

## ON CHARACTERIZATION OF UNIQUELY 3-LIST COLORABLE COMPLETE MULTIPARTITE GRAPHS\*

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### Abstract

For each vertex  $v$  of a graph  $G$ , if there exists a list of  $k$  colors,  $L(v)$ , such that there is a unique proper coloring for  $G$  from this collection of lists, then  $G$  is called a *uniquely  $k$ -list colorable graph*. Ghebleh and Mahmoodian characterized uniquely 3-list colorable complete multipartite graphs except for nine graphs:  $K_{2,2,r}$   $r \in \{4, 5, 6, 7, 8\}$ ,  $K_{2,3,4}$ ,  $K_{1*4,4}$ ,  $K_{1*4,5}$ ,  $K_{1*5,4}$ . Also, they conjectured that the nine graphs are not U3LC graphs. After that, except for  $K_{2,2,r}$   $r \in \{4, 5, 6, 7, 8\}$ , the others have been proved not to be U3LC graphs. In this paper we first prove that  $K_{2,2,8}$  is not U3LC graph, and thus as a direct corollary,  $K_{2,2,r}$  ( $r = 4, 5, 6, 7, 8$ ) are not U3LC graphs, and then the uniquely 3-list colorable complete multipartite graphs are characterized completely.

**Keywords:** list coloring, complete multipartite graph, uniquely 3-list colorable graph.

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## 1. INTRODUCTION

We consider simple graphs which are finite, undirected, with no loops or multiple edges. For the necessary definitions and notations we refer the reader to standard texts, such as [2].

By a  $k$ -list assignment  $L$  to a graph  $G$  we mean a map which assigns to each vertex  $v$  of  $G$  a set  $L(v)$  of size  $k$ . A *list coloring* for  $G$  from  $L$ , or an  $L$ -coloring for short, is a proper coloring  $c$ , in which for each vertex  $v$ ,  $c(v)$  is chosen from  $L(v)$ . The idea of list coloring is due independently to Vizing [11] and to Erdős, Rubin, and Taylor [4]. For a recent survey on list coloring we refer the interested reader to Alon [1].

For each vertex  $v$  in  $G$ , if there exists a list of  $k$  colors  $L(v)$ , such that there exists a uniquely  $L$ -coloring for  $G$ , then  $G$  is called a *uniquely  $k$ -list colorable graph* or a  $UkLC$  graph for short. The idea of uniquely colorable graph was introduced independently by Dinitz and Martin [3] and by Mahmoodian and Mahdian [8].

If a graph  $G$  is not uniquely  $k$ -list colorable, we also say that  $G$  has property  $M(k)$ . So  $G$  has the property  $M(k)$  if and only if for any collection of lists assigned to its vertices, each of size  $k$ , either there is no list coloring for  $G$  or there exist at least two list colorings. The least integer  $k$  such that  $G$  has the property  $M(k)$  is called the  $m$ -number of  $G$ , denoted by  $m(G)$ . This conception was originally introduced by Mahmoodian and Mahdian in [9].

Mahdian and Mahmoodian characterized uniquely 2-list colorable graphs as follows:

**Theorem 1.1** ([8]). *A connected graph  $G$  has the property  $M(2)$  if and only if every block of  $G$  is either a cycle, a complete graph, or a complete bipartite graph.*

Ghebleh and Mahmoodian studied uniquely 3-list colorability about complete multipartite graphs, and they gave the following results:

**Theorem 1.2** ([6]). *Graphs  $K_{3,3,3}$ ,  $K_{2,4,4}$ ,  $K_{2,3,5}$ ,  $K_{2,2,9}$ ,  $K_{1,2,2,2}$ ,  $K_{1,1,2,3}$ ,  $K_{1,1,1,2,2}$ ,  $K_{1*4,6}$ ,  $K_{1*5,5}$ ,  $K_{1*6,4}$  are uniquely 3-list colorable graphs.*

Here,  $K_{s*r,t}$  denote a complete  $(r+1)$ -partite graph in which each part of the  $r$  parties is of size  $s$ , one party is of size  $t$ .

