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HAMILTON CYCLES IN SPLIT GRAPHS WITH LARGE MINIMUM DEGREE

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Abstract

A graph G is called a split graph if the vertex-set V of G can be partitioned into two subsets V_1 and V_2 such that the subgraphs of G induced by V_1 and V_2 are empty and complete, respectively. In this paper, we characterize hamiltonian graphs in the class of split graphs with minimum degree δ at least $|V_1| - 2$.

Keywords: Hamilton cycle, split graph, bipartite graph.

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1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ (or V and E in short) will denote its vertex-set and its edge-set, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). For a vertex $v \in V(G)$, the degree of v, denoted by $\deg(v)$, is $|N_G(v)|$. The minimum degree of a graph $G = (V, E)$, denoted by $\delta(G)$ or δ in short, is the number $\min\{\deg(v) \mid v \in V\}$. The subgraph of G

induced by $W \subseteq V(G)$ is denoted by $G[W]$. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

One of the fundamental problems in graph theory is the hamiltonian problem. Although this is an old one, the amount of papers dealing with this subject does not decrease nowadays (see [3], [8], [11]). Most of these works give sufficient conditions for the existence of a Hamilton cycle in graphs. Only a few of them deal with necessary ones. The history of development of the problem shows that there is a very little hope that an useful and simple characterization of all hamiltonian graphs exists. However, this does not exclude the availability of such a characterization of hamiltonian graphs in some particular classes of graphs, e.g., in [14] hamiltonian self-complementary graphs have been characterized by Rao and in [10] hamiltonian threshold graphs have been characterized by Harary and Peled.

A graph $G = (V, E)$ is called a split graph if there exists a partition $V = V_1 \cup V_2$ such that $G[V_1]$ and $G[V_2]$ are empty and complete graphs, respectively. We will denote such a graph by $S(V_1 \cup V_2, E)$. The notion of split graphs was introduced in [6] by Foldes and Hammer. These graphs have been paid attention because of their connection with many combinatorial problems (see [5], [7], [13]).

In this paper, we consider the hamiltonian problem for split graphs. It is clear that if $|V_1| > |V_2|$ then a split graph $G = S(V_1 \cup V_2, E)$ has no Hamilton cycles. So without loss of generality we may consider the hamiltonian problem only for split graphs $G = S(V_1 \cup V_2, E)$ with $|V_1| \leq |V_2|$. The main result here is Theorem 1 below. The condition for the existence of a Hamilton cycle in a split graph obtained here is similar to Hall's condition for the existence of a complete matching in a bipartite graph [9].

Theorem 1. *Let $G = S(V_1 \cup V_2, E)$ be a split graph with $|V_1| = m \leq n = |V_2|$ and the minimum degree $\delta(G) \geq m-2$. Then G has a Hamilton cycle if and only if $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$ with $m-2 \leq |S| \leq min\{m, n-1\}$, except the following graphs for which the sufficiency does not hold:*

(i) *$m = 3 < n$ and G is the graph G_n^3,*

(ii) *$m = 4 < n$ and G is a spanning subgraph of D_n^4 or G_n^4,*

(iii) *$m = 4 \leq n$ and $G - u$ is the graph G_n^3 for some $u \in V_1$,*

(iv) *$m = 5 < n$ and G is the graph F_n^5 or a spanning subgraph of G_n^5,*

(v) *$6 \leq m < n$ and G is a spanning subgraph of G_n^m.*

The graphs G_n^m, D_n^4 and F_n^5 will be defined in Section 2. It will be also proved there that these graphs are split graphs $S(V_1 \cup V_2, E)$ satisfying $|V_1| = m < n = |V_2|$, $\delta(G) \geq m-2$ and $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$ with $|S| \geq m-2$, but they have no Hamilton cycles. Every graph in (iii) also has no Hamilton cycles.

We note that there are in literature some papers dealing with the hamiltonian problem for split graphs [2], [12]. But the conditions obtained there for the existence of a Hamilton cycle in split graphs are only necessary, but not sufficient. In [2] the authors also asked if the conditions obtained there can be sharpened to a necessary and sufficient one.

From Theorem 1 we have the following corollary.

Corollary 2. *Let $G = B(V_1 \cup V_2, E)$ be a bipartite graph with bipartition $V = V_1 \cup V_2$, where $|V_1| = m \leq n = |V_2|$ and $\delta(V_1) = min\{deg(v) \mid v \in V_1\} \geq m-2$. Then G has a Hamilton cycle if and only if $m = n$ and $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$ with $|S| = m-2$ or $m-1$, unless $m = 4$ and $G - u$ is the graph BG_4^3 for some $u \in V_1$.*

The graph BG_4^3 is obtained from G_4^3 by deleting all edges, the both endvertices of which are in V_2. The sufficiency does not hold for these exceptional graphs.

Thus, we have got in this paper a characterization of hamiltonian split graphs $G = S(V_1 \cup V_2, E)$ with $\delta(G) \geq |V_1| - 2$. We note that although many sufficient conditions for the existence of a Hamilton cycle in a graph are known (see [3]), almost all they involve the order $|V|$ of G and all they are not necessary. Meanwhile, our condition is also necessary and involves only the cardinality $|V_1|$ of the subset V_1. Therefore, it is not a consequence of former conditions.

2. PRELIMINARIES

Let C be a cycle in a graph $G = (V, E)$. By $\overrightarrow{C}$ we denote the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices in the reverse order are given by $v\overleftarrow{C}u$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. If $u \in V(C)$, then u^+ denotes the successor of u on $\overrightarrow{C}$, and u^- denotes its predecessor. Similar notation as described above for cycles is used for paths.

If $W \subseteq V(G)$ and $v \in W$, then v is called a W-vertex. Also, by $N_{G,W}(u)$ or $N_W(u)$ in short we denote the set $W \cap N_G(u)$.

Lemma 3. *If a split graph* $G = S(V_1 \cup V_2, E)$ *has a Hamilton cycle, then* $|N(S)| > |S|$ *for any* $\emptyset \neq S \subseteq V_1$ *with* $|S| \leq min\{m, n-1\}$.

