

## ON LOCAL STRUCTURE OF 1-PLANAR GRAPHS OF MINIMUM DEGREE 5 AND GIRTH 4

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### Abstract

A graph is 1-planar if it can be embedded in the plane so that each edge is crossed by at most one other edge. We prove that each 1-planar graph of minimum degree 5 and girth 4 contains

- (1) a 5-vertex adjacent to an  $\leq 6$ -vertex,
- (2) a 4-cycle whose every vertex has degree at most 9,
- (3) a  $K_{1,4}$  with all vertices having degree at most 11.

**Keywords:** light graph, 1-planar graph, star, cycle.

**2000 Mathematics Subject Classification:** 05C10.

### 1. INTRODUCTION

Throughout this paper, we consider connected graphs without loops or multiple edges. We use the standard graph terminology by [6]. By a  $k$ -path (a  $k$ -cycle) we mean a path  $P_k$  (a cycle  $C_k$ ) on  $k$  vertices. A  $k$ -star  $S_k$  is the complete bipartite graph  $K_{1,k}$ . A vertex of degree  $k$  is called a  $k$ -vertex, a vertex of degree  $\geq k$  ( $\leq k$ ) an  $\geq k$ -vertex (an  $\leq k$ -vertex, respectively). For a plane graph  $G$ , the size of a face  $\alpha \in G$  is the length of the minimal boundary walk of  $\alpha$ ; a face of size  $k$  (or  $\geq k$ ) is called a  $k$ -face (an  $\geq k$ -face, respectively).

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\*This work was supported by Science and Technology Assistance Agency under the contract No. APVV-0007-07, and by Slovak VEGA Grant 1/3004/06.

A graph is *1-planar* if it can be embedded in the plane so that each edge is crossed by at most one other edge. 1-planar graphs were first considered by Ringel [11] in connection with the simultaneous vertex/face colouring of plane graphs (note that the graph of adjacency/incidence of vertices and faces of a plane graph is 1-planar); in the mentioned paper, he proved that each 1-planar graph is 7-colourable (in [5], a linear time algorithm for 7-colouring of 1-planar graphs is presented). Borodin [2, 3] proved that each 1-planar graph is 6-colourable (and the bound 6 is best possible), and in [4], it was proved that each 1-planar graph is acyclically 20-colourable. The global structural aspects of 1-planar graphs were studied in [12] and [10]. The local structure of 1-planar graphs was studied in detail in [7].

But, comparing to the family of all plane graphs, the family of 1-planar graphs is still only little explored. In particular, the complete information on dependence of girth of 1-planar graphs according to their minimum degree is not known. From Euler polyhedral formula, one obtains that each planar graph of minimum degree  $\delta \geq 3$  has the girth at most 5, and if  $\delta \geq 4$ , then the girth is 3. For 1-planar graphs, it was proved in [7] that each 1-planar graph of minimum degree  $\delta \geq 5$  has girth at most 4 (and there are graphs that reach this bound) and if  $\delta \geq 6$ , then the girth is 3. In addition, R. Soták (personal communication) found an example of 1-planar graph with  $\delta = 4$  and girth 5, and in [7], there is an example of 1-planar graph of minimum degree 3 and girth 7; we conjecture that the values 5 and 7 are best possible.

The aim of this paper is to explore in deeper details the local properties of 1-planar graphs of minimum degree 5. Our motivation comes from the research of structure of plane graphs of minimum degree 5. It is known that such graphs contain a variety of small configurations having vertices of small degrees. For example, Borodin [1] proved that each plane graph of minimum degree 5 contains a triangular face of weight (that is, the sum of degrees of its vertices) at most 17 (the bound being sharp). Other similar results may be found in [8] or [9]. It appears that analogical results hold also for 1-planar graphs with sufficiently high minimum degree. In [7], it was proved that each 1-planar graph of minimum degree  $\geq 6$  contains a 3-cycle with all vertices of degree at most 10 as well as 4-star with all vertices of degrees at most 23, and if the minimum degree is 7, then it contains also a 6-star with all vertices of degree at most 15. On the other hand, such result do not apply in general on 1-planar graphs of minimum degree  $\geq 5$ : in [7], there are examples of 1-planar graphs of minimum degree 5 such that all their 3-cycles have arbitrarily high degree-sum of vertices.

In this paper, we show that, under the additional requirement of girth being 4, 1-planar graphs of minimum degree  $\geq 5$  also contain certain small subgraphs with vertices of small degrees. We prove

**Theorem 1.** *Each 1-planar graph of minimum degree 5 and girth 4 contains a 5-vertex which is adjacent to an  $\leq 6$ -vertex.*

**Theorem 2.** *Each 1-planar graph of minimum degree 5 and girth 4 contains a 4-cycle such that each its vertex has degree at most 9.*

**Theorem 3.** *Each 1-planar graph of minimum degree 5 and girth 4 contains a 4-star such that each its vertex has degree at most 11.*

In Theorems 1 and 2, the requirement of girth 4 cannot be avoided: in [7], it was shown that 4-cycles and 4-stars in 1-planar graphs of minimum degree 5 may reach arbitrarily high degrees. This requirement is also substantial for the first result, since there is an example of 1-planar graph of minimum degree 5 such that each its 5-vertex is adjacent only with  $\geq 8$ -vertices (see Figure 1). An open question is whether each 1-planar graph of minimum degree 5 and girth 4 contains a pair of adjacent 5-vertices.

