$ ) has two neighbours in  $B_M$  (resp. in  $A_M$ ) since  $G$  is a cubic graph. Consequently,  $A_M$  and  $B_M$  are two disjoint exact doubly dominating sets of  $G$ . This completes the proof. ■

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