**Theorem 1.3** ([6]). *Let  $G$  be a complete multipartite graph that is not  $K_{2,2,r}$ ,  $r \in \{4, 5, 6, 7, 8\}$ ,  $K_{2,3,4}$ ,  $K_{1*4,4}$ ,  $K_{1*4,5}$ ,  $K_{1*5,4}$ , then  $G$  is U3LC if and only if it has one of the ten graphs in Theorem 1.2 as an induced subgraph.*

It can be seen from Theorem 1.2 and Theorem 1.3 that, if we can determine whether the nine graphs exempted in Theorem 1.3 are U3LC or not, we can simplify Theorem 1.3. About the nine graphs, Ref. [6] gave an open problem.

**Problem 1** ([6]). Verify the property  $M(3)$  for the graphs  $K_{2,2,r}$ ,  $r \in \{4, 5, 6, 7, 8\}$ ,  $K_{2,3,4}$ ,  $K_{1*4,4}$ ,  $K_{1*4,5}$ , and  $K_{1*5,4}$ .

Recently, except for  $K_{2,2,r}$ ,  $r \in \{4, 5, 6, 7, 8\}$ , the other graphs in the above problem have been proved to have the property  $M(3)$ . They are shown below:

**Theorem 1.4** ([10]). *The graph  $K_{1*5,4}$  has property  $M(3)$ .*

**Theorem 1.5** ([7]). *Graphs  $K_{1*4,5}$  and  $K_{1*4,4}$  have property  $M(3)$ .*

**Theorem 1.6** ([12]). *The graph  $K_{2,3,4}$  has property  $M(3)$ .*

In the next section of this paper, we will prove that  $K_{2,2,8}$  has the property  $M(3)$ . As its a direct corollary, we show that  $K_{2,2,r}$ ,  $r \in \{4, 5, 6, 7\}$  have the property  $M(3)$ , and thus based on the results we prove together with previous results, we completely characterize the U3LC complete multipartite graphs.

## 2. $K_{2,2,r}$ , $r \in \{4, 5, 6, 7, 8\}$ HAVE PROPERTY $M(3)$

First we give three results as lemmas.

**Lemma 2.1** ([5]). *If  $G$  is a U3LC complete tripartite graph, then in its uniquely 3-list coloring, at least two colors are used to color all vertices in each part of  $G$ .*

**Lemma 2.2** ([9]). *If  $L$  is a  $k$ -list assignment to the vertices in a graph  $G$ , and  $G$  has a unique  $L$ -coloring, then  $|\bigcup_v L(v)| \geq k + 1$  and all these colors are used in the unique  $L$ -coloring of  $G$ .*

**Lemma 2.3** ([6]). *If  $G$  is a complete multipartite graph which has an induced  $UkLC$  subgraph, then  $G$  is  $UkLC$ .*

In the graph  $K_{2,2,8}$ , we denote its three parts by  $X_1 = \{v_1, v_2\}$ ,  $X_2 = \{v_3, v_4\}$ ,  $X_3 = \{v_5, v_6, \dots, v_{12}\}$ , and denote 3-list assigned to the vertex  $v_i$  of  $K_{2,2,8}$  by  $L(v_i) = \{c_{i1}, c_{i2}, c_{i3}\}$ ,  $1 \leq i \leq 12$ . Now, we suppose that  $K_{2,2,8}$  has a unique  $L$ -coloring  $c$  such that  $c(v_i) = c_{i1}$ ,  $i \in \{1, 2, \dots, 12\}$ . For convenience, let  $S = \{c_{51}, c_{61}, \dots, c_{12,1}\}$ . To obtain our main result, we give more lemmas as follows.

**Lemma 2.4.** *Colors  $c_{11}, c_{21}, c_{31}, c_{41}$  are different from each other.*

**Proof.** The result follows from Lemma 2.1. ■

**Lemma 2.5.** *If  $v_i \neq v_j$  are in the same part of  $K_{2,2,8}$ , then  $c_{i1} \notin \{c_{j2}, c_{j3}\}$ .*

**Proof.** Otherwise, if  $c_{i1} = c_{jk}$ ,  $k \in \{2, 3\}$ , then by letting  $c'(v_j) = c_{jk} = c_{i1}$  and letting  $c'(v_i) = c(v_i)$  for  $i \neq j$ , we obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$ , which contradicts the fact that  $K_{2,2,8}$  is  $U3LC$ . ■

