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GRAPHS WITH SMALL ADDITIVE STRETCH NUMBER

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Abstract

The additive stretch number $s_{\mathrm{add}}(G)$ of a graph G is the maximum difference of the lengths of a longest induced path and a shortest induced path between two vertices of G that lie in the same component of G.

We prove some properties of minimal forbidden configurations for the induced-hereditary classes of graphs G with $s_{\mathrm{add}}(G) \leq k$ for some $k \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$. Furthermore, we derive characterizations of these classes for $k = 1$ and $k = 2$.

Keywords: stretch number, distance hereditary graph, forbidden induced subgraph.

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1. Introduction

Let $G = (V, E)$ be a finite and simple graph. A path $P : x_0x_1x_2 \ldots x_l$ in G is called induced, if for $0 \leq i < j \leq l$ we have $x_ix_j \in E$ if and only if $j - i = 1$. For vertices x and y in G that lie in the same component of G let $P_G(x, y)$ and $p_G(x, y)$ denote a longest and a shortest induced path in G from x to y, respectively. Let $D_G(x, y)$ and $d_G(x, y)$ denote the lengths of $P_G(x, y)$ and $p_G(x, y)$, respectively.

In [3] Cicerone, D'Ermiliis and Di Stefano define the *additive stretch number* $s_{\mathrm{add}}(G)$ of G as the maximum of $D_G(x, y) - d_G(x, y)$ over all pairs of vertices x and y of G that lie in the same component of G. A multiplicative version of this parameter was introduced and studied in [2], [4] (cf. also [6]). Note that $s_{\text{add}}(G) = 0$ holds for a graph G, if and only if G is *distance hereditary* [1, 5].

It is obvious from the definitions that the class of graphs G with $s_{\text{add}}(G) \leq k$ for some $k \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$ is induced-hereditary, i.e., it is closed under forming induced subgraphs and can therefore be characterized in terms of minimal forbidden induced subgraphs. The final result of [3] is such a characterization of the class of graphs G with $s_{\text{add}}(G) \leq 1$. Since Cicerone et al. derive this result from the main result of [4], their proof is long and indirect.

The purpose of the present paper is to provide a direct approach, a simpler proof of their result and an extension of it. In the next section we collect some properties of 'forbidden configurations'. In Section 3, we derive characterizations of the induced-hereditary classes of graphs G with $s_{\text{add}}(G) \leq k$ for $k \in \{1, 2\}$.

For plenty of references to related work and motivating comments on this concept we refer the reader to [2], [3] and [4].

2. Forbidden Configurations

Throughout this section let $G = (V, E)$ be a graph such that $s_{\text{add}}(G) > k$ for some $k \in \mathbf{N}_0$. Let $x, y \in V$ be such that

(i) $D_G(x, y) - d_G(x, y) > k$,

(ii) $d_G(x, y)$ is minimum subject to (i) and

(iii) $D_G(x, y)$ is minimum subject to (i) and (ii).

Clearly, $d_G(x, y) \geq 2$ and thus $D_G(x, y) + d_G(x, y) > 2d_G(x, y) + k \geq 4 + k$.

Let $P_G(x, y) : x = u_0 u_1 u_2 \ldots u_{D-1} u_D = y$ be a longest induced path from x to y and let $p_G(x, y) : x = v_0 v_1 v_2 \ldots v_{d-1} v_d = y$ be a shortest induced path from x to y.

Since the paths are induced, $u_i u_j \notin E$ for $0 \leq i, j \leq D$ with $j - i \geq 2$ and $v_i v_j \notin E$ for $0 \leq i, j \leq d$ with $j - i \geq 2$. By Condition (ii) of the choice of x and y, we have $v_1, v_{d-1} \notin \{u_1, u_2, \ldots, u_{D-1}\}$ and $u_1, u_{D-1} \notin \{v_1, v_2, \ldots, v_{d-1}\}$.