Proof. Suppose that $G = S(V_1 \cup V_2, E)$ has a Hamilton cycle C. Let $u_1, \ldots, u_k$ $(1 \leq k \leq \min\{m, n-1\})$ be the vertices of $\emptyset \neq S \subseteq V_1$, occurring on $\overrightarrow{C}$ in the order of their indices. It is not difficult to see that $u_i^+ \neq u_j^+$ if $i \neq j$. Since $|S| = k < n$, there exists $i \in \{1, \ldots, k\}$ such that there are at least two V_2-vertices in $u_i^+ \overrightarrow{C} u_{i+1}^-$(indices mod k). Therefore, $u_i^+ \neq u_{i+1}^-$. So $N(S) \supseteq \{u_i^+, \ldots, u_k^+, u_{i+1}^-\}$ and $|N(S)| \geq k+1 > k = |S|$. ■

Lemma 4. *Let* $G = S(V_1 \cup V_2, E)$ *be a split graph with* $|V_1| < |V_2|$. *Then* G *has a Hamilton cycle if and only if* $|N_G(V_1)| > |V_1|$ *and the subgraph* $G' = G[V_1 \cup N_G(V_1)]$ *has a Hamilton cycle.*

Proof. Suppose that G has a Hamilton cycle C. By Lemma 3, $|N_G(V_1)| > |V_1|$. If $v \in V_2 - N_G(V_1)$, then both v^- and v^+ are in V_2. So $v^-v^+ \in E(G-v)$. This means that $C' = C - v + v^-v^+$ is a Hamilton cycle of $G-v$. By backward induction on $|V_2 - N_G(V_1)|$ we can show that G' has a Hamilton cycle.

Conversely, let $|N_G(V_1)| > |V_1|$ and $G' = G[V_1 \cup N_G(V_1)]$ have a Hamilton cycle C'. Futher, let $V_2 - N_G(V_1) = \{y_1, \ldots, y_l\}$. Since $|N_G(V_1)| > |V_1|$, there exists $v \in N_G(V_1)$ such that both v and v^+ (with respect to C') are in $N_G(V_1) \subseteq V_2$. It follows that $C = vy_1 \ldots y_l v^+ \overrightarrow{C}' v$ is a Hamilton cycle of G. ■

In Table 1 we define the split graphs G_n^m, D_n^4 and F_n^5. The conditions that m and n must be satisfied for the corresponding graph are indicated in parentheses under its name in Column 1. The subsets V_1 and V_2 of the vertex-set V for each of these graphs are indicated in Column 2. Finally, in Column 3, we present the edges of the corresponding graph.

Lemma 5. (a) *Let* $G = S(V_1 \cup V_2, E)$ *with* $|V_1| = m$ *and* $|V_2| = n$ *be one of the split graphs* G_n^m, D_n^4 *and* F_n^5. *Then* $m < n$, $\delta(G) \geq m-2$ *and* $|N(S)| > |S|$ *for any* $\emptyset \neq S \subseteq V_1$ *with* $|S| \geq m-2$, *but* G *has no Hamilton cycles.*

(b) *Every graph $G = S(V_1 \cup V_2, E)$, for which $G - u$ is the graph G_n^3 for some $u \in V_1$, also has no Hamilton cycles.*

Table 1. The graphs G_n^m, D_n^4 and F_n^5.

The graph $G = (V, E)$	The vertex-set $V = V_1 \cup V_2$	The edge-set $E = E_1 \cup E_2 \cup E_3$
G_n^m $(3 \le m < n)$	$V_1 = \{u_1, \ldots, u_m\}$, $V_2 = \{v_1, \ldots, v_n\}$.	$E_1 = \{u_1v_1, u_2v_2, u_3v_3\}$, $E_2 = \{u_iv_j \mid i = 1, \ldots, m;\ j = 4, \ldots, m+1\}$, $E_3 = \{v_iv_j \mid i \neq j;\ i, j = 1, \ldots, n\}$.
D_n^4 $(4 < n)$	$V_1 = \{u_1, \ldots, u_4\}$, $V_2 = \{v_1, \ldots, v_n\}$.	$E_1 = \{u_1v_2, u_2v_1, u_iv_i \mid i = 1, 2, 3, 4\}$, $E_2 = \{u_iv_5 \mid i = 1, 2, 3, 4\}$, $E_3 = \{v_iv_j \mid i \neq j;\ i, j = 1, \ldots, n\}$.
F_n^5 $(6 < n)$	$V_1 = \{u_1, \ldots, u_5\}$, $V_2 = \{v_1, \ldots, v_n\}$.	$E_1 = \{u_iv_i \mid i = 1, \ldots, 5\}$, $E_2 = \{u_iv_j \mid i = 1, \ldots, 5; j = 6, 7\}$, $E_3 = \{v_iv_j \mid i \neq j; i, j = 1, \ldots, n\}$.

Proof. (a) It is not difficult to verify that for each of these graphs the inequalities $m < n$, $\delta(G) \ge m - 2$ and $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$ with $|S| \ge m - 2$ are true.

Suppose that $G = G_n^m$ has a Hamilton cycle C. Set $R = \{v_4, \ldots, v_{m+1}\}$, where $v_4, \ldots, v_{m+1}$ occur on $\overrightarrow{C}$ in the order of their indices. Then $G - R$ can be covered by $m - 2 = |R|$ vertex-disjoint paths $P_1 = v_4^+ \overrightarrow{C} v_5^-$, $P_2 = v_5^+ \overrightarrow{C} v_6^-$,...,$P_{m-2} = v_{m+1}^+ \overrightarrow{C} v_4^-$. On the other hand, it is not difficult to see that $G - R$ need at least $m - 1$ vertex-disjoint paths to cover it, a contradiction.

The proofs of the fact that D_n^4 and F_n^5 are non-hamiltonian are left to the reader.

(b) Let G have a Hamilton cycle C. Then both u^- and u^+ are in V_2. So $u^-u^+ \in E(G-u)$ and therefore $C' = u^-u^+\overrightarrow{C}u^-$ is a Hamilton cycle of $G-u$, contradicting the fact that G_n^3 is non-hamiltonian by (a). ■

By Lemma 4, we assume from now on that all considered split graphs $S(V_1 \cup V_2, E)$ have $N(V_1) = V_2$.