As any 1-planar graph  $G$  with girth at least 5 has  $\delta(G) \leq 4$ , we obtain that  $G$  is 5-colourable. It is not known whether an analogical result holds for 1-planar graphs of girth at least 4. The example of infinite 1-planar graph of minimum degree 4 and girth 6 (constructed from hexagonal tiling of the plane, see Figure 2) together with the example of Soták concerning 1-planar graph of minimum degree 4 and girth 5 suggest to conjecture that each 1-planar graph with  $\delta \geq 4$  has girth at most 5; in this case, any 1-planar graph of girth at least 6 would be of minimum degree at most 3, hence, it would be 4-colourable.

## 2. PROOFS

### 2.1. Basic terms

The following definitions are taken from [7].

Let  $G$  be a 1-planar graph and let  $D(G)$  be its 1-planar diagram (a drawing of  $G$  in which every edge is crossed at most once). Given two nonadjacent edges  $xy, uv \in E(G)$ , the *crossing* of  $xy, uv$  is the common point of two arcs  $\widehat{xy}, \widehat{uv} \in D(G)$  (corresponding to edges  $xy$  and  $uv$ ). Denote by  $C = C(D(G))$  the set of all crossings in  $D(G)$  and by  $E_0$  the set of all

non-crossed edges in  $D(G)$ . The *associated plane graph*  $D^\times(G)$  of  $D(G)$  is the plane graph such that  $V(D^\times(G)) = V(D(G)) \cup C$  and  $E(D^\times(G)) = E_0 \cup \{xz, yz \mid xy \in E(D(G)) - E_0, z \in C, z \in xy\}$ . Thus, in  $D^\times(G)$ , the crossings of  $D(G)$  become new vertices of degree 4; we call these vertices *crossing vertices*. The vertices of  $D^\times(G)$  which are also vertices of  $G$  are called *true*.

Note that a 1-planar graph may have different 1-planar diagrams, which lead to nonisomorphic associated plane graphs. Among all possible 1-planar diagrams of a 1-planar graph  $G$ , we denote by  $M(G)$  such a diagram that has the minimum number of crossings (it is not necessarily unique), and by  $M^\times(G)$  its associated plane graph.

All results of this paper are proved in a common way: we proceed by contradiction, thus, we consider a hypothetical counterexample  $G$  which does not contain the specified graph. Further, on the plane graph  $M^\times(G) = (V^\times, E^\times, F^\times)$ , the Discharging method is used. We define the charge  $c : V^\times \cup F^\times \rightarrow \mathbb{Z}$  by the assignment  $c(v) = \deg_G(v) - 6$  for all  $v \in V^\times$  and  $c(\alpha) = 2\deg_G(\alpha) - 6$  for all  $\alpha \in F^\times$ . From the Euler polyhedral formula, it follows that  $\sum_{x \in V^\times \cup F^\times} c(x) = -12$ . Next, we define the local redistribution of charges between the vertices and faces of  $M^\times(G)$  such that the sum of charges remain the same. This is performed by certain rules which specify the charge transfers from one element to other elements in specific situations. After such redistribution, we obtain a new charge  $\tilde{c} : V^\times \cup F^\times \rightarrow \mathbb{Q}$ . To get the contradiction, we prove that for any element  $x \in V^\times \cup F^\times$ ,  $\tilde{c}(x) \geq 0$  (hence,  $\sum_{x \in V^\times \cup F^\times} \tilde{c}(x) \geq 0$ ).

For the purposes of these proofs, we introduce some specialized notations. Given a  $d$ -vertex  $x \in M^\times(G)$ , by  $x_1, \dots, x_d$  we denote its neighbours in  $M^\times(G)$  in the clockwise order. By  $f_i, i = 1, \dots, d$ , we denote the face of  $M^\times(G)$  which contains the path  $x_i x x_{i+1}$  (index modulo  $d$ ) as a part of its boundary walk. If  $f_i$  is a 4-face, then  $x'_i$  will denote the common neighbour of  $x_i$  and  $x_{i+1}$  which is different from  $x$ .

## 2.2. Proof of Theorem 1

The proof proceeds in the way described in Subsection 2.1 under the following discharging rules:

**Rule 1.** Each face  $f \in F^\times$  sends  $\frac{c(f)}{m(f)}$  to each incident 4- and 5-vertex, where  $m(f)$  is the number of 4- and 5-vertices incident with  $f$ . If  $m(f) = 0$ , no charge is transferred.

**Rule 2.** Each  $\geq 7$ -vertex  $v$  sends  $\frac{c(v)}{\deg_G(v)}$  to each 5-vertex which is incident with  $v$  in  $G$ .

Let  $\bar{c}(x)$  be the charge of a vertex  $x \in V^\times$  after application of Rules 1 and 2.

**Rule 3.** Each 5-vertex  $v$  with  $\bar{c}(v) > 0$  sends  $\frac{\bar{c}(v)}{\bar{m}(v)}$  to each adjacent 4-vertex;  $\bar{m}(v)$  is the number of 4-vertices adjacent to  $v$  (if  $\bar{m}(v) = 0$ , no charge is transferred).

We will check the final charge of vertices and faces after the charge redistribution. From Rule 1, it follows that  $\tilde{c}(f) \geq 0$  for each face  $f \in F^\times$ . Similarly, the Rule 2 ensures that  $\tilde{c}(v) \geq 0$  for any  $\geq 7$ -vertex  $v \in V^\times$ . Since, for a 6-vertex  $v$ ,  $\tilde{c}(v) = c(v) = 0$ , it is enough to check the final charge of 5- and 4-vertices.