**Lemma 2.6.** *If  $v_i, v_j \in \{v_1, v_2, v_3, v_4\}$  are in two different parts of  $K_{2,2,8}$  such that  $c_{i1} \in \{c_{j2}, c_{j3}\}$ , then  $c_{j1} \notin \{c_{i2}, c_{i3}\}$ .*

**Proof.** Otherwise, by letting  $c'(v_i) = c_{j1}$ ,  $c'(v_j) = c_{i1}$  and  $c'(v_k) = c(v_k)$  for  $k \neq i, j$ , we obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$ , a contradiction. ■

**Lemma 2.7.** *There exists an  $i \in \{1, 2, 3, 4\}$  such that  $\{c_{i2}, c_{i3}\} \subseteq S$ .*

**Proof.** Otherwise, for each  $i \in \{1, 2, 3, 4\}$ ,  $|\{c_{i1}, c_{i2}, c_{i3}\} \setminus S| \geq 2$ . Let  $L'(v_i) = \{c_{i1}, c_{i2}, c_{i3}\} \setminus S$  for each  $i \in \{1, 2, 3, 4\}$ . Note that the subgraph of  $K_{2,2,8}$  induced by  $v_1, v_2, v_3$  and  $v_4$  is  $K_{2,2}$ . Then by Theorem 1.1,  $K_{2,2}$  has the property  $M(2)$ . So, there is another  $L$ -coloring  $c'$  of  $K_{2,2}$  with  $c'(v_i) \in L'(v_i)$ ,  $i \in \{1, 2, 3, 4\}$ , which can be easily extended to another proper coloring of  $K_{2,2,8}$  by letting  $c'(v_i) = c(v_i)$  for  $i \in \{5, 6, \dots, 12\}$ , a contradiction. ■

**Lemma 2.8.**  *$\{c_{i2}, c_{i3}\} \subseteq \{c_{11}, c_{21}, c_{31}, c_{41}\}$  for every  $i \in \{5, 6, \dots, 12\}$ .*

**Proof.** By Lemma 2.5, if  $v_i \neq v_j$  are in the same part of  $K_{2,2,8}$ , then  $c_{i1} \notin \{c_{j2}, c_{j3}\}$ . Also obviously  $c_{i1} \notin \{c_{i2}, c_{i3}\}$ . So  $c_{i2}, c_{i3} \notin S$  for  $i \in \{5, 6, \dots, 12\}$ . Again by Lemma 2.2,  $\{c_{i2}, c_{i3}\} \subseteq \{c_{11}, c_{21}, c_{31}, c_{41}\}$ . ■

**Lemma 2.9.** *If  $X_1$  and  $X_2$  are unconsidered, then any three colors in  $\{c_{11}, c_{21}, c_{31}, c_{41}\}$  can be used to  $L$ -color  $X_3$ .*

**Proof.** For convenience, we can suppose  $c_{i1} = i$ ,  $i \in \{1, 2, 3, 4\}$  by Lemma 2.4. Now without loss of generality, suppose to the contrary that three colors  $\{1, 2, 3\}$  cannot be used to  $L$ -color  $X_3$  without considering  $X_1$  and  $X_2$ , then  $\{c_{i2}, c_{i3}\} \cap \{1, 2, 3\} = \emptyset$  for some  $i \in \{5, 6, \dots, 12\}$  and thus  $|\{c_{i2}, c_{i3}\} \cap \{1, 2, 3, 4\}| \leq 1$ , contradicting the conclusion of Lemma 2.8. ■

**Lemma 2.10.** *Take any three colors from  $\{1, 2, 3, 4\}$ , for example  $\{2, 3, 4\}$ . If any two colors in  $\{2, 3, 4\}$  can not be used to  $L$ -color seven vertices of  $X_3$  with  $X_1$  and  $X_2$  unconsidered, then there must exist two colors  $\{1, 2\}$ , or  $\{1, 3\}$ , or  $\{1, 4\}$  which can be used to  $L$ -color  $X_3$  with  $X_1$  and  $X_2$  unconsidered.*