Lemma 6. *Let $G = S(V_1 \cup V_2, E)$ with $|V_1| = m$ and $|V_2| = n$ be a maximal non-hamiltonian split graph with $\delta(G) \geq m-k$ $(0 \leq k \leq m)$. Then for any $v \in V_2$, either $|N_{V_1}(v)| \leq k$ or $|N_{V_1}(v)| = m$.*

Proof. Suppose that there exists $v \in V_2$ such that $k < |N_{V_1}(v)| < m$. Since $|N_{V_1}(v)| < m$, there exists $u \in V_1$ such that $uv \notin E$. Therefore, $G + uv$ has a Hamilton cycle C, which must contain the edge uv because G is maximal non-hamiltonian. Without loss of generality we may assume $u^- = v$. Let $x_1, \dots, x_t$ be the neighbours in G of u, occurring on $u\overrightarrow{C}v$ in the order of their indices. Then $t \geq m-k$ by the assumption and x_i^- is not adjacent to v in G for every $i = 1, \dots, t$ because otherwise, $C' = u\overrightarrow{C}x_i^-v\overleftarrow{C}x_iu$ is a Hamilton cycle of G, a contradiction. So $x_1^-, \dots, x_t^-$ are in V_1 because all V_2-vertices are adjacent to v. Hence, $|N_{V_1}(v)| \leq m-t \leq m-(m-k) = k$, contradicting $|N_{V_1}(v)| > k$. ■

Proposition 7. *Let $G = S(V_1 \cup V_2, E)$ be a split graph with $|V_1| < |V_2|$ and $|N_{V_1}(v)| \leq 2$ for each $v \in V_2$. Then G has a Hamilton cycle if and only if $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$.*

Proof. The necessity follows from Lemma 3. Now we prove the sufficiency by induction on $|V_1|$. If $|V_1| = 1$, then G trivially has a Hamilton cycle. Suppose that the sufficiency has been proved when $|V_1| < t$ and G is a split graph such that $|V_1| = t < |V_2|, |N_{V_1}(v)| \leq 2$ for any $v \in V_2$ and $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$. By the induction hypothesis, for any $u \in V_1$, the graph $G_u = G - u$ has a Hamilton cycle C.

First assume that there exists $v_1 \in V_2$ such that $|N_{V_1}(v_1)| = 1$, say $N_{V_1}(v_1) = \{u\}$. Since $|N_G(u)| > |\{u\}| = 1$, there exists $v_2 \in N(u)$ with $v_2 \neq v_1$. If $v_1^+ = v_2$, then $C' = v_1uv_2\overrightarrow{C}v_1$ is a Hamilton cycle of G. So we assume that $v_1^+ \neq v_2$. Since $N_{V_1}(v_1) = \{u\}$, $|N_{V_1}(v_2)| \leq 2$ and $v_2 \in N(u)$, either both v_1^- and v_2^- or both v_1^+ and v_2^+ are in V_2, say v_1^+ and v_2^+. Then $C' = v_1uv_2\overleftarrow{C}v_1^+v_2^+\overrightarrow{C}v_1$ is a Hamilton cycle of G.

Now assume that for any $v \in V_2$, $|N_{V_1}(v)| = 2$. If for any $u \in V_1$, $|N(u)| \leq 2$, then

$$2|V_1| \geq \sum_{u \in V_1} |N(u)| = \sum_{v \in V_2} |N_{V_1}(v)| = 2|V_2|,$$

contradicting $|V_1| < |V_2|$. Thus, there exists $u \in V_1$ such that $|N_G(u)| \geq 3$. Let $v_1, v_2, v_3 \in N(u)$. Since $|N_{V_1}(v_i)| = 2$ for each $i = 1, 2, 3$, either $\{v_1^-, v_2^-, v_3^-\}$ or $\{v_1^+, v_2^+, v_3^+\}$ contains two V_2-vertices, say v_1^+ and v_2^+. If $v_1^+ = v_2$, then $C' = v_1 u v_2 \overrightarrow{C} v_1$ is a Hamilton cycle of G. Otherwise, $C' = v_1 u v_2 \overleftarrow{C} v_1^+ v_2^+ \overrightarrow{C} v_1$ is a Hamilton cycle of G. ■

Proposition 8. *Let $G = S(V_1 \cup V_2, E)$ be a split graph with $|V_1| < |V_2|$ and $\delta(G) \geq |V_1|$. Then G has a Hamilton cycle if and only if $|N(V_1)| > |V_1|$.*

Proof. The necessity follows from Lemma 3. Now we prove the sufficiency. It is not difficult to verify that $|V(G)| \geq 3$, $\alpha(G) = |V_1| \leq \kappa(G)$, where $\alpha(G)$ and $\kappa(G)$ are the independence number and the connectivity of G, respectively. By [4] G has a Hamilton cycle. ■

A graph G is called to have the property ($\bullet$) if the following conditions are satisfied:

1. G is a split graph $S(V_1 \cup V_2, E)$ with $m = |V_1| < |V_2| = n$ and $\delta(G) \geq |V_1| - k$ where $1 \leq k \leq 2$;
2. G is a maximal non-hamiltonian, but for any $u \in V_1$ the graph $G_u = G - u$, which is the split graph $S(W_1 \cup V_2, E_u)$ with $W_1 = V_1 - u$, $E_u = E - \{uv \in E \mid v \in V_2\}$, has a Hamilton cycle C;
3. For any $\emptyset \neq S \subseteq V_1$ with $|S| \geq m - k$, $|N(S)| > |S|$;
4. G has a V_1-vertex u such that u has a neighbour v_1 with $v_1^+ \in V_2$ and a neighbour v_2 with $v_2^- \in V_2$ (with respect to C). The vertex v_1 may coincide with v_2.

We note that if the vertices $v_1 \neq v_2$, then $v_2 \notin \{v_1^-, v_1^+\}$ because otherwise $C' = v_1 u v_2 \overrightarrow{C} v_1$ or $C' = v_2 u v_1 \overrightarrow{C} v_2$ is a Hamilton cycle of G, a contradiction.