*Case 1.* Let  $x$  be a 5-vertex. Then all its neighbours in  $G$  are  $\geq 7$ -vertices, thus, by Rule 2,  $x$  receives at least  $5 \cdot \frac{1}{5}$

$$\bar{c}(x) \geq -1 + 5 \cdot \frac{1}{5} \quad - \quad -$$

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$$\bar{c}(x_2) \geq -1 + \frac{5}{5} \quad - \quad - \quad \bar{c}(x_3) \geq \frac{17}{5} \\ - \quad - \quad \frac{17}{5} = \frac{67}{210} > 0.$$

Now, let one of  $x_1, x_4$  be a 5-vertex, say  $x_1$ . Then  $x_3, x_4$  are  $\geq 7$ -vertices and  $\tilde{c}(x) \geq -2 + 1 + \frac{2}{5} \quad - \quad -$

*Case 2.3.* Let  $x$  be incident with exactly two  $\geq 4$ -faces. Note that the remaining 3-faces in the neighbourhood of  $x$  cannot be adjacent (otherwise their vertices form a 3-cycle); thus, assume that  $f_4$  and  $f_2$  are 3-faces. Moreover, at most two neighbours of  $x$  in  $M^\times(G)$  are 5-vertices.

*Case 2.3.1.* If all vertices  $x_1, \dots, x_4$  are  $\geq 6$ -vertices, then  $\tilde{c}(x) \geq -2 + 2 \cdot 1 = 0$ .

*Case 2.3.2.* Let exactly one of  $x_1, \dots, x_4$  be a 5-vertex, say  $x_1$ . Then  $x_3, x_4$  are  $\geq 7$ -vertices,  $x_2$  is a  $\geq 6$ -vertex. If  $f_1$  is an  $\geq 5$ -face, then  $\tilde{c}(x) \geq -2 + 1 + \frac{2 \cdot 5 - 6}{5 - 1} = 0$ ; hence, assume that  $f_1$  is a 4-face.

If  $x'_1$  is a true vertex, then it is an  $\geq 7$ -vertex and  $\tilde{c}(x) \geq -2 + 1 + 1 = 0$ ; thus, assume that  $x'_1$  is a crossing-vertex. Denote by  $w$  and  $y$  the remaining neighbours of  $x_1$  (such that the neighbours of  $x_1$  in  $M^\times(G)$  in clockwise ordering are  $w, y, x'_1, x, x_4$ ) and let  $f', f'', f'''$  be faces that contain, as a part of the boundary, the paths  $x'_1xy, yxw$  and  $wxx_4$ , respectively.

Suppose first that  $f'''$  is an  $\geq 4$ -face. If  $w$  is a true vertex, then it is  $\geq 7$ -vertex and  $\bar{c}(x_1) \geq -1 + 1 + \frac{2}{7} = \frac{29}{7} - \frac{1}{3} = \frac{8}{63} > 0$ . Thus, let  $w$  be a crossing-vertex. Now, if  $y$  is a true vertex, then it is also an  $\geq 7$ -vertex and  $\bar{m}(x_1) \leq 3$ ; we have  $\bar{c}(x_1) \geq -1 + 5 \cdot \frac{1}{7} - \frac{1}{3} = \frac{1}{63} > 0$ . Hence, both  $y$  and  $w$  are crossing-vertices; but then  $f''$  is necessarily an  $\geq 4$ -face. In this case, we obtain  $\bar{c}(x_1) \geq -1 + 5 \cdot \frac{1}{7} - \frac{1}{4} = \frac{3}{56} > 0$ .

Now, let  $f'''$  be a 3-face. Then  $w$  is a crossing-vertex and  $f''$  is an  $\geq 4$ -face (otherwise either 3-cycle appears or 1-planarity of  $G$  is violated). If  $y$  is a true vertex, then it is an  $\geq 7$ -vertex and we have  $\bar{c}(x_1) \geq -1 + 5 \cdot \frac{1}{7} - \frac{1}{3} = \frac{1}{63} > 0$ ; so, suppose that  $y$  is a crossing-vertex. But then  $f'$  is necessarily an  $\geq 4$ -face and we obtain  $\bar{c}(x_1) \geq -1 + 5 \cdot \frac{1}{7} + \frac{2}{3} + 2 \cdot \frac{1}{4} - \frac{1}{4} = \frac{1}{84} > 0$ .

*Case 2.3.3.* Let two of  $x_1, \dots, x_4$  be 5-vertices, say  $x_1$  and  $x_2$ . Then  $x_3, x_4$  are  $\geq 7$ -vertices. If  $f_1$  is an  $\geq 6$ -face, then  $\tilde{c}(x) \geq -2 + 1 + \frac{2 \cdot 6 - 6}{6} = 0$ . If  $f_1$  is a 5-face, then it is incident with at most two crossing-vertices, which implies that there exists an  $\geq 7$ -vertex incident with  $f_1$  which is adjacent with  $x_1$  or  $x_2$ ; this yields  $m(f_1) \leq 4$  and  $\tilde{c}(x) \geq -2 + 1 + \frac{2 \cdot 5 - 6}{4} = 0$ . So, let  $f_1$  be a 4-face.

If  $x'_1$  is a true vertex, then it is an  $\geq 7$ -vertex; in this case,  $\bar{c}(x_1) \geq -1 + 5 \cdot \frac{1}{7} - \frac{1}{4} - \frac{1}{4} = \frac{37}{28} - \frac{1}{4} = \frac{3}{28} > 0$ .

Hence, assume that  $x'_1$  is a crossing-vertex. Now, consider the charge of the vertex  $x_1$  after application of Rules 1 and 2 (the situation for the vertex  $x_2$  is symmetric). Let  $w, y, f', f'', f'''$  be the same elements as defined in the case 2.3.2.