**Proof.** If any two colors in  $\{2, 3, 4\}$  can not be used to  $L$ -color seven vertices of  $X_3$  with  $X_1$  and  $X_2$  unconsidered, then, noting Lemma 2.8, since  $\{3, 4\}$  can not be used to  $L$ -color seven vertices of  $X_3$  with  $X_1$  and  $X_2$  unconsidered, there must be at least two  $i \in \{5, 6, \dots, 12\}$  such that  $\{c_{i2}, c_{i3}\} = \{1, 2\}$ ; since  $\{2, 4\}$  can not be used to  $L$ -color seven vertices of  $X_3$  with  $X_1$  and  $X_2$  unconsidered, there must be at least two  $i \in \{5, 6, \dots, 12\}$  such that  $\{c_{i2}, c_{i3}\} = \{1, 3\}$ ; since  $\{2, 3\}$  can not be used to  $L$ -color seven vertices of  $X_3$  with  $X_1$  and  $X_2$  unconsidered, there must be at least two  $i \in \{5, 6, \dots, 12\}$  such that  $\{c_{i2}, c_{i3}\} = \{1, 4\}$ . Without loss of generality, suppose the six sets above are  $\{c_{i2}, c_{i3}\}$ ,  $i \in \{7, 8, \dots, 12\}$ . Then we check the remaining two sets  $\{c_{52}, c_{53}\}, \{c_{62}, c_{63}\}$ . If  $\{c_{52}, c_{53}\} \cap \{c_{62}, c_{63}\} \neq \emptyset$ , suppose  $s \in \{c_{52}, c_{53}\} \cap \{c_{62}, c_{63}\}$ . Then  $1, s$  can color  $X_3$ . If  $\{c_{52}, c_{53}\} \cap \{c_{62}, c_{63}\} = \emptyset$ , then  $\{c_{52}, c_{53}\} \cup \{c_{62}, c_{63}\} = \{1, 2, 3, 4\}$  by Lemma 2.8. Without loss of generality, assume  $\{c_{52}, c_{53}\} = \{1, 2\}$  and  $\{c_{62}, c_{63}\} = \{3, 4\}$ . Then  $1, 3$ , or  $1, 4$  can be used to color  $X_3$ . ■

**Theorem 2.1.**  $K_{2,2,r}$ ,  $r \in \{4, 5, 6, 7, 8\}$  have property  $M(3)$ .

**Proof.** First we prove  $K_{2,2,8}$  has property  $M(3)$  by contradiction. Assume to the contrary that  $c$  is a unique 3-list coloring of  $K_{2,2,8}$  and  $L(v_i) = \{c_{i1}, c_{i2}, c_{i3}\}$ ,  $c(v_i) = c_{i1}$ ,  $i \in \{1, 2, \dots, 12\}$ . By Lemma 2.3, suppose  $c_{i1} = i$ ,  $i \in \{1, 2, 3, 4\}$ . By Lemma 2.7, there exists an  $i \in \{1, 2, 3, 4\}$  such that  $\{c_{i2}, c_{i3}\} \subseteq S = \{c_{51}, c_{61}, \dots, c_{12,1}\}$ . Without loss of generality, suppose such an  $i = 1$  and  $c_{12} = 5, c_{13} = 6$ , that is,  $L(v_1) = \{1, 5, 6\}$ . To proceed the

proof, we distinguish the following three cases. In each case, we will obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$ .

*Case 1.*  $|\{c_{22}, c_{23}\} \cap S| = 2$ , for example,  $c_{22} = a, c_{23} = b, a \neq b, a, b \in S$ . If  $\{c_{32}, c_{33}, c_{42}, c_{43}\} \cap S \neq \emptyset$ , for example,  $c_{32} = d \in S$ , we can let  $c'(v_3) = d, c'(v_1) = m, m \in \{5, 6\} \setminus d, c'(v_2) = n, n \in \{a, b\} \setminus d, c'(v_4) = 4$ . By Lemma 2.9, we can let  $c'(v_i) = l_i$ , where,  $l_i \in \{1, 2, 3\} \cap L(v_i), i \in \{5, 6, \dots, 12\}$ . Then we obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  (which is exhibited below), a contradiction. Thus assume  $\{c_{32}, c_{33}, c_{42}, c_{43}\} \cap S = \emptyset$ . Then, by Lemmas 2.4–2.5, it must be that  $L(v_3) = \{3, 1, 2\}, L(v_4) = \{4, 1, 2\}$ . Let  $c'(v_1) = 5, c'(v_2) = a, c'(v_3) = 1, c'(v_4) = 1$ , and by Lemma 2.9, we can let  $c'(v_i) = l_i$ , where,  $l_i \in \{2, 3, 4\} \cap L(v_i), i \in \{5, 6, \dots, 12\}$ . Then another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  occurs (which is exhibited below), a contradiction.