Let G be a graph with the property ($\bullet$) and u, v_1 and v_2 be the vertices of G chosen as in its definition above. Set

$$B_i = \{v \in V_2 \mid |N_{V_1}(v)| = i\},$$

$$A_i = \bigcup_{v \in B_i} N_{V_1}(v).$$

Many assertions below can be proved easily by contradiction. So we omit their detailed proofs and give in parentheses only a Hamilton cycle C' of G if we assume the contrary.

Claim 2.1. $B_m \neq \emptyset$.

(C' exists by Lemma 6 and Proposition 7 if $B_m = \emptyset$.)

Claim 2.2. v_1^+ and v_2^- are not in B_m.

($C' = v_1 u v_1^+ \overrightarrow{C} v_1$ if $v_1^+ \in B_m$.)

Claim 2.3. For any $x \in B_m$, x^+ and x^- are in W_1.

($C' = v_1 u x \overleftarrow{C} v_1^+ x^+ \overrightarrow{C} v_1$ if $x^+ \in V_2$.)

By Claim 2.1 and Claim 2.3 there exists a positive integer t such that C possesses t disjoint paths $P_1 = x_1 \overrightarrow{C} y_1, \ldots, P_t = x_t \overrightarrow{C} y_t$, which occur in $v_1^+ \overrightarrow{C} v_1^-$ in the order of their indices and have the following properties:

(a) Vertices of W_1 and B_m occur alternatively in P_i for every $i = 1, \ldots, t$,
(b) The endvertices x_i and y_i of P_i are in W_1 for every $i = 1, \ldots, t$,
(c) x_i^- and y_i^+ are not in B_m for every $i = 1, \ldots, t$,
(d) Every vertex of B_m is in one of $P_1, \ldots, P_t$.

Let l_i be the number of B_m-vertices in P_i. Then it is clear that the number of W_1-vertices in P_i is $l_i + 1$. By (d), $l_1 + \ldots + l_t = |B_m|$. So in total, the number of W_1-vertices in all paths $P_1, \ldots, P_t$ is $|B_m| + t$. It follows that $|B_m| + t \leq |W_1| = m - 1$. Thus, we have proved the following.

Claim 2.4. $|B_m| \leq m - 1 - t$.

Set $Q_1 = v_1^+ \overrightarrow{C} v_2^-$ and $Q_2 = v_2^+ \overrightarrow{C} v_1^-$. Thus, if $v_1 = v_2$, then $Q_1 = Q_2$. But if $v_1 \neq v_2$, then Q_1 and Q_2 are disjoint and each of them has at least one vertex because $v_2 \notin \{v_1^-, v_1^+\}$ as we have noted before. Let among $P_1, \ldots, P_t$ there be l paths in Q_1 ($0 \leq l \leq t$). Since $P_1, \ldots, P_t$ occur in $v_1^+ \overrightarrow{C} v_1^-$ in the order of their indices, these l paths in Q_1 are $P_1, \ldots, P_l$. Then the following assertions are also true.

Claim 2.5. All W_1-neighbours of v_1^+ and v_2^- are in Q_1.

($C' = v_1 u v_2 \overrightarrow{C} u_1 v_1^+ \overrightarrow{C} v_2^- u_1^+ \overrightarrow{C} v_1$ if u_1 is a W_1-neighbour in Q_2 of v_1^+.)

Claim 2.6. If among $P_1, \dots, P_t$ there are $l \geq 1$ paths in Q_1 and w is a W_1-neighbour in some P_i of v_1^+ (resp. v_2^-), then $w = x_1$ (resp. $w = y_l$).

Suppose the otherwise that $w \neq x_1$. If $w^- \in B_m$, then $C' = v_1 u w^- \overleftarrow{C} v_1^+ w \overrightarrow{C} v_1$ is a Hamilton cycle of G, a contradiction. So $w^- \notin B_m$. It follows $w = x_i$ for some $i \geq 2$. Then $C' = v_1 u x_1^+ \overleftarrow{C} v_1^+ x_i \overleftarrow{C} x_1^{++} x_i^+ \overrightarrow{C} v_1$ is a Hamilton cycle of G because both x_1^+ and x_i^+ are in B_m, a contradiction again. By symmetry, we can show the assertion for v_2^-.

Claim 2.7. If among $P_1, \dots, P_t$ there are $l \geq 1$ paths in Q_1 and v_1^+ (resp. v_2^-) has a W_1-neighbour in some P_i, then $v_1^{++} \in W_1$ (resp. $v_2^{--} \in W_1$).

Suppose the otherwise that $v_1^{++} \in V_2$. Let w be a W_1-neighbour of v_1^+ in P_i. By Claim 2.6, $w = x_1$. Therefore, $C' = v_1 u x_1^+ \overrightarrow{C} v_2^- v_1^+ x_1 \overleftarrow{C} v_1^{++} v_2 \overrightarrow{C} v_1$ is a Hamilton cycle of G, a contradiction. By symmetry, we can show the assertion for v_2^-.

Proposition 9. *Let $G = S(V_1 \cup V_2, E)$ be a split graph with $m = |V_1| < |V_2| = n$ and $\delta(G) \geq m - 1$. Then G has a Hamilton cycle if and only if $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$ with $|S| \geq m - 1$, except the graph G_n^3 for which the sufficiency does not hold.*

Proof. The necessity follows from Lemma 3. Now, we prove the sufficiency. Let $G = S(V_1 \cup V_2, E)$ be a maximal non-hamiltonian split graph satisfying $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$ with $|S| \geq m - 1$. By Lemma 6, $B_2 = B_3 = \cdots = B_{m-1} = \emptyset$. Set $A = V_1 \setminus A_1$.

For any $u \in V_1$ denote by G_u the graph $G - u$. Thus, $G_u = S(W_1 \cup V_2, E_u)$ with $W_1 = V_1 - u$. By Proposition 8, G_u has a Hamilton cycle C. The following assertions are easily proved by contradiction. (We again indicate in parentheses a Hamilton cycle C' of G if we assume the contrary.)

Claim 2.8. $B_1 \neq \emptyset$.

(C' exists by Proposition 8 if $B_1 = \emptyset$.)

Claim 2.9. Each $u \in A_1$ has only one B_1-neighbour.