If  $f''$  is a 3-face, then  $w$  is a crossing-vertex; subsequently,  $f''$  has to be an  $\geq 4$ -face. Independently of this, if  $f'$  is a 3-face, then  $y$  is not a crossing-vertex, hence, it is an  $\geq 7$ -vertex. Thus, we consider four possibilities:

- (a)  $f''', f'$  are 3-faces. Then  $\overline{m}(x_1) = 3$  and  $\overline{c}(x_1) \geq -1 + 5 \cdot \frac{1}{126}$  — — — —  
 $\overline{m}(x_1) \leq 4$   
 and  $\overline{c}(x_1) \geq -1 + 5 \cdot \frac{1}{126}$  — — — —  
 $\frac{1}{126}$  — — — —  $\overline{m}(x_1) = 3$  and  $\overline{c}(x_1) \geq -1 + 5 \cdot \frac{1}{126}$   
 $\frac{1}{126}$  — — — —  $\overline{m}(x_1) \leq 4$  and  $\overline{c}(x_1) \geq -1 + 5 \cdot \frac{1}{126}$

$\frac{1}{126}$  (the bound being attained in the first and third case). The same considerations apply for the vertex  $x_2$ . Hence, in total,  $x$  receives at least  $1 + \frac{1}{126} = \frac{263}{126}$  ■

### 2.3. Proof of Theorem 2

The proof proceeds in the way described in Subsection 2.1 under the following discharging rules (for the purpose of this proof, a big vertex is one of degree  $\geq 10$ ):

**Rule 1.** Each face  $f \in F^\times$  sends  $\frac{c(f)}{m(f)}$  to each incident 4- and 5-vertex, where  $m(f)$  is the number of 4- and 5-vertices incident with  $f$ . If  $m(f) = 0$ , no charge is transferred.

**Rule 2.** Each big vertex sends  $\frac{2}{10}$

$\overline{c}(x)$  be the charge of a vertex  $x \in V^\times$  after application of Rules 1 and 2.

**Rule 3.** Each 5-vertex  $v$  with  $\overline{c}(v) > 0$  sends  $\frac{\overline{c}(v)}{\overline{m}(v)}$  to each adjacent 4-vertex;  $\overline{m}(v)$  is the number of 4-vertices adjacent to  $v$  (if  $\overline{m}(v) = 0$ , no charge is transferred).

To show that the final charges of elements of  $M^\times(G)$  are nonnegative, we consider several cases; due to the formulation of discharging rules, it is enough to check the final charges of 4-, 5- and big vertices.

*Case 1.* Let  $x$  be a 5-vertex. Then  $x$  is incident with at most three 3-faces (otherwise it is incident with two 3-faces with the common edge having one endvertex of degree 4; thus,  $x$  belongs to a 3-cycle of  $G$ , a contradiction). By Rule 1,  $\tilde{c}(x) \geq -1 + 2 \cdot \frac{2 \cdot 4 - 6}{4} = -1 + 2 \cdot \frac{1}{4}$

if at least two of them are  $\geq 5$ -faces, then  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{2 \cdot 5 - 6}{5} + \frac{1}{6} = 0$ ; similarly,

then  $\tilde{c}(x) \geq -2 + \frac{4}{4-1} = \frac{2}{15} > 0$ . Hence, we may assume that both  $x_2$  and  $x_3$  are 5-vertices. Then each of them is incident with at least three  $\geq 4$ -faces, so  $\bar{c}(x_2) \geq -1 + \frac{4}{4-1} = \frac{3}{3} = 1$  and  $\bar{c}(x_3) \geq -1 + 3 \cdot \frac{1}{5} = -\frac{2}{5}$ . Thus,  $\tilde{c}(x) \geq -\frac{2}{5} + \frac{1}{5} = -\frac{1}{5} > 0$ .

Suppose next that  $f_2$  is a 5-face. If at least one of  $x_2, x_3$  is an  $\geq 6$ -vertex, then  $\tilde{c}(x) \geq -2 + \frac{1}{4-1} + \frac{2 \cdot 5 - 6}{5-1} > 0$ ; thus, assume that both  $x_2, x_3$  are 5-vertices. Then again,  $x_2$  and  $x_3$  are incident with at least three  $\geq 4$ -faces and  $\bar{c}(x_2) \geq -1 + 2 \cdot \frac{1}{5} = -\frac{3}{5}$  and  $\bar{c}(x_3) \geq -1 + 2 \cdot \frac{1}{5} = -\frac{3}{5}$ . Thus,  $\tilde{c}(x) \geq -\frac{3}{5} - \frac{3}{5} = -\frac{6}{5} > 0$ .

*Case 2.2.2.* Let each of  $f_1, f_2, f_3$  be a 4-face.

If both vertices  $x_2, x_3$  are of degree  $\geq 6$ , then  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{1}{2} = 0$ ; also, it is easy to check that  $\tilde{c}(x) \geq 0$  if two of  $x_1, \dots, x_4$  (except of  $x_1, x_4$ )



are of degree  $\geq 6$ . If both  $x_1, x_4$  are  $\geq 6$ -vertices, then  $\bar{c}(x_2) \geq -1 + \frac{2}{5} - \frac{1}{5} = \frac{1}{5} > 0$ .  
 Moreover, if at least one of  $x_1, \dots, x_4$  is big, then  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{1}{5} = -\frac{8}{5}$ .