$$\begin{array}{cc} \underline{156} & \underline{2ab} \\ 3\underline{d} \times & \underline{4} \times \times \end{array} \qquad \begin{array}{cc} \underline{156} & \underline{2ab} \\ 3\underline{12} & \underline{412} \end{array}$$

$$\{c_{32}, c_{33}, c_{42}, c_{43}\} \cap S \neq \emptyset \quad \{c_{32}, c_{33}, c_{42}, c_{43}\} \cap S = \emptyset$$

*Case 2.*  $|\{c_{22}, c_{23}\} \cap S| = 1$ , for example,  $c_{22} = a \in S$ . By Lemma 2.5,  $1 \notin \{c_{22}, c_{23}\}$ . So  $3 \in \{c_{22}, c_{23}\}$  or  $4 \in \{c_{22}, c_{23}\}$ , for example,  $3 \in \{c_{22}, c_{23}\}$ . That is,  $L(v_2) = \{2, a, 3\}$ . We consider two subcases.

*Subcase 2.1.*  $\{c_{32}, c_{33}\} \subseteq S$  or  $\{c_{42}, c_{43}\} \subseteq S$ , for example,  $\{c_{32}, c_{33}\} \subseteq S$ . For convenience, write  $c_{32} = b, c_{33} = d, \{b, d\} \subseteq S$ . Then, by letting  $c'(v_2) = a, c'(v_3) = n$ , where,  $n \in \{b, d\} \setminus a, c'(v_1) = m, m \in \{5, 6\} \setminus n, c'(v_4) = 4$  and letting  $c'(v_i) = l_i$ , where,  $l_i \in \{1, 2, 3\} \cap L(v_i), i \in \{5, 6, \dots, 12\}$ , we obtain a new  $L$ -coloring  $c'$  of  $K_{2,2,8}$  (which is exhibited below), a contradiction.

$$\begin{array}{cc} \underline{156} & \underline{2a3} \\ 3\underline{bd} & \underline{4} \times \times \end{array}$$

*Subcase 2.2.*  $\{c_{32}, c_{33}\} \not\subseteq S$  and  $\{c_{42}, c_{43}\} \not\subseteq S$ . By Lemma 2.5,  $4 \notin \{c_{32}, c_{33}\}$ . By Lemma 2.6,  $2 \notin \{c_{32}, c_{33}\}$ . Noting Lemma 2.2, we have  $L(v_3) = \{3, 1, b\}, b \in S$ .

If  $1 \in \{c_{42}, c_{43}\}$ , then by letting  $c'(v_3) = c'(v_4) = 1, c'(v_2) = a, c'(v_1) = 5, c'(v_i) = l_i$ , where,  $l_i \in \{2, 3, 4\} \cap L(v_i), i \in \{5, 6, \dots, 12\}$  we obtain another

$L$ -coloring  $c'$  of  $K_{2,2,8}$  (which is exhibited below), a contradiction. So assume  $1 \notin \{c_{42}, c_{43}\}$ . Then noting that  $3 \notin L(v_4)$  by Lemma 2.5, there must be  $\{c_{42}, c_{43}\} = \{2, d\}, d \in S$ .

If  $b = d$ , then  $5 \neq b$  or  $6 \neq b$ , for example  $5 \neq b$ . By letting  $c'(v_3) = c'(v_4) = b, c'(v_1) = m, m \in \{5, 6\} \setminus b, c'(v_2) = 2, c'(v_i) = l_i$ , where,  $l_i \in \{1, 3, 4\} \cap L(v_i), i = 5, 6, \dots, 12$  we obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  (which is exhibited below), a contradiction. So suppose  $b \neq d$ . We discuss color  $a$  in  $L(v_2)$ .

(1) If  $a \in \{5, 6\}$ , for example  $a = 5$ , then  $5 \neq b$  or  $5 \neq d$ , for example  $5 \neq b$ . Let  $c'(v_1) = c'(v_2) = 5, c'(v_3) = b, c'(v_4) = 4, c'(v_i) = l_i$ , where,  $l_i \in \{1, 2, 3\} \cap L(v_i), i \in \{5, 6, \dots, 12\}$ , another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  occurring (which is exhibited below), a contradiction.