If for some $1 \leq j \leq d - 1$ the vertex v_j has a neighbour in $\{u_1, u_2, \ldots, u_{D-1}\}$, then we define

$$l_j = \min\{j' \mid 1 \leq j' \leq D - 1 \text{ and } v_j u_{j'} \in E\}$$

and

$$r_j = \max\{j' \mid 1 \leq j' \leq D-1 \text{ and } v_j u_{j'} \in E\}$$

and say that r_j *and* l_j *are defined*. Note that if $v_j \in \{u_1, u_2, \ldots, u_{D-1}\}$ for some $1 \leq j \leq d-1$, then $2 \leq j \leq d-2$, v_j has a neighbour in $\{u_1, u_2, \ldots, u_{D-1}\}$ and r_j and l_j are defined. Furthermore, by Condition (ii), if $d_G(x,y) \geq 3$, then the indices r_1, l_1, r_{d-1} and l_{d-1} are defined. We collect some properties of $P_G(x,y)$ and $p_G(x,y)$.

Lemma 1.

(i) *If r_j is defined for some $1 \leq j \leq d-1$, then $r_j \leq k+j+1$.*

(ii) *If r_j is defined for some $1 \leq j \leq d-2$, then $r_j \geq (D-d-k)+j+1$.*

(iii) $r_{d-1} \geq D-k-2$.

(iv) *If r_j is defined for some $1 \leq j \leq d-2-\lceil \frac{k}{2} \rceil$, then at least one of $r_{j+1}, r_{j+2}, \ldots, r_{j+\lceil \frac{k}{2} \rceil}$ is defined.*

Proof. (i) For contradiction we assume that $r_j > j+k+1$ for some $1 \leq j \leq d-1$. $xu_1u_2 \ldots u_{r_j}$ is an induced path from x to u_{r_j} and $xv_1v_2 \ldots v_j u_{r_j}$ is a path from x to u_{r_j}. Note that the existence of a path of length l between two vertices always implies the existence of an induced path of length at most l between these vertices.

Hence $D_G(x, u_{r_j}) - d_G(x, u_{r_j}) \geq r_j - (j+1) > k$. Since either $d_G(x, u_{r_j}) < d$ or $d_G(x, u_{r_j}) = d$ and $D_G(x, u_{r_j}) < D$, we obtain a contradiction to the choice of x and y. This implies (i).

(ii) For contradiction we assume that $r_j \leq (D-d-k)+j$ for some $1 \leq j \leq d-2$.

$v_j u_{r_j} u_{r_j+1} \ldots u_{D-1} y$ is an induced path from v_j to y and $v_j v_{j+1} \ldots v_{d-1} y$ is an induced path from v_j to y. Hence $D_G(v_j, y) - d_G(v_j, y) \geq (D - r_j + 1) -(d-j) > k$. Since $d_G(v_j, y) < d$, we obtain a contradiction to the choice of x and y. This implies (ii).

(iii) For contradiction we assume that $r_{d-1} \leq D-k-3$. $u_{r_{d-1}} u_{r_{d-1}+1} \ldots u_{D-1} y$ is an induced path from $u_{r_{d-1}}$ to y and $u_{r_{d-1}} v_{d-1} y$ is an induced path from $u_{r_{d-1}}$ to y. Hence $D_G(u_{r_{d-1}}, y) - d_G(u_{r_{d-1}}, y) \geq (D - r_{d-1}) - 2 > k$. Since either $d_G(u_{r_{d-1}}, y) < d$ or $d_G(u_{r_{d-1}}, y) = d$ and $D_G(u_{r_{d-1}}, y) < D$, we obtain a contradiction to the choice of x and y. This implies (iii).

(iv) For contradiction we assume that r_j is defined and that r_{j+1}, $r_{j+2}, \ldots, r_{j+\lceil \frac{k}{2} \rceil}$ are not defined for some $1 \leq j \leq d-2-\lceil \frac{k}{2} \rceil$.