$\bar{c}(x_1) \geq -1 + \frac{1}{3} - \frac{1}{3} = -\frac{1}{3}$ ,  $\bar{c}(x_2) \geq -1 + 2 \cdot \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$ ,  $\bar{m}(x_1) \leq 3$ ,  $\bar{m}(x_2) \leq 4$  and  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{1}{3} - \frac{16}{3} + \frac{16}{4} = \frac{11}{4}$ .

$$\bar{c}(x_3) \geq -1 + 2 \cdot \frac{1}{3} - \frac{1}{3} = \frac{1}{3}, \quad \bar{m}(x_3) \leq 4 \text{ and } \tilde{c}(x) \geq -2 + 2 \cdot \frac{1}{3} - \frac{16}{3} + \frac{16}{4} = \frac{17}{4}.$$

$\bar{c}(x_3) \geq -1 + 3 \cdot \frac{1}{4} - \frac{1}{4} = \frac{1}{4}$ ,  $\bar{m}(x_3) \leq 4$  and  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{1}{4} - \frac{1}{4} = -\frac{3}{4} > 0$ . So, it remains to resolve the case when  $x'_2$  is not big and both  $x_2, x_3$  are 5-vertices. Consider the charge of  $x_2$  after application of the first two discharging rules (the case of  $x_3$  is analogous). If  $x_2$  is incident with at most one 3-face, then  $\bar{c}(x_2) \geq -1 + 4 \cdot \frac{1}{10} - \frac{1}{10} = -\frac{9}{10}$ ,  $\bar{m}(x_2) \leq 4$ , thus,  $x_2$  contributes at least  $\frac{1}{10}$ .

$\bar{c}(x_2) \geq -1 + 3 \cdot \frac{1}{10} - \frac{1}{10} = -\frac{9}{10}$ ,  $\bar{m}(x_2) \leq 3$  and  $x_2$  may contribute at least  $\frac{9}{10} > \frac{1}{10}$ .

$\bar{c}(x_2) \geq -1 + 4 \cdot \frac{1}{10} - \frac{1}{10} = -\frac{6}{10}$ ,  $\bar{m}(x_2) \leq 4$  and  $x_2$  contributes at least  $\frac{1}{10}$ .

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Suppose now that  $x'_2$  is a big vertex. If some of  $x_2, x_3$  is an  $\geq 6$ -vertex, then  $\tilde{c}(x) \geq -2 + \frac{1}{10} - \frac{1}{10}$   
 $\bar{c}(x_2) \geq -1 + 3 \cdot \frac{1}{10} - \frac{1}{10}, \bar{c}(x_3) \geq -1 + 3 \cdot \frac{1}{10} - \frac{1}{10}$  and  
 $\tilde{c}(x) \geq -2 + 2 \cdot \frac{1}{10} - \frac{1}{10} = \frac{7}{60} > 0$ .

*Case 2.2.2.2.2.* Let each of  $x'_1, x'_2, x'_3$  be a crossing-vertex.

*Case 2.2.2.2.2.1.* Suppose first that exactly one of  $x_1, \dots, x_4$  is an  $\geq 6$ -vertex; due to the symmetry, we will consider  $x_1$  or  $x_2$ . Let  $x_1$  be an  $\geq 6$ -vertex. If  $x_4$  is incident with at most two 3-faces, then  $\bar{c}(x_4) \geq -1 + 3 \cdot \frac{1}{15} - \bar{m}(x_4) \leq 4$  and  $x_4$  may contribute to  $x$  at least  $\frac{1}{15}$

$\bar{c}(x_4) \geq -1 + 3 \cdot \frac{1}{15} - \bar{m}(x_4) \leq 3$  and  $x_4$  may contribute to  $x$  at least  $\frac{2}{15} > \frac{1}{15}$   
 $\bar{c}(x_2) \geq -1 + 3 \cdot \frac{1}{15} - \bar{c}(x_3) \geq -1 + 3 \cdot \frac{1}{15} - \frac{1}{15} + \frac{1}{10} = \frac{1}{40} > 0$ .

Now, let  $x_2$  be an  $\geq 6$ -vertex. Then  $\bar{c}(x_1) \geq -1 + 3 \cdot \frac{1}{15} - \bar{c}(x_3) \geq -1 + 2 \cdot \frac{1}{15} - \frac{1}{15} + \frac{2}{5} = 0$ .

*Case 2.2.2.2.2.2.* Let all neighbours of  $x$  be 5-vertices. Note that each of  $x_2, x_3$  is incident with at least three  $\geq 4$ -faces; this implies that each of them may contribute at least  $\frac{-1+3 \cdot \frac{1}{5}}{4} = \frac{1}{5}$

$\frac{1}{5} - \frac{1}{5} = 0$   
 $\frac{1}{5} - \frac{1}{5} = 0$

$\bar{m}(x_1) = 3, \bar{c}(x_1) \geq -1 + 2 \cdot \frac{1}{5} - \bar{c}(x_2) \geq -1 + 2 \cdot \frac{1}{5} - \frac{1}{5} = \frac{2}{5} > \frac{1}{5}$   
 $\frac{1}{5} - \frac{1}{5} = 0$

*Case 2.3.* Let  $x$  be incident with exactly two 3-faces (note that they are not adjacent). Then (to avoid a light 4-cycle) one of  $x_1, \dots, x_4$  is big, say  $x_1$ . If  $x$  has also another big neighbour then  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{2}{5} - \frac{1}{5} = \frac{1}{5} > 0$

$\frac{1}{5} - \frac{1}{5} = 0$   
 $\frac{1}{5} - \frac{1}{5} = 0$

$\frac{1}{5} - \frac{1}{5} = 0$   
 $\frac{1}{5} - \frac{1}{5} = 0$

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$\frac{1}{5} - \frac{1}{5} = 0$   
 $\frac{1}{5} - \frac{1}{5} = 0$