(2) If  $a \notin \{5, 6\}$ , then there exists at least one color in  $\{5, 6, a\}$  such that it does not belong to  $\{b, d\}$ , for example  $5 \notin \{b, d\}$ . Then, by letting  $c'(v_1) = 5, c'(v_2) = 2, c'(v_3) = b, c'(v_4) = d, c'(v_i) = l_i$ , here,  $l_i \in \{1, 3, 4\} \cap L(v_i), i \in \{5, 6, \dots, 12\}$ , we obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  (which is exhibited below), a contradiction.

$$\begin{array}{cc}
 \begin{array}{cc}
 1\underline{5}6 & 2\underline{a}3 \\
 3\underline{1}b & 4\underline{1}\times \\
 1 \in \{c_{42}, c_{43}\} \\
 1\underline{5}6 & 2\underline{5}3 \\
 3\underline{1}b & 4\underline{2}d
 \end{array}
 &
 \begin{array}{cc}
 1\underline{5}6 & 2\underline{a}3 \\
 3\underline{1}b & 4\underline{2}d \\
 1 \notin \{c_{42}, c_{43}\}, b = d \\
 1\underline{5}6 & 2\underline{a}3 \\
 3\underline{1}b & 4\underline{2}d
 \end{array}
 \end{array}$$

$$1 \notin \{c_{42}, c_{43}\}, b \neq d \text{ (1)} \quad 1 \notin \{c_{42}, c_{43}\}, b \neq d \text{ (2)}$$

Case 3.  $\{c_{22}, c_{23}\} \cap S = \emptyset$ .

By Lemma 2.5,  $1 \notin L(v_2)$ . So  $L(v_2) = \{2, 3, 4\}$ . By Lemma 2.6,  $2 \notin L(v_3), 2 \notin L(v_4)$ . By Lemma 2.5,  $3 \notin L(v_4), 4 \notin L(v_3)$ . So there exist  $a \in S$  and  $b \in S$  such that  $a \in L(v_3), b \in L(v_4)$ , for example,  $c_{32} = a, c_{42} = b$ .

Subcase 3.1. If  $a, b \in S \setminus \{5, 6\}$ , then by letting  $c'(v_1) = 5, c'(v_2) = 2, c'(v_3) = a, c'(v_4) = b, c'(v_i) = l_i$ , where,  $l_i \in \{1, 3, 4\} \cap L(v_i), i \in \{5, 6, \dots, 12\}$  we obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  (which is exhibited below), a contradiction.

*Subcase 3.2.* If  $a \in \{5, 6\}$  and  $b \notin \{5, 6\}$ , or  $a \notin \{5, 6\}$  and  $b \in \{5, 6\}$ , for example  $a = 5, b \in S \setminus \{5, 6\}$ , then by letting  $c'(v_3) = 5, c'(v_1) = 6, c'(v_4) = b, c'(v_2) = 2, c'(v_i) = l_i$ , where,  $l_i \in \{1, 3, 4\} \cap L(v_i), i \in \{5, 6, \dots, 12\}$  we obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  (which is exhibited below), a contradiction.

*Subcase 3.3.* If  $a, b \in \{5, 6\}$  and  $a = b$ , for example,  $a = b = 5$ , then by letting  $c'(v_3) = c'(v_4) = 5, c'(v_1) = 6, c'(v_2) = 2, c'(v_i) = l_i$ , where,  $l_i \in \{1, 3, 4\} \cap L(v_i), i \in \{5, 6, \dots, 12\}$  we obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  (which is exhibited below), a contradiction.

$\underline{156} \quad \underline{234}$	$\underline{156} \quad \underline{234}$	$\underline{156} \quad \underline{234}$
$\underline{3a} \times \quad \underline{4b} \times$	$\underline{35} \times \quad \underline{4b} \times$	$\underline{35} \times \quad \underline{45} \times$
Subcase 3.1	Subcase 3.2	Subcase 3.3

*Subcase 3.4.* If  $a, b \in \{5, 6\}$  and  $a \neq b$ , for example,  $a = 5, b = 6$ , then we assert that  $\{c_{33}, c_{43}\} \cap S = \emptyset$  holds. Otherwise, without loss of generality, suppose  $c_{33} = d \in S$ . Noting  $d \neq 5$ , we can let  $c'(v_4) = 6, c'(v_3) = d, c'(v_1) = 5, c'(v_2) = 2, c'(v_i) = l_i$ , where,  $l_i \in \{1, 3, 4\} \cap L(v_i), i = 5, 6, \dots, 12$ , another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  appearing, a contradiction. So  $L(v_3) = \{3, 5, 1\}, L(v_4) = \{4, 6, 1\}$ . Then, we consider the following two cases.