$v_{j+\lceil \frac{k}{2} \rceil} v_{j+\lceil \frac{k}{2} \rceil - 1} \dots v_j u_{r_j} u_{r_j+1} \dots u_{D-1} y$ is an induced path from $v_{j+\lceil \frac{k}{2} \rceil}$ to y and $v_{j+\lceil \frac{k}{2} \rceil} v_{j+\lceil \frac{k}{2} \rceil + 1} \dots v_{d-1} y$ is an induced path from $v_{j+\lceil \frac{k}{2} \rceil}$ to y. Hence, by (i),

$$\begin{aligned} D_G(v_{j+\lceil \frac{k}{2} \rceil}, y) - d_G(v_{j+\lceil \frac{k}{2} \rceil}, y) &\geq \left(D - r_j + \left\lceil \frac{k}{2} \right\rceil + 1 \right) - \left(d - j - \left\lceil \frac{k}{2} \right\rceil \right) \\ &\geq D - d - r_j + k + j + 1 \\ &\geq D - d > k. \end{aligned}$$

Since $d_G(v_{j+\lceil \frac{k}{2} \rceil}, y) < d$, we obtain a contradiction to the choice of x and y. This implies (iv) and the proof is complete. ∎

By symmetry, we obtain.

Corollary 2.

(i) *If l_j is defined for some $1 \leq j \leq d-1$, then $l_j \geq (D-d-k) + j - 1$.*

(ii) *If l_j is defined for some $2 \leq j \leq d-1$, then $l_j \leq k + j - 1$.*

(iii) *$l_1 \leq k + 2$.*

(iv) *If l_j is defined for some $2 + \lceil \frac{k}{2} \rceil \leq j \leq d-1$, then at least one of $l_{j-1}, l_{j-2}, \dots, l_{j-\lceil \frac{k}{2} \rceil}$ is defined.*

Using Lemma 1, we can bound $D_G(x,y) - d_G(x,y)$.

Corollary 3. *If $d_G(x,y) = 2$ and r_1 is defined, then $k+1 \leq D_G(x,y) - d_G(x,y) \leq 2k+2$ and if $d_G(x,y) \geq 3$, then $k+1 \leq D_G(x,y) - d_G(x,y) \leq 2k$.*

Proof. If $d_G(x,y) = 2$ and r_1 is defined, then (i) and (iii) of Lemma 1 imply $D - k - 2 \leq r_{d-1} = r_1 \leq k + 1 + 1$ and hence $k + 1 \leq D_G(x,y) - d_G(x,y) = D - 2 \leq 2k + 2$.

If $d_G(x,y) \geq 3$, then r_1 is defined and $1 < d - 1$. Now (i) and (ii) of Lemma 1 imply $(D-d-k) + 1 + 1 \leq r_1 \leq k + 1 + 1$ and hence $k + 1 \leq D - d \leq 2k$. ∎

The next lemma analyses the situation when the two paths $P_G(x,y)$ and $p_G(x,y)$ 'meet in reverse order'.

Lemma 4. *There are no* $k_1, k_2 \in \mathbf{N} = \{1, 2, \ldots\}$ *with* $k_1 + k_2 \geq k$ *and* $u_{j_1} = v_{j_2+k_2}$ *and* $u_{j_1+k_1} = v_{j_2}$ *for some* $1 \leq j_1 \leq D - 1 - k_1$ *and some* $1 \leq j_2 \leq d - 1 - k_2$ (*cf. Figure* 1 *for an illustration*).

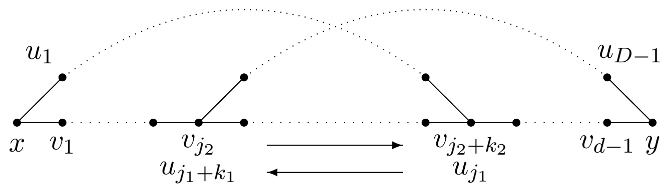

Figure 1. Parts of $P_G(x, y)$ and $p_G(x, y)$

Proof. For contradiction, we assume that k_1, k_2, j_1 and j_2 exist as in the statement.

If $j_1 = 1$, then $xv_{j_2+k_2} \in E$ with $j_2 + k_2 \geq 2$ which is a contradiction. This implies $j_1 \geq 2$. By symmetry, we obtain $2 \leq j_1 \leq (D - 1 - k_1) - 1$ and $2 \leq j_2 \leq (d - 1 - k_2) - 1$.