($C' = v_1uv_2\overleftarrow{C}v_1^+v_2^+\overrightarrow{C}v_1$ if v_1 and v_2 are two different B_1-neighbours of u.)
Let $u \in A_1$ and v be the B_1-neighbour of u. Then both v^- and v^+ are in V_2. Thus G is a graph having the property ($\bullet$) with $k = 1$. By Claim 2.1 $B_m \neq \emptyset$. By Claim 2.9, all neighbours of u but v are in B_m. It follows that $|B_m| = |N(u)| - 1 \geq (m-1) - 1 = m - 2$. Together with Claim 2.4 we have $m - 2 \leq |B_m| \leq m - 1 - t$, where t is the number of paths $P_i = x_i\overrightarrow{C}y_i$ defined for a graph with property ($\bullet$) as above. So $|B_m| = m - 2$, $t = 1$ because t is a positive integer. Therefore, the number of W_1-vertices in $P_1 = x_1\overrightarrow{C}y_1$ is $m - 1 = |W_1|$. This means that all vertices of W_1 are in P_1. So if $R = y_1^+\overrightarrow{C}x_1^-$, then $V(R) = B_1$. By Claim 2.2 both v^- and v^+ are in B_1. Hence, all v^-, v and v^+ are in R. Let w be the unique W_1-neighbour of v^+, then $w = x_1 = v^{++}$ by Claim 2.6 and Claim 2.7. By symmetry, if w is the unique W_1-neighbour of v^-, then $w = y_1 = v^{--}$. This means that $v^+ = x_1^-$ and $v^- = y_1^+$ and therefore $R = v^-vv^+$ with all v^-, v, v^+ in B_1. Since $|B_m| = m - 2$ and $\delta(G) \geq m - 1$, the set A must be empty. Thus, $B_1 = \{v^-, v, v^+\}$, $A = \emptyset$. Using Claim 2.9 it is not difficult to see that G must be G_4^3. ■

3. Proof of the Results

First we prove the following two propositions 10 and 11.

Proposition 10. *Let $G = S(V_1 \cup V_2, E)$ be a split graph with $m = |V_1| < |V_2| = n$ and $\delta(G) \geq m - 2$. Then G has a Hamilton cycle if and only if $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$ with $|S| \geq m - 2$, except the following graphs for which the sufficiency does not hold :*

(i) *$m = 3$ and G is the graph G_n^3,*
(ii) *$m = 4$ and G is a spanning subgraph of D_n^4 or G_n^4,*
(iii) *$m = 4$ and $G - u$ is the graph G_n^3 for some $u \in V_1$,*
(iv) *$m = 5$ and G is the graph F_n^5 or a spanning subgraph of G_n^5,*
(v) *$m \geq 6$ and G is a spanning subgraph of G_n^m.*

Proof. The necessity follows from Lemma 3. Now, we prove the sufficiency. If $m = 1$ or 2, then by Proposition 7, G has a Hamilton cycle. For any $3 \leq m < n$, let $G = S(V_1 \cup V_2, E)$ be a maximal non-hamiltonian split graph satisfying $\delta(G) \geq m - 2$ and $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$ with $|S| \geq m - 2$. By Lemma 6, $B_3 = B_4 = \cdots = B_{m-1} = \emptyset$. Set $A = V_1 \setminus (A_1 \cup A_2)$.

For any $u \in V_1$, denote by G_u the graph $G - u = S(W_1 \cup V_2, E_u)$, where $W_1 = V_1 - u$. By Proposition 9, either $|W_1| = m - 1 = 3$ and G_u is the graph G_4^3 or G_u has a Hamilton cycle. In the former case, G is a graph in (iii). So we assume from now on that

Claim 3.1. For any $u \in V_1$, G_u has a Hamilton cycle C with a fixed orientation $\overrightarrow{C}$.

Below we omit the detailed proofs of many assertions which can be easily proved by contradiction. In these cases, as before, we indicate in parentheses a Hamilton cycle C' of G if we assume the contrary.

Claim 3.2. Each $u \in A_1$ has only one B_1-neighbour.

($C' = v_1 u v_2 \overleftarrow{C} v_1^+ v_2^+ \overrightarrow{C} v_1$ if v_1 and v_2 are two different B_1-neighbours of u.)

Claim 3.3. $A_1 \cap A_2 = \emptyset$.

Suppose the otherwise that $u \in A_1 \cap A_2$. Let v_1 and v_2 be a B_1-neighbour and a B_2-neighbour of u, respectively. Then v_1^- and v_1^+ are in V_2 and at least one of v_2^- and v_2^+ is in V_2, say v_2^+. So $C' = v_1 u v_2 \overleftarrow{C} v_1^+ v_2^+ \overrightarrow{C} v_1$ is a Hamilton cycle of G, a contradiction.

Claim 3.4. Each $u \in A_2$ has at most two B_2-neighbours.

Suppose the otherwise that some $u \in A_2$ has three B_2-neighbours v_1, v_2 and v_3. Then either $\{v_1^-, v_2^-, v_3^-\}$ or $\{v_1^+, v_2^+, v_3^+\}$ has at least two vertices in V_2, say $v_1^+ \in V_2$ and $v_2^+ \in V_2$. Then $C' = v_1 u v_2 \overleftarrow{C} v_1^+ v_2^+ \overrightarrow{C} v_1$ is a Hamilton cycle of G, a contradiction.

Claim 3.5. At least one of B_1 and B_2 is not empty.

(C' exists by Proposition 8 if both B_1 and B_2 are empty.)

Now we consider separately two cases.

Case 1. $B_1 \neq \emptyset$.
Let $v \in B_1$ and u be the V_1-neighbour of v. By Claim 3.1, G_u has a Hamilton cycle C with a fixed orientation $\overrightarrow{C}$. Since $v \in B_1$, both v^- and v^+ are in V_2. Thus, G is a graph having the property ($\bullet$) with $k = 2$.