*Case 2.3.1.* Let  $x_2, x_3, x_4$  be 5-vertices. If  $f_1$  is an  $\geq 5$ -face, then  $\bar{c}(x_2) \geq -1 + 1 + \frac{1}{10}$  -  $\bar{m}(x_2) \leq 4, \bar{c}(x_4) \geq -1 + 2 \cdot \frac{1}{10}$  -  $\bar{m}(x_4) \leq 4$  and  $\tilde{c}(x) \geq -2 + 1 + \frac{1}{4}$  -  $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$   $\bar{c}(x_4) \geq -1 + \frac{4}{10}$  -  $\bar{m}(x_4) \leq 4, \bar{c}(x_2) \geq -1 + \frac{2}{10}$  -  $\bar{m}(x_2) \leq 4$  and  $\tilde{c}(x) \geq -2 + \frac{2}{10}$  -  $\frac{10}{4} + \frac{1}{4} = \frac{1}{12} > 0$ . Hence, we may suppose that both  $f_1$  and  $f_3$  are 4-faces. In addition, we may also suppose that  $x'_3$  is a crossing vertex (otherwise  $x'_3$  is big, since  $x'_3 x_3 x_2 x_4$  is a 4-cycle; then  $\bar{c}(x_3) \geq -1 + \frac{1}{10}$  -  $\bar{c}(x_4) \geq -1 + \frac{1}{10}$  -  $\bar{m}(x_3) \leq 3, \bar{m}(x_4) \leq 3$  and  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{2}{10}$  -  $\frac{10}{3} + \frac{29}{3} = \frac{56}{3}$

$\bar{c}(x_3) \geq -1 + 3 \cdot \frac{1}{10}$  -  $\bar{c}(x_4) \geq -1 + 3 \cdot \frac{1}{10}$  -  $\bar{m}(x_3) \leq 4, \bar{m}(x_4) \leq 4$ . Now, if  $\bar{m}(x_4) \leq 3$ , then  $\tilde{c}(x) \geq -2 + \frac{1}{10}$  -  $\frac{10}{3} + \frac{1}{4} + \frac{1}{4} = \frac{1}{30} > 0$ . Thus, let  $\bar{m}(x_4) = 4$ . Further, if  $\bar{m}(x_3) \leq 3$ , then  $\tilde{c}(x) \geq -2 + \frac{1}{10}$  -  $\frac{10}{4} + \frac{1}{3} + \frac{1}{4} = 0$ ; hence, we may assume that also  $\bar{m}(x_3) = 4$ . Consider now the faces  $\alpha, \beta, \gamma, f_2, f_3$  that appear around  $x_3$  in the counter-clockwise order. We may assume that  $\gamma$  is a 3-face (otherwise  $\tilde{c}(x) \geq -2 + \frac{1}{10}$  -  $\frac{10}{4} + \frac{1}{4} = \frac{1}{12} > 0$ ). If some of  $\alpha, \beta$  is an  $\geq 5$ -face, then  $\bar{c}(x_3) \geq -1 + 2 \cdot \frac{1}{2} + \frac{4}{2}$  -  $\frac{10}{2} + \frac{2}{3} + \frac{2}{3} = \frac{10}{3} + \frac{4}{4} + \frac{1}{4} = \frac{1}{30} > 0$ . Hence, let  $\alpha = [x_3 x'_3 y w], \beta = [x_3 w z q]$  be 4-faces. As  $x_2 x_4 y z$  is a 4-cycle in  $G$ , one of  $y, z$  must be big. Then  $\bar{c}(x_3) \geq -1 + 2 \cdot \frac{1}{10}$  -  $\frac{10}{10} + \frac{2}{4} + \frac{1}{4} = 0$ .

*Case 2.3.1.2.* Let  $x'_3 x_4$  be incident with exactly one  $\geq 4$ -face  $f_3$ . Then there exists a 3-face  $[x'_3 x_4 w]$ . Since  $x_4 w x_3 x_2$  is a 4-cycle in  $G$ ,  $w$  is a big vertex. Moreover, at least one of edges  $w x_4, x_1 x_4$  is incident with an  $\geq 4$ -face. This yields  $\bar{c}(x_4) \geq -1 + \frac{1}{10}$  -  $\bar{m}(x_4) \leq 3$ . Now, if the edge  $x_3 x'_3$  is incident with two  $\geq 4$ -faces, then  $\bar{c}(x_3) \geq -1 + 3 \cdot \frac{1}{10}$  -  $\bar{m}(x_3) \leq 4$  and  $\tilde{c}(x) \geq -2 + \frac{1}{10}$  -  $\frac{10}{3} + \frac{2}{4} + \frac{1}{4} = \frac{1}{18} > 0$ . So, let  $x_3 x'_3$  be incident with exactly one  $\geq 4$ -face  $f_3$ . Then there exists a 3-face  $[x_3 x'_3 w']$ . As  $x_4 x_2 x_3 w$  is a 4-cycle in  $G$ ,  $w'$  is big. Therefore, we have  $\bar{c}(x_3) \geq -1 + 2 \cdot \frac{1}{10}$  -  $\bar{m}(x_3) \leq 3$  and  $\tilde{c}(x) \geq -2 + \frac{1}{10}$  -  $\frac{10}{3} + \frac{2}{3} + \frac{1}{4} = \frac{23}{360} > 0$ .

*Case 2.3.1.3.* Let  $x'_3 x_3$  be incident with exactly one  $\geq 4$ -face  $f_3$ . Then there exists a 3-face  $[x'_3 x_3 w]$ . Again,  $w x_4 x_2 x_3$  is a 4-cycle in  $G$ , so  $w$  is big.