We assume that $j_1 - j_2 < (D - d - k) + k_2$. $u_{j_1}u_{j_1+1} \ldots u_{D-1}y$ is an induced path from $u_{j_1} = v_{j_2+k_2}$ to y and $v_{j_2+k_2}v_{j_2+k_2+1} \ldots v_{d-1}y$ is an induced path from $u_{j_1} = v_{j_2+k_2}$ to y. Hence $D_G(u_{j_1}, y) - d_G(u_{j_1}, y) \geq (D - j_1) - (d - j_2 - k_2) > k$. Since $d_G(u_{j_1}, y) < d$, we obtain a contradiction to the choice of x and y. Hence $j_1 - j_2 \geq (D - d - k) + k_2$.

$xu_1u_2 \ldots u_{j_1+k_1}$ is an induced path from x to $u_{j_1+k_1} = v_{j_2}$ and $xv_1v_2 \ldots v_{j_2}$ is an induced path from x to $u_{j_1+k_1} = v_{j_2}$. Hence $D_G(x, v_{j_2}) - d_G(x, v_{j_2}) \geq (k_1 + j_1) - j_2 \geq D - d - k + k_1 + k_2 \geq D - d > k$. Since $d_G(x, v_{j_2}) < d$, we obtain a contradiction to the choice of x and y and the proof is complete. ■

3. $\{G \mid s_{\text{add}}(G) \leq k\}$ for $k \in \{1, 2\}$

Let $G = (V, E)$ be a graph. If $\tilde{V} \subseteq V$, then $G[\tilde{V}]$ denotes the subgraph of G induced by $\tilde{V}$. A *chord* of a cycle C of G is an edge of G that joins two non-consecutive vertices of C. The *chord distance* $cd(C)$ of a cycle C of G is the minimum number of consecutive vertices of C such that each chord of C is incident with one of these vertices.

In order to facilitate the statement of our main result we introduce some more notation. For some $\nu \geq 2$ let $n_1, n_2, \ldots, n_\nu \geq 5$, $c_1, c_2, \ldots, c_\nu \geq 1$ and

$m_1, m_2, \ldots, m_{\nu-1} \geq 1$ be integers. For $1 \leq i \leq \nu$ let $G_i : x_{1,i}x_{2,i}\ldots x_{n_i,i}x_{1,i}$ be a cycle of order n_i such that all chords of G_i are incident with a vertex in $\{x_{1,i}, x_{2,i}, \ldots, x_{c_i,i}\}$, i.e., G_i has chord distance at most c_i. For $1 \leq i \leq \nu-1$ let $H_i : y_{1,i}y_{2,i}\ldots y_{m_i,i}$ be an induced path of order m_i. Let the graph

$$G((n_1, c_1), m_1, (n_2, c_2), m_2, \ldots, (n_{\nu-1}, c_{\nu-1}), m_{\nu-1}, (n_\nu, c_\nu))$$

arise by identifying the two vertices $x_{c_i+1,i}$ and $y_{1,i}$ and the two vertices $x_{n_{i+1},i+1}$ and $y_{m_i,i}$ for $1 \leq i \leq \nu - 1$. (Note that if $m_i = 1$ for some $1 \leq i \leq \nu - 1$, then $y_{1,i} = y_{m_i,i}$ and the three vertices $x_{c_i+1,i}$, $y_{1,i}$ and $x_{n_{i+1},i+1}$ are identified.) See Figure 2 for an illustration of two examples.

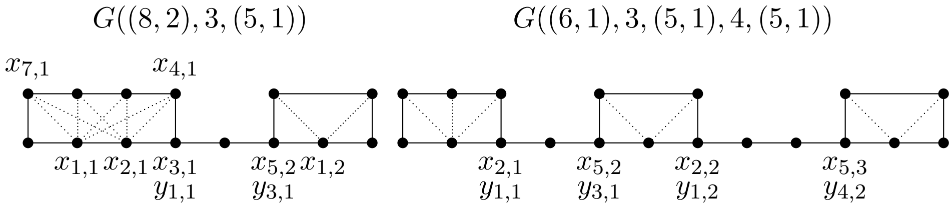

Figure 2. $G((8,2),3,(5,1))$ and $G((6,1),3,(5,1),4,(5,1))$

We proceed to our main result of this section.