(Here $v_1 = v_2 = v$ and therefore $Q_1 = v_1^+ \overrightarrow{C} v_2^- = v_2^+ \overrightarrow{C} v_1^- = Q_2$.) By Claim 3.2 and Claim 3.3, all neighbours of u but v are in B_m. It follows that $|B_m| = |N(u)| - 1 \geq (m-2) - 1 = m-3$. Together with Claim 2.4 we have $m - 3 \leq |B_m| \leq m - 1 - t$, where t is the number of paths $P_i = x_i \overrightarrow{C} y_i$ defined for a graph with the property $(\bullet)$ as in Section 2. So $|B_m| = m-2$ or $m-3$ because t is a positive integer.

If $|B_m| = m-2$, then $t = 1$ and the number of W_1-vertices in P_1 is $m - 1 = |W_1|$. It follows that all vertices of W_1 are in P_1. So if $R = y_1^+ \overrightarrow{C} x_1^-$, then $V(R) = B_1 \cup B_2$. In particular, v^-, v and v^+ are in R by Claim 2.2. Let w be a W_1- neighbour of v^+. Then w is in P_1. By Claim 2.6, $w = x_1$ and therefore $v^+ \in B_1$. By Claim 2.7, $v^+ = x_1^-$. By symmetry, $v^- \in B_1$ and $v^- = y_1^+$. Therefore, $R = v^- v v^+$ with all v^-, v and v^+ in B_1 and $B_2 = \emptyset$. Using Claim 3.2 it is not difficult to see that G is G_n^m in this subcase.

If $|B_m| = m-3$, then $t = 1$ or 2 and $A = \emptyset$ because $\delta(G) \geq m-2$.

First assume that $|B_m| = m-3$, $t = 1$ and $A = \emptyset$. Then P_1 contains $m - 2$ W_1-vertices. Therefore, $R = y_1^+ \overrightarrow{C} x_1^-$ contains exactly one W_1-vertex, say u_1. All the other vertices of R are in $B_1 \cup B_2$. By symmetry, without loss of generality we may assume that u_1 is in $v^{++} \overrightarrow{C} x_1^-$. From Claim 3.2, Claim 3.3 and the fact that both u_1^- and u_1^+ are in R, we see that $u_1 \in A_2$. If $v^+ \neq u_1^-$, then v^+ is not adjacent to u_1 because otherwise u_1 has three neighbours in $B_1 \cup B_2$, contradicting Claim 3.3 and Claim 3.4. Thus v^+ has a W_1-neighbour in P_1. Now by Claim 2.7, $v^{++} \in W_1$, contradicting the fact that there are no W_1-vertices in $v^+ \overrightarrow{C} u_1^-$. Thus $v^+ = u_1^-$. So $v^+ \in B_2$ and v^+ has another W_1-neighbour in P_1, namely, x_1 by Claim 2.6. Now if $u_1^+ \neq x_1^-$, then $C' = vux_1^+ \overrightarrow{C} v^- x_1^- x_1 v^+ \overrightarrow{C} x_1^{--} v$ is a Hamilton cycle of G, a contradiction. It follows that $u_1^+ = x_1^-$. Further, consider v^-. If w is a W_1-neighbour of v^-, then $w \neq u_1$ because otherwise u_1 has three neighbours in $B_1 \cup B_2$. So w is in P_1. By Claim 2.6 and Claim 2.7, $v^- = y_1^+$ and $v^- \in B_1$. Thus, $R = v^- v v^+ u_1 x_1^-$, $B_1 = \{v^-, v\}$, $B_2 = \{v^+, x_1^-\}$, $N_{W_1}(v^+) = N_{W_1}(x_1^-) = \{u_1, x_1\}$ and $A = \emptyset$. Using Claims 3.2–3.4, it is not difficult to see that G is D_5^4 in this subcase.

Now assume that $|B_m| = m-3$, $t = 2$ and $A = \emptyset$. Since the total number of W_1-vertices in P_1 and P_2 is $m - 1 = |W_1|$, every vertex of W_1 is in P_1 or P_2. Set $R_1 = y_1^+ \overrightarrow{C} x_2^-$ and $R_2 = y_2^+ \overrightarrow{C} x_1^-$. Then R_1 has at least one vertex, R_2 contains v^-, v, v^+ and $V(R_1 \cup R_2) = B_1 \cup B_2$. It follows that all W_1-neighbours of v^+ are in P_1 or P_2. So by Claim 2.6, the only W_1-neighbour of v^+ is x_1. This means that $v^+ \in B_1$. By Claim 2.7, $v^+ = x_1^-$.

By symmetry, we can show that $v^- \in B_1$ and $v^- = y_2^+$. Thus, $R_2 = v^- v v^+$ with all v^-, v, v^+ in B_1.

Consider R_1. If $y_1^{++} \notin R_1$, then $y_1^+ = x_2^-$ and $R_1 = y_1^+$. So $y_1^+ \in B_2$. Thus, $B_1 = \{v^-, v, v^+\}$, $B_2 = \{y_1^+\}$ and $A = \emptyset$. Since $y_1^+ \in B_2$, $y_1^+ u$, $y_1^+ x_1$ and $y_1^+ y_2$ are not edges of G. From this, Claim 3.2 and Claim 3.3 it is not difficult to see that G is a proper spanning subgraph of G_6^5, contradicting the choise of G because G_6^5 is non-hamiltonian by Lemma 5. Thus $y_1^{++} \in R_1$. If $y_1^{++} \neq x_2^-$, then y_1^{++} has a W_1-neighbour u_1. By symmetry, without loss of generality we may assume that u_1 is in P_1. If $u_1 \neq y_1$, then $C' = vuu_1^+ \overrightarrow{C} y_1^{++} u_1 \overleftarrow{C} v^+ y_1^{+++} \overrightarrow{C} v$ is a Hamilton cycle of G. If $u_1 = y_1$, then $C' = vuy_1^- \overleftarrow{C} v^+ y_1^+ y_1 y_1^{++} \overleftarrow{C} v$ is a Hamilton cycle of G. These contradictions show that $y_1^{++} = x_2^-$ and therefore $R_1 = y_1^+ x_2^-$. If $y_1^+ \in B_2$, then y_1^+ has a W_1-neighbour $u_1 \neq y_1$. If u_1 is in P_1, then $C' = vuu_1^+ \overrightarrow{C} y_1^+ u_1 \overleftarrow{C} v^+ x_2^- \overrightarrow{C} v$ is a Hamilton cycle of G, a contradiction. If u_1 is in P_2, then $u_1 \neq y_2$ because otherwise $y_2 \in A_2$ and therefore $v^- \in B_2$, contradicting the fact that $v^- \in B_1$ as shown in the preceding paragraph. It follows that $u_1^+ \in B_m$ and $C' = vuu_1^+ \overrightarrow{C} v^- x_2^- \overrightarrow{C} u_1 y_1^+ \overleftarrow{C} v$ is a Hamilton cycle of G, a contradiction. So $y_1^+ \in B_1$. Similarly, $x_2^- \in B_1$. Thus, $B_1 = \{v^-, v, v^+, y_1^+, x_2^-\}$, $B_2 = \emptyset$ and $A = \emptyset$. Using Claim 3.2 it is not difficult to show that G is F_7^5 in this subcase.