Now, if the edge  $x_3w$  is incident with an  $\geq 4$ -face, then  $\bar{c}(x_3) \geq -1 + \frac{1}{10}$ ,  $\bar{m}(x_3) \leq 3$ ,  $\bar{c}(x_4) \geq -1 + 3 \cdot \frac{1}{10}$ ,  $\bar{m}(x_4) \leq 4$  and  $\tilde{c}(x) \geq -2 + \frac{1}{3} + \frac{\frac{9}{10}}{4} + \frac{1}{4} = \frac{1}{45} > 0$ . In the opposite case, consider the face  $\gamma \neq f_2$  incident with the edge  $x_2x_3$ . If  $\gamma$  is an  $\geq 5$ -face, then  $\bar{c}(x_3) \geq -1 + \frac{1}{10}$ ,  $\bar{c}(x_2) \geq -1 + \frac{1}{10}$ ,  $\bar{m}(x_3) \leq 3$ ,  $\bar{m}(x_4) \leq 4$  and  $\tilde{c}(x) \geq -2 + \frac{1}{4} + \frac{\frac{7}{10}}{3} + \frac{\frac{3}{10}}{4} = \frac{1}{10} > 0$ . Hence, suppose that  $\gamma$  is a 4-face. If  $\bar{m}(x_3) = 2$ , then  $\tilde{c}(x) \geq -2 + \frac{1}{4} + \frac{\frac{2}{10}}{2} + \frac{1}{4} = \frac{1}{30} > 0$ ; so, we may assume that  $\bar{m}(x_3) = 3$ . Now, if  $\bar{m}(x_2) \leq 2$ , then  $\tilde{c}(x) \geq -2 + \frac{1}{4} + \frac{\frac{2}{10}}{3} + \frac{1}{2} = \frac{1}{120} > 0$ . So, assume that  $\bar{m}(x_2) = 3$ . Then  $x_2$  is adjacent to two crossing vertices lying in the face  $\omega \neq f_1$  that is incident with the edge  $x'_1x_2$ , which implies that  $\omega$  is an  $\geq 4$ -face. Then we have  $\bar{c}(x_2) \geq -1 + 2 \cdot \frac{1}{4} + \frac{\frac{2}{10}}{3} + \frac{2}{3} = \frac{53}{360} > 0$ .

*Case 2.3.2.* Let  $x_4$  be an  $\geq 6$ -vertex and  $x_2, x_3$  be 5-vertices; moreover, we can suppose that both  $f_1, f_3$  are 4-faces (otherwise  $\tilde{c}(x) \geq -2 + 1 + \frac{2}{4}$ ).

$\bar{c}(x_2) \geq -1 + 2 \cdot \frac{1}{4} + \frac{\frac{2}{10}}{3} + \frac{2}{3} = \frac{53}{360} > 0$ ; the same holds for  $x_3$  and we get  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{2}{4} + \frac{1}{4} > 0$ .

*Case 2.3.2.2.* Let  $x_3$  be incident with precisely two  $\geq 4$ -faces. Then there exists a 3-face  $[x_3x'_3w]$ . As  $x_4x_2x_3w$  is a 4-cycle in  $G$ ,  $w$  is a big vertex; consequently,  $\bar{m}(x_1) = 3$ . Now, if the edge  $x_3w$  is incident with an  $\geq 4$ -face, then  $\bar{c}(x_3) \geq -1 + 2 \cdot \frac{2}{4} + \frac{1}{4} = \frac{7}{4} > 0$  and  $\bar{m}(x_2) \leq 4$  and  $\bar{c}(x_2) \geq -1 + \frac{1}{3} + \frac{1}{4} = \frac{7}{360} > 0$ . Thus, assume that  $x_3w$  is incident only with 3-faces  $[x_3x'_3w], [x_3wy]$ . If  $y$  is a true vertex, then  $\bar{m}(x_3) = 2$ ,  $\bar{c}(x_3) \geq -1 + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} > 0$ . From  $x_2$ ,  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{2}{4} + \frac{1}{2} + \frac{1}{4} = \frac{7}{120} > 0$ . So, let  $y$  be a crossing-vertex. Consider now the local neighbourhood of vertex  $x_2$ ; let  $\alpha, \beta, \gamma, f_1, f_2$  be faces incident with  $x_2$  in the counter-clockwise order. If  $\alpha$  is an  $\geq 5$ -face, then  $\bar{c}(x_3) \geq -1 + \frac{2}{3} + \frac{1}{4} = \frac{5}{12} > 0$  and  $\bar{c}(x_2) \geq -1 + \frac{2}{3} + \frac{1}{4} = \frac{5}{12} > 0$ ; hence, let  $\alpha = [x_2x_3yz]$  be a 4-face. Then  $z$  is a true vertex, which implies

$\overline{m}(x_2) \leq 3$ . If  $x_2$  is incident with at least three  $\geq 4$ -faces, the  $\tilde{c}(x_2) \geq -1 + 2 \cdot \frac{1}{3} - \frac{2}{3} = \frac{13}{3} > 0$ .

$\overline{m}(x_2) \leq 2$ . In this case, we obtain  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{2}{3} - \frac{1}{3} + \frac{1}{2} = \frac{1}{180} > 0$ .

*Case 2.3.3.* Let  $x_3$  be an  $\geq 6$ -vertex and  $x_2, x_4$  be 5-vertices; again, we can suppose that both  $f_1, f_3$  are 4-faces (otherwise  $\tilde{c}(x) \geq -2 + 1 + \frac{2}{3} - \frac{1}{3} = 0$ ).