Theorem 5. *Let $k \in \{1, 2\}$. A graph $G = (V, E)$ satisfies $s_{\text{add}}(G) \leq k$ if and only if*

(a) (cf. [3]) *for $k = 1$ the graph G does not contain one of the following graphs as an induced subgraph.*

(i) *A chordless cycle C of length $l \geq 6$.*

(ii) *A cycle C of length $l \in \{6, 7, 8\}$ and chord distance $cd(C) = 1$.*

(iii) *A cycle C of length 8 and chord distance $cd(C) = 2$.*

(iv) *The graph $G((5, 1), m_1, (5, 1))$ for some $m_1 \geq 1$.*

(b) *for $k = 2$ the graph G does not contain one of the following graphs as an induced subgraph.*

(i) *A chordless cycle C of length $l \geq 7$.*

(ii) *A cycle C of length $l \in \{7, 8, 9, 10\}$ and chord distance $cd(C) = 1$.*

(iii) *A cycle C of length 9 or 10 and chord distance $cd(C) = 2$.*

(iv) *A cycle C of length* 11 *and chord distance $cd(C) = 3$.*

(v) *The graph that arises from $G((5,1),1,(6,1))$ by adding the edge $x_{1,1}x_{5,2}$* (*cf. Figure* 3).

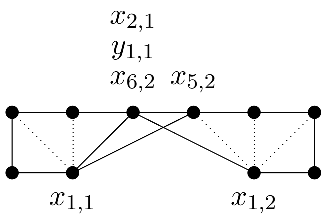

Figure 3

(vi) *The graph $G((6,1),m_1,(5,1))$ for some $m_1 \geq 1$.*

(vii) *The graph $G((8,2),m_1,(5,1))$ for some $m_1 \geq 1$.*

(viii) *The graph $G((6,1),m_1,(6,1))$ for some $m_1 \geq 1$.*

(ix) *The graph $G((5,1),m_1,(5,1),m_2,(5,1))$ for some $m_1, m_2 \geq 1$.*

Proof. The '*only if*'-part can easily be checked by calculating s_{add} for the described graphs and we leave this task to the reader. For the '*if*-part, we assume that $s_{\text{add}}(G) > k$ and prove that G has an induced subgraph as described in (a) or (b), respectively.

Let x, y, $P_G(x,y) : x = u_0u_1 \ldots u_{D-1}u_D = y$, $p_G(x,y) : x = v_0v_1 \ldots v_{d-1}v_d = y$, r_j and l_j be exactly as in Section 2, i.e., the Conditions (i) to (iii) are satisfied.

If $d = d_G(x,y) = 2$, then $C : xv_1yu_{D-1} \ldots u_1x$ is a cycle of length $D+d \geq 2d+k+1 = 5+k$ in G. If C has no chords, then C is as in (i) of (a) and (b), respectively. If C has chords, then all chords of C are incident with v_1 and Corollary 3 implies that C is as in (ii) of (a) and (b), respectively.

We can assume now that $d \geq 3$. Since r_1 and l_1 are defined and since $\lceil \frac{k}{2} \rceil = 1$, (iv) of Lemma 1 and Corollary 2 imply that r_j and l_j are defined for all $1 \leq j \leq d-1$. Furthermore, the estimations given in Lemma 1, Corollary 2 and Corollary 3 hold. (Note that in what follows we often use these estimations without explicit reference.)

If $d = 3$, then $C : xv_1v_2yu_{D-1} \ldots u_1x$ is a cycle of G. By the above properties, C is as in (iii) of (a) and (b), respectively.

From now on we assume that $d \geq 4$.

If $k = 1$, then $r_1 = 3$ and $l_{d-1} = d - 1$ and the graph $G[\{x, y, v_1, v_{d-1}, u_1, u_2, \ldots, u_{d+1}\}]$ is as in (iv) of (a). (Note that the proof for the case $k = 1$ is already complete at this point.)

From now on we assume that $k = 2$.