Case 2. $B_1 = \emptyset$.
By Claim 3.5, $B_2 \neq \emptyset$. Then $A_2 \neq \emptyset$. Let $u \in A_2$ and C be a Hamilton cycle of G_u. We again divide this case into two subcases.

Subcase 2.1. Every A_2-vertex has exactly one B_2-neighbour.
Let v be the only B_2-neighbour of u. Then at least one of v^- and v^+ is in V_2, say v^+. Then v^+ must be in B_2 and therefore it has a W_1-neighbour u_1 such that $u_1^- \neq v^+$. If $u_1^- \in B_m$, then $C' = vuu_1^- \overleftarrow{C} v^+ u_1 \overrightarrow{C} v$ is a Hamilton cycle of G, a contradiction. So $u_1^- \in B_2$. It follows that u_1 is an A_2-vertex which has two B_2-neighbours, namely, v^+ and u_1^-. This contradicts the assumption of this subcase. Thus, Subcase 2.1 cannot occur.

Subcase 2.2. There exists an A_2-vertex u such that u has two B_2-neighbours v_1 and v_2.
Since v_1 has only one W_1-neighbour in G_u, one of v_1^- and v_1^+ is in V_2. Similarly, one of v_2^- and v_2^+ is in V_2. The following assertions are easily proved by contradiction.

Claim 3.6. $v_1^+ \neq v_2$ and $v_2^+ \neq v_1$.

($C' = v_1 u v_2 \overrightarrow{C} v_1$ if $v_1^+ = v_2$.)

Claim 3.7. v_1^+ is not adjacent to v_2^+ and v_1^- is not adjacent to v_2^-.

($C' = v_1 u v_2 \overleftarrow{C} v_1^+ \; v_2^+ \overrightarrow{C} v_1$ if v_1^+ is adjacent to v_2^+.)

In particular, v_1^+ and v_2^+ (resp. v_1^- and v_2^-) cannot be in V_2 simultaneously. It follows that either v_1^+ and v_2^- are in V_2 or v_1^- and v_2^+ are in V_2. For definiteness, assume that v_1^+ and v_2^- are in V_2. Then v_1^- and v_2^+ must be in W_1. Thus G has the property ($\bullet$) with $k = 2$. Set $Q_1 = v_1^+ \overrightarrow{C} v_2^-$ and $Q_2 = v_2^+ \overrightarrow{C} v_1^-$. If $v_1^+ = v_2^-$, then $Q_1 = v_1^+$. So Q_1 contains no W_1-vertices. Therefore, all W_1-neighbours of v_1^+ are in Q_2, contradicting Claim 2.5. Thus,

Claim 3.8. $v_1^+ \neq v_2^-$.

By Claim 3.4, all neighbours of u but v_1 and v_2 are in B_m. So we have $|B_m| = |N(u)| - 2 \geq (m-2) - 2 = m - 4$. Together with Claim 2.4 we have $m - 4 \leq |B_m| \leq m - 1 - t$, where t is the number of paths $P_i = x_i \overrightarrow{C} y_i$ defined for a graph with the property ($\bullet$) as in Section 2. It follows that the ordered pair $(|B_m|, t)$ is equal to one of $(m-2, 1)$, $(m-3, 1)$, $(m-3, 2)$, $(m-4, 1)$, $(m-4, 2)$ and $(m-4, 3)$.

If $(|B_m|, t)$ is one of $(m-2, 1)$, $(m-3, 2)$ and $(m-4, 3)$, then the number of W_1-vertices in $P_1 \cup \ldots \cup P_t$ is $m - 1 = |W_1|$. So all W_1-vertices are in $P_1 \cup \ldots \cup P_t$. By Claim 2.5 and Claim 2.6, v_1^+ has at most one W_1-neighbour, contradicting $v_1^+ \in B_2$.

If $(|B_m|, t)$ is one of $(m-3, 1)$ and $(m-4, 2)$, then the number of W_1-vertices in $P_1 \cup \ldots \cup P_t$ is $m - 2$ and there is exactly one W_1-vertex outside $P_1 \cup \ldots \cup P_t$, say u_1. If u_1 is in Q_2 or u_1 is in Q_1 but all $P_1, \ldots, P_t$ are in Q_2, then again by Claim 2.5 and Claim 2.6 v_1^+ has at most one W_1-neighbour, contradicting $v_1^+ \in B_2$. If u_1 and some P_i are in Q_1, then at least one of inequalities $v_1^+ \neq u_1^-$ and $u_1^+ \neq v_2^-$ is true. By symmetry, without loss of generality we may assume that $u_1^+ \neq v_2^-$. If v_2^- is adjacent to u_1, then u_1 has three B_2-neighbours, namely, u_1^-, u_1^+ and v_2^-. This contradicts Claim 3.4. It follows by Claim 2.5 and Claim 2.6 that v_2^- has only one W_1-neighbour, contradicting $v_2^- \in B_2$.