$\tilde{c}(x_4) \geq -1 + 2 \cdot \frac{1}{4} - \frac{1}{4} = \frac{1}{4} > 0$ .  
 - - -  $\overline{m}(x_4) \leq 4$  and  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{2}{4} - \frac{1}{4} + \frac{1}{4} = \frac{1}{24} > 0$ ; so, suppose that  $x_4$  is incident with precisely two  $\geq 4$ -faces. Then there exists a 3-face  $[x'_3 x_4 w]$ . As  $x_4 x_2 x_3 w$  is a 4-cycle,  $w$  must be big. Hence,  $\tilde{c}(x_4) \geq -1 + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = -\frac{1}{4} > 0$ .  
 - - -  $\overline{m}(x_4) \leq 3$  and  $\tilde{c}(x) \geq -2 + 2 \cdot \frac{2}{3} - \frac{1}{3} + \frac{1}{4} = \frac{7}{72} > 0$ .

*Case 3.* Let  $x$  be a big  $d$ -vertex. Then  $\tilde{c}(x) \geq d - 6 - \frac{2}{5} - 6 \geq 0$  for  $d \geq 10$ . ■

## 2.4. Proof of Theorem 3

The proof proceeds in the way described in Subsection 2.1 under the following discharging rules (here, the big vertex is one of degree  $\geq 12$ ):

**Rule 1.** Each face  $f \in F^\times$  sends  $\frac{c(f)}{m(f)}$  to each incident 4- and 5-vertex, where  $m(f)$  is the number of 4- and 5-vertices incident with  $f$ . If  $m(f) = 0$ , no charge is transferred.

**Rule 2.** Each big vertex  $v$  sends  $\frac{1}{5}$

$\tilde{c}(x)$  be the charge of a vertex  $x \in V^\times$  after application of Rules 1 and 2.

**Rule 3.** Each 5-vertex  $v$  with  $\tilde{c}(v) > 0$  sends  $\frac{\tilde{c}(v)}{\overline{m}(v)}$  to each adjacent 4-vertex;  $\overline{m}(v)$  is the number of 4-vertices adjacent to  $v$  (if  $\overline{m}(v) = 0$ , no charge is transferred).

We will check final charges of elements of  $M^\times(G)$ . Due to the formulation of Rule 1, the final charge of each face of  $M^\times(G)$  is nonnegative. Also,

if  $v$  is a big  $d$ -vertex, then it has at most  $d$  neighbours of degree 5 in  $G$ , thus  $\tilde{c}(v) \geq d - 6 - \frac{d}{2} \geq 0$  since  $d \geq 12$ . As vertices of degree between 6 and 11 do not change their initial charge, it is enough to check just the final charges of 5- and 4-vertices.

*Case 1.* Let  $x$  be a 5-vertex. Recall that  $x$  is incident with at least two  $\geq 4$ -faces. Also, in  $G$ ,  $x$  has at least two big neighbours (otherwise a light 4-star is found). Thus,  $\bar{c}(x) \geq -1 + 2 \cdot \frac{1}{5}$  –

$\frac{\bar{c}(x)}{m(x)}$  from Rule 3 according to

type of the neighbourhood of  $x$ .

(a) If  $x$  is incident only with  $\geq 4$ -faces, then  $\bar{c}(x) \geq -1 + 5 \cdot \frac{1}{5}$  – –

$$\frac{-}{5} = \frac{1}{5}$$

$$\bar{c}(x) \geq -1 + 4 \cdot \frac{1}{5} \quad -$$

– –

$$\bar{c}(x) \geq -1 + 3 \cdot \frac{1}{5}$$

$$- \quad - \quad \frac{-}{3} = \frac{1}{3}$$

$$\bar{c}(x) \geq -1 + 3 \cdot \frac{1}{5}$$

$$\frac{1}{5} \quad - \quad - \quad \frac{-}{4} = \frac{3}{4}$$

$$\bar{c}(x) \geq -1 + 2 \cdot \frac{1}{5} \quad -$$

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if both these vertices are 5-vertices, then  $f_1$  contributes  $\frac{1}{2}$

–

■

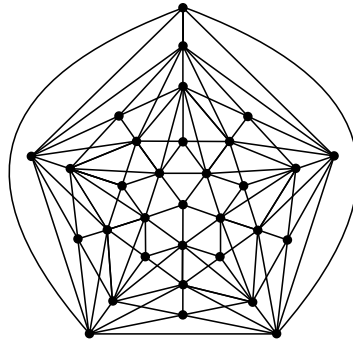


Figure 1

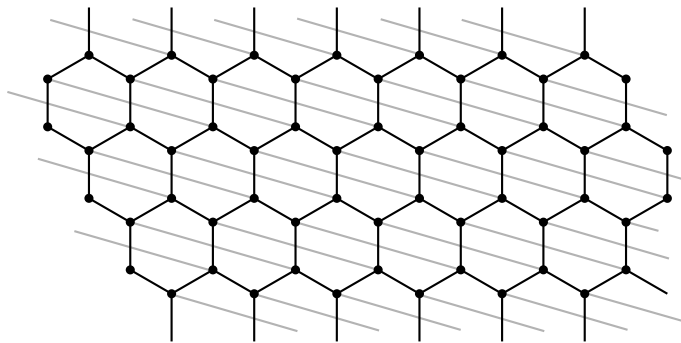


Figure 2

### Acknowledgement

The authors wish to thank anonymous referee for valuable comments and suggestions which helped to decrease the upper bound in Theorem 2.

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Received 13 November 2007

Revised 28 July 2008

Accepted 28 July 2008