Case 1. $r_1 = 4$ *or* $l_{d-1} = D - 4$.
If $D \geq 8$ or $(D, r_1) = (7, 3)$ or $(D, l_{d-1}) = (7, D - 3)$, then the graph $G[\{x, y, v_1, v_{d-1}, u_1, u_2, \ldots, u_{D-1}\}]$ is as in (vi) or (viii) of (b). Hence we assume $d = 4$, $D = 7$, $r_1 = 4$ and $l_3 = 3$. Since $v_1u_5, v_3u_2 \notin E$, we have $v_2 \notin \{u_1, u_2, u_5, u_6\}$. If $v_2 \notin \{u_3, u_4\}$, then the graph $G[\{x, y, v_1, v_2, v_3, u_1, u_2, \ldots, u_6\}]$ is as in (iv) of (b). If $v_2 \in \{u_3, u_4\}$, then, by symmetry, we can assume that $v_2 = u_3$ and the graph $G[\{x, y, v_1, v_3, u_1, u_2, \ldots, u_6\}]$ is as in (v) of (b). This completes the case.

From now on we assume that $r_1 = 3$ and $l_{d-1} = D - 3$. By (ii) of Lemma 1, we obtain $(D - d - 2) + 1 + 1 \leq r_1 = 3$. As $D - d \geq 3$, this implies $D = d + 3$ and thus $l_{d-1} = d$.

Case 2. $d = 4$.
Since $v_1u_4, v_1u_5, v_3u_2, v_3u_3 \notin E$, we have $v_2 \notin \{u_1, u_2, u_3, u_4, u_5, u_6\}$. The graph $G[\{x, y, v_1, v_2, v_3, u_1, u_2, \ldots, u_6\}]$ is as in (iv) of (b). This completes the case.

From now on we assume that $d \geq 5$.

Case 3. $r_2 = 5$.
Since $v_1u_4 \notin E$, we have $v_2 \notin \{u_1, u_2, \ldots, u_{d+2}\}$. The graph $G[\{x, y\} \cup \{v_1, v_2, v_{d-1}\} \cup \{u_1, u_2, \ldots, u_{d+2}\}]$ is as in (vii) of (b). This completes the case.

From now on we assume that $r_2 = 4$ and, by symmetry, $l_{d-2} = d - 1$.

Case 4. $l_3 = 3$.
Note that Lemma 1 implies that $j + 2 \leq r_j \leq j + 3$ for $2 \leq j \leq d - 2$. First, we assume that there is an index j with $2 \leq j \leq d - 3$ such that $r_j = j + 2$ and $r_{j+1} = j + 4$. Let j be minimal with these properties. Since $v_ju_{j+3}, v_ju_{j+4} \notin E$, we have $|\{v_j, v_{j+1}, u_{j+2}, u_{j+3}, u_{j+4}\}| = 5$.

If $j = 2$ and $d = 5$, then the graph $G[\{x, y\} \cup \{v_1, v_3, v_4\} \cup \{u_1, u_2, \ldots, u_7\}]$ is as in (vii) of (b). If $j = 2$ and $d \geq 6$, then the graph $G[\{x, y\} \cup$

$\{v_1, v_3, v_{d-1}\} \cup \{u_1, u_2, \ldots, u_{d+2}\}]$ is as in (ix) of (b). If $3 \leq j \leq d-4$, then the graph

$$G[\{x,y\} \cup \{v_1, v_3, v_{d-1}\} \cup \{u_1, u_2, u_3\} \cup \{v_4, v_5, \ldots, v_j, v_{j+1}\} \\ \cup \{u_{j+2}, u_{j+3}, \ldots, u_{d+2}\}]$$

is as in (ix) of (b). If $3 \leq j = d-3$, then the graph

$$G[\{x,y\} \cup \{v_1, v_3\} \cup \{u_1, u_2, u_3\} \cup \{v_4, v_5, \ldots, v_{d-1}\} \cup \{u_{d-1}, u_d, \ldots, u_{d+2}\}]$$

is as in (vii) of (b). Hence we can assume that no such index exists. Since $r_2 = 4$, this implies, by an inductive argument, that $r_j = j+2$ for $2 \leq j \leq d-2$ and thus $r_{d-2} = d$. Now the graph

$$G[\{x,y\} \cup \{v_1\} \cup \{v_3, v_4, \ldots, v_{d-1}\} \cup \{u_1, u_2, u_3\} \cup \{u_d, u_{d+1}, u_{d+2}\}]$$

is as in (vi) of (b). This completes the case.

From now on we assume that $l_3 = 4$ and, by symmetry, $r_{d-3} = d-1$.