If $(|B_m|, t)$ is $(m-4, 1)$, then the number of W_1-vertices in P_1 is $m - 3$ and there are exactly two W_1-vertices outside P_1, say u_1 and u_2. If P_1 is in

Q_2 and $\{v_1^-, v_2^+\} \not\subseteq V(P_1)$, then Q_1 has at most one W_1-vertex. By Claim 2.5, v_1^+ has at most one W_1-neighbour, a contradiction. If P_1 is in Q_2 and $\{v_1^-, v_2^+\} \subseteq V(P_1)$, then both u_1 and u_2 are in Q_1. For definiteness without loss of generality we may assume that u_1 is in $v_1^+ \overrightarrow{C} u_2^-$. Then $u_1^+ \neq v_2^-$. If v_2^- is adjacent to u_1, then u_1 has three B_2-neighbours, namely, u_1^-, u_1^+ and v_2^-. This contradicts Claim 3.4. It follows that v_2^- has at most one W_1-neighbour by Claim 2.5, contradicting again $v_2^- \in B_2$. Finally, let P_1 be in Q_1. Then at most one W_1-vertex from $\{u_1, u_2\}$ may be in Q_1 because v_1^- and v_2^+ are in W_1. If none of u_1 and u_2 is in Q_1, then by Claim 2.5 and Claim 2.6, v_1^+ has only one W_1-neighbour, contradicting $v_1^+ \in B_2$. If there is one W_1-vertex from $\{u_1, u_2\}$ in Q_1, then by arguments similar to those for the last situation in the preceding paragraph we can get a contradiction.

Thus, Subcase 2.2 also cannot occur. ■

Proposition 11. *Let $G = S(V_1 \cup V_2, E)$ be a split graph with $|V_1| = |V_2| = m$ and $\delta(G) \geq m-2$. Then G has a Hamilton cycle if and only if $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$ with $|S| = m-2$ or $m-1$, unless $m = 4$ and $G - u$ is the graph G_4^3 for some $u \in V_1$.*

Proof. The necessity follows from Lemma 3. Now we prove the sufficiency. Let $G = S(V_1 \cup V_2, E)$ be a maximal non-hamiltonian split graph satisfying $|V_1| = |V_2| = m$, $\delta(G) \geq m-2$ and $|N(S)| > |S|$ for any $\emptyset \neq S \subseteq V_1$ with $|S| = m-2$ or $m-1$. By Lemma 6, $B_3 = B_4 = \ldots = B_{m-1} = \emptyset$. The following assertions are true.

Claim 3.9. $B_1 = \emptyset$.

Suppose the otherwise that $B_1 \neq \emptyset$. Let $v \in B_1$ and $N_{V_1}(v) = \{u\}$. Then $|N_G(V_1 - u)| \leq |V_2 - v| = |V_2| - 1 = |V_1| - 1 = |V_1 - u|$, contradicting $|N_G(V_1 - u)| > |V_1 - u|$.

Claim 3.10. $B_2 \neq \emptyset$.

Suppose the otherwise that $B_2 = \emptyset$. Since B_1 is also empty by Claim 3.9, we have $V_2 = B_m$. Therefore, G contains the complete bipartite graph $K_{m,m}$ with the bipartition $V = V_1 \cup V_2$. So G has a Hamilton cycle, a contradiction.

For any $u \in V_1$, by Proposition 9, $G_u = G - u = S(W_1 \cup V_2, E_u)$ where $W_1 = V_1 - u$ has a Hamilton cycle C, unless $|W_1| = m - 1 = 3$ and G_u is the graph G_4^3.

First assume that for any $u \in V_1$ the graph G_u has a Hamilton cycle C with a fixed orientation $\overrightarrow{C}$. Let $w_1, \ldots, w_{m-1}$ be the vertices of W_1, occurring on $\overrightarrow{C}$ in the order of their indices. Since $|V_2| = m$, it is not difficult to see that the following assertion is true.

Claim 3.11. There exists exactly one of $T_1 = w_1\overrightarrow{C}w_2, \ldots\ldots, T_{m-1} = w_{m-1}\overrightarrow{C}w_1$, which contains exactly two V_2-vertices. Each of the others T_i contains exactly one V_2-vertex.

Now we consider separately two cases.

Case 1. There exists $u \in A_2$ which has two different B_2-neighbours v_1 and v_2.
Then $v_1^+ \neq v_2$ because otherwise $C' = v_1uv_2\overrightarrow{C}v_1$ is a Hamilton cycle of G. Since v_1 (resp. v_2) has only one W_1-neighbour, one of v_1^- and v_1^+ (resp. v_2^- and v_2^+) is in V_2. For definiteness, without loss of generality we may assume that $v_1^+ \in V_2$. Then v_2^+ cannot be in V_2 because otherwise $C' = v_1uv_2\overleftarrow{C}v_1^+v_2^+\overrightarrow{C}v_1$ is a Hamilton cycle of G, a contradiction. So $v_2^- \in V_2$. If $v_1^+\overrightarrow{C}v_2^-$ has no W_1-vertices, then $v_1\overrightarrow{C}v_2$ has at least three V_2-vertices. This contradicts Claim 3.11 because $v_1\overrightarrow{C}v_2$ must be contained in some T_i. If $v_1^+\overrightarrow{C}v_2^-$ has a W_1-vertex, then $v_1^+ \neq v_2^-$ and there exist two different T_i and T_j such that T_i contains $\{v_1, v_1^+\}$ and T_j contains $\{v_2^-, v_2\}$. This contradicts Claim 3.11 again.

Case 2. Every A_2-vertex has exactly one B_2-neighbour.
By arguments similar to those used for Subcase 2.1 of Proposition 10, we can get a contradiction in this case. So Case 2 also cannot occur.
Thus, there exists $u \in V_1$ such that G_u does not have a Hamilton cycle. So $|W_1| = m - 1 = 3 \Leftrightarrow m = 4$ and G_u is G_4^3. ■

***Proof of Theorem* 1.** The necessity follows from Lemma 3 and the sufficiency follows from Propositions 10 and 11. ■

***Proof of Corollary* 2.** If a bipartite graph $G = B(V_1 \cup V_2, E)$ with $|V_1| = m$ and $|V_2| = n$ has a Hamilton cycle C, then vertices of V_1 and V_2 occur on C alternatively. It follows that m must be equal to n. So we may assume further that $|V_1| = |V_2| = m$. Let $G' = S(V_1 \cup V_2, E')$ be the split graph obtained from G by adding to E all edges joining any two different vertices of V_2. It is not difficult to show that G has a Hamilton cycle if and only if G' does. Now Corollary 2 follows from Proposition 11. ■

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