(1) There exist two colors in  $\{2, 3, 4\}$  which can be used to color seven vertices of  $X_3$  with  $X_1$  and  $X_2$  unconsidered. Without loss of generality, suppose  $\{3, 4\}$  can be used to color  $v_i, i = 5, 6, \dots, 11$ , then by letting  $c'(v_{12,1}) = c_{12,1}, c'(v_3) = c'(v_4) = 1, c'(v_2) = 2, c'(v_1) = m, m \in \{5, 6\} \setminus c_{12,1}, c'(v_i) = l_i$ , where,  $l_i \in \{3, 4\} \cap L(v_i), i \in \{5, 6, \dots, 11\}$  we obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  (which is exhibited below), a contradiction.

(2) If any two colors in  $\{2, 3, 4\}$  can not be used to color seven vertices of  $X_3$  with the other two parts unconsidered, then by Lemma 2.10, one set of  $\{1, 2\}, \{1, 3\}, \{1, 4\}$  can be used to  $L$ -color  $X_3$  with the other two parts unconsidered. Without loss of generality, suppose  $\{1, 2\}$  can be used to color  $X_3$ . Then by letting  $c'(v_2) = 3, c'(v_3) = 5, c'(v_1) = 6, c'(v_4) = 4, c'(v_i) = l_i$ , here  $l_i \in \{1, 2\} \cap L(v_i), i \in \{5, 6, \dots, 12\}$ , we obtain another  $L$ -coloring  $c'$  of  $K_{2,2,8}$  (which is exhibited below), a contradiction.



$$\begin{array}{cc} \underline{156} & \underline{234} \end{array}$$

$$\begin{array}{cc} \underline{156} & \underline{234} \end{array}$$

$$\begin{array}{cc} \underline{351} & \underline{461} \end{array}$$

$$\begin{array}{cc} \underline{351} & \underline{461} \end{array}$$

Subcase 3.4 (1)

Subcase 3.4 (2)

Consequently,  $K_{2,2,8}$  has property  $M(3)$ . Then, we can easily prove  $K_{2,2,r}$ ,  $r \in \{4, 5, 6, 7\}$  have property  $M(3)$ . For otherwise, assume  $K_{2,2,r}$  is U3LC for some  $r \in \{4, 5, 6, 7\}$ . Since  $K_{2,2,r}$ ,  $r \in \{4, 5, 6, 7\}$  is an induced subgraph of  $K_{2,2,8}$ , by Lemma 2.3,  $K_{2,2,8}$  is U3LC, a contradiction. So,  $K_{2,2,r}$ ,  $r \in \{4, 5, 6, 7\}$  have property  $M(3)$ . This completes the proof of Theorem 2.1. ■

**Corollary 2.1.**  $m(K_{2,2,r}) = 3$ ,  $r \in \{4, 5, 6, 7, 8\}$ .

**Proof.** It follows from Theorem 1.1 and Theorem 2.1. ■

Now all the graphs in Problem 1 are proved to have the property  $M(3)$ . Therefore, we can completely characterize the U3LC complete multipartite graphs as follows.

**Theorem 2.2.** *A complete multipartite graph is U3LC if and only if it has one of the ten graphs  $K_{3,3,3}$ ,  $K_{2,4,4}$ ,  $K_{2,3,5}$ ,  $K_{2,2,9}$ ,  $K_{1,2,2,2}$ ,  $K_{1,1,2,3}$ ,  $K_{1,1,1,2,2}$ ,  $K_{1*4,6}$ ,  $K_{1*5,5}$ , and  $K_{1*6,4}$  as an induced subgraph.*

**Proof.** It follows from Theorem 1.3, Theorem 1.4, Theorem 1.5, Theorem 1.6 and Theorem 2.1. ■

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