Case 5. $l_2 = 3$.
Since $r_2 = 4$, we have $v_2 \notin \{u_1, u_2, \ldots, u_{d+2}\}$. The graph

$$G[\{x,y\} \cup \{v_1, v_2, v_3, v_{d-1}\} \cup \{u_1, u_2, u_3\} \cup \{u_{r_3}, u_{r_3+1}, \ldots, u_{d+2}\}]$$

is as in (vi) of (b). This completes the case.

From now on we assume that $l_2 = 2$ and, by symmetry, $r_{d-2} = d+1$.

Case 6. $r_3 = 6$.
Since $v_2v_3 \in E$, $l_2 = 2$ and $l_3 = 4$, we have $v_2, v_3 \notin \{u_1, u_2, \ldots, u_{d+2}\}$. If $d = 5$, then the graph $G[\{x, y, v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4, u_6, u_7\}]$ is as in (vii) of (b). If $d \geq 6$, then the graph

$$G[\{x,y\} \cup \{v_1, v_2, v_3, v_{d-1}\} \cup \{u_1, u_2\} \cup \{u_4, u_5, \ldots, u_{d+2}\}]$$

is as in (ix) of (b). This completes the case.

From now on we assume that $r_3 = 5$ and, by symmetry, $l_{d-3} = d-2$.

Case 7. There is an index j with $3 \leq j \leq d-3$ such that $r_j = j+2$ and $r_{j+1} = j+4$.

Let j be minimal with these properties. As in Case 4, we obtain $|\{v_j, v_{j+1}, u_{j+2}, u_{j+3}, u_{j+4}\}| = 5$. The graph $G[\{x,y\} \cup \{u_1,u_2\} \cup \{v_1, v_2, \ldots, v_j, v_{j+1}\} \cup \{v_{d-1}\} \cup \{u_{j+2}, u_{j+3}, \ldots, u_{d+2}\}]$ is as in (vii) or (ix) of (b). This completes the case.

From now on we assume that no such index exists. Since $r_3 = 5$, this implies, by an inductive argument, that $r_j = j+2$ for $3 \leq j \leq d-2$ and thus $r_{d-2} = d$. Now the graph $G[\{x,y\} \cup \{u_1,u_2\} \cup \{v_1, v_2, \ldots, v_{d-1}\} \cup \{u_d, u_{d+1}, u_{d+2}\}]$ is as in (vi) of (b). This completes the proof. ■

4. Concluding Remarks

Using Theorem 5 it is now a simple but tedious task to determine an explicit list of all minimal forbidden induced subgraphs for the class of graphs G with $s_{\text{add}}(G) \leq 2$.

In [3] it was shown that the recognition of graphs G with $s_{\text{add}}(G) \leq k$ is a co-NP-complete problem, if k is part of the input. At the end of [3] a polynomial time recognition algorithm for the class of graphs G with $s_{\text{add}}(G) \leq 1$ was described. It is obvious how to extend the '*brute force*'-approach of this algorithm to obtain a polynomial time recognition algorithm for the class of graphs G with $s_{\text{add}}(G) \leq 2$.

It is easy to see that for $k \geq 1$ the graphs $G((n_1,c_1), m_1, (n_2,c_2), m_2, \ldots, m_{\nu-1}, (n_\nu, c_\nu))$ such that $c_i \geq 1$ for $1 \leq i \leq \nu$, $n_i \geq 2c_i + 3$ for $1 \leq i \leq \nu$ and $\sum_{i=1}^{\nu}(n_i - 2c_i - 2) > k$ are forbidden induced subgraphs for the graphs G with $s_{\text{add}}(G) \leq k$. Nevertheless, in view of the graph in (v) of (b) in Theorem 5, we believe that there is no regular pattern for the minimal forbidden induced subgraphs for $k \geq 2$. The graph in Figure 4 shows that for $k \geq 3$ the two paths $P_G(x,y)$ and $p_G(x,y)$ may even use edges in reverse order (in such a situation Lemma 4 can be used to bound the number of these edges).

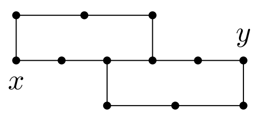

Figure